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SUMMABILITY AND FOURIER ANALYSIS

GEORGE ULRICH BRAUER

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An integration on βN , the Stone-Cech compactification of the natural numbers N , is defined such that if s is a bounded sequence and ϕ is a summation method evaluating s to σ , $\int s d\phi = \sigma$. The Fourier transform $\hat{\phi}$ of a summation method ϕ is defined as a linear functional on a space of test functions analytic in the unit disc: if

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n, \quad |z| < 1, \quad \text{then } \phi(f) = \int \hat{f}(n)d\phi.$$

A functional which agrees with the Fourier transform of a regular summation method must annihilate the Hardy space H_1 . Our space of test functions is often the space M_p of functions $f = \sum \hat{f}(n)z^n$, analytic in the unit disc, such that

$$\|f\|_{M_p} = \limsup [(1-r) \int_0^{2\pi} |f(re^{i\theta})|^p d\theta / 2\pi]^{1/p}$$

is finite for some $p > 1$. A functional L which is well defined on a space M_p for some $p \geq 2$ such that $L(1/(1-z)) = 1$ agrees with the Fourier transform of a summation method which is slightly stronger than convergence.

Let $s = \{s_n\}$ be an infinite sequence of complex numbers, that is, a continuous function on the discrete additive semigroup of natural numbers N . The sequence s has a continuous extension s^β to βN , the Stone-Cech compactification of N (s^β takes the value ∞ if s is unbounded). Throughout the paper, the symbol βZ denotes the Stone-Cech compactification of the space Z , and the continuous extension of a function f defined on Z to βZ will be denoted by f^β ; for a description of the Stone-Cech compactification we refer the reader to [2, pp. 82-93]. We impose the norm

$$\begin{aligned} \|s\| &= \limsup |s_n| \\ &= LUB |s^\beta(\gamma)|, \quad \gamma \in \beta N - N \end{aligned}$$

on the space m_0 of bounded sequences. Thus m_0 is isometric to $C(\beta N - N)$, the ring of continuous complex functions on $\beta N - N$; sequences differing by a null sequence are identified in m_0 .

Let ϕ denote a summation method—that is, a linear functional on a subspace of m_0 . We assume that the ϕ -transform of every sequence s to which ϕ is applicable is either a continuous function on N or else a continuous function on the half open unit interval $I: [0, 1)$.

For example, if ϕ is representable by a summation matrix $A = (a_{nk})$, then the ϕ -transform of a sequence s is the sequence t given by

$$t_n = \sum_{k=0}^{\infty} a_{nk} s_k \quad n = 0, 1, \dots,$$

which is continuous function on N ; if ϕ is the Abel method \mathcal{A} , then the ϕ transform of s is the continuous function on I given by

$$t(r) = (1 - r) \sum_{n=0}^{\infty} s_n r^n \quad 0 \leq r < 1.$$

If ϕ is a regular and nonnegative summation method, then $\underline{\phi}$ is a functional of norm one on a closed subspace of m_0 . Moreover if we denote the $\underline{\phi}$ -transform of s by t then $\limsup |t|$ is a semi-norm on m_0 . Thus by the Hahn Banach theorem, the linear functional $\underline{\phi}$ may be extended to a nonnegative linear functional on m_0 which satisfies

$$(1) \quad |\underline{\phi}(s)| \leq \limsup |t|,$$

for each bounded sequence s ; we shall denote this extension of ϕ also by $\underline{\phi}$; *throughout the paper we will assume that ϕ has been extended to m_0 in such a way that (1) is fulfilled.* Such an extension is never unique, and the results to be described hold for each such extension $\underline{\phi}$:

As a linear functional on m_0 , $\underline{\phi}$ gives rise to a nonnegative measure on βN which we also denote by $\underline{\phi}$. Since $\underline{\phi}$ is a regular summation method, the measure $\underline{\phi}$ is concentrated on $\beta N - N$ - we have $\int_{\beta N} d\underline{\phi} = 1$. We shall write $\int s d\underline{\phi}$ for $\int s^{[\beta]} d\underline{\phi}$.

Using (1) we can show

REMARK. *If s is a bounded sequence and $\underline{\phi}$ is a regular non-negative summation method which is representable by either a summation matrix or a sequence-to-function transformation, then*

$$\liminf t \leq \int_{\beta N} s d\underline{\phi} \leq \limsup t,$$

where t denotes the $\underline{\phi}$ -transform of s .

The Abel summation method \mathcal{A} induces translation-invariant measures on βN . This summation method will play a vital role in our discussion of Fourier transforms of sequences.

1. L^p Spaces. If $p \geq 1$ and ϕ is a regular summation method which is representable either by a summation matrix or by a sequence-

to-function transformation, we define $L^p(\phi)$ as the space of sequences s with the property that for each $\varepsilon > 0$ there is a bounded sequence $s^{(\varepsilon)}$ such that the sequence $|s - s^{(\varepsilon)}|^p$ has a ϕ transform whose limit superior is bounded in absolute value by ε ; this definition is more restrictive than the usual definition of L^p spaces. If s is a sequence in an L^p space we define

$$\int_{\beta.N} s d\phi = \lim_{\varepsilon \rightarrow 0} \int_{\beta.N} s^{(\varepsilon)} d\phi ,$$

where $\{s^{(\varepsilon)}\}$ is a set of bounded sequences which approximate s in the sense that for each $\varepsilon > 0$, there is a bounded sequence $s^{(\varepsilon)}$ such that the limit superior of the ϕ -transform of $|s - s^{(\varepsilon)}|^p$ is less than ε in absolute value. We norm L^p by:

$$\|s\|_p = \left(\int |s|^p d\phi \right)^{1/p} = \lim_{\varepsilon \rightarrow 0} \left[\int |s|^{(\varepsilon)p} d\phi \right]^{1/p} .$$

(Clearly the limit is independent of the choice of $\{s^{(\varepsilon)}\}$).

By Holder's inequality we have that for $1 \leq q \leq p$, $L^p(\phi) \subseteq L^q(\phi)$, and moreover $\|s\|_q \leq \|s\|_p$.

As usual we identify two sequences s and t in $L^p(\phi)$ if

$$\|s - t\|_p = 0 .$$

THEOREM. *Let ϕ be a regular nonnegative summation method and let s be a sequence in $L^p(\phi)$, $p \geq 1$. Let t denote the ϕ -transform of $|s|^p$. Then*

$$\liminf t \leq \int |s|^p d\phi \leq \limsup t < \infty .$$

In particular if ϕ evaluates the sequence $|s_n|^p$ to σ , then

$$\int |s|^p d\phi = \sigma .$$

Proof. We deal only with the case where ϕ is represented by a summation matrix $A = (a_{nk})$ — the case where ϕ is representable by a sequence-to-function may be dealt with in a similar fashion. Let $s^{(\varepsilon)}$ be a set of bounded sequences approximating s , that is, for each $\varepsilon > 0$ there is a bounded sequence $s^{(\varepsilon)}$ such that

$$\limsup \sum_{k=0}^{\infty} a_{nk} |s_k - s_k^{(\varepsilon)}|^p \leq \varepsilon .$$

If we take $\varepsilon = 1$,

$$\begin{aligned}
& \limsup \sum_{k=0}^{\infty} a_{nk} |s_k|^p \\
& \leq 2^p \left[\limsup \sum_{k=0}^{\infty} a_{nk} |s_k^{(\varepsilon)}|^p \right. \\
& \quad \left. + \limsup \sum_{k=0}^{\infty} a_{nk} |s_k - s_k^{(s)}|^p \right] \\
& \leq 2^p [\limsup \sum a_{nk} |s_k^{(\varepsilon)}|^p + 1] .
\end{aligned}$$

Hence $\limsup |t|$ is finite.

Also

$$\int |s|^p dA = \lim_{\varepsilon \rightarrow 0} \int |s^{(\varepsilon)}|^p dA .$$

Since each $s^{(\varepsilon)}$ is a bounded sequence,

$$\begin{aligned}
\liminf t_n & \leq \liminf \sum a_{nk} |s_k^{(\varepsilon)}|^p + C_1 \varepsilon^{1/p} \\
& \leq \int |s^{(\varepsilon)}|^p dA + C_1 \varepsilon^{1/p} \\
& \leq \limsup \sum_{k=0}^{\infty} a_{nk} |s_k^{(\varepsilon)}|^p + C_1 \varepsilon^{1/p} \\
& \leq \limsup t_p + C_2 \varepsilon^{1/p} ,
\end{aligned}$$

where C_1 and C_2 are numbers not depending on ε . If we let ε tend to zero we have the theorem.

Holder's inequality together with the technique of the above proof may be used to yield:

THEOREM. *Let ϕ be a regular nonnegative summation method and let s be a sequence in $L^p(\phi)$ $p \geq 1$. If t denotes the ϕ -transform of s , then*

$$\liminf t \leq \int s d\phi \leq \limsup t .$$

In particular if ϕ evaluates s to σ , then $\int_{\hat{\rho}, \lambda} s d\phi = \sigma$.

2. Fourier transforms. The Fourier transform $\hat{\phi}$ of a summation method ϕ is defined as a functional on a space M of test functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ analytic in the unit disc $D: |z| < 1$, given by

$$\begin{aligned}
\hat{\phi}(f) & = \int_{\hat{\rho}, \lambda} (\hat{f}(n))^{\hat{\rho}} d\hat{\phi} \\
& = \int_{\hat{\rho}, \lambda} \hat{f}(n) d\hat{\phi} ;
\end{aligned}$$

the Fourier transform \hat{s} of a sequence $s = \{s_n\}$ is defined as the linear

functional on M given by

$$\begin{aligned} \hat{s}(f) &= \int_{\beta N} s_n^{\beta} (\hat{f}(n))^{\beta} d.\underline{\mathcal{A}} , \\ &= \int s_n \hat{f}(n) d.\underline{\mathcal{A}} , \quad f \in M , \end{aligned}$$

where $\underline{\mathcal{A}}$ is any measure on $\beta N - N$ induced by the Abel method.

The more customary definition of the Fourier transform, namely as the function of $[0, 2\pi]$ given by

$$\int_N \exp(-i n \alpha) s_n d.\underline{\mathcal{A}} , \quad 0 \leq \alpha < 2\pi ,$$

is insufficient; S. P. Lloyd has given examples of sequences s such that $|s_n| = 1$ for all α and such that $\int_N \exp(-i n \alpha) s_n d.\underline{\mathcal{A}}$ vanishes for all α cf [6]. Later we shall make some remarks about sequences s which may be written

$$s_k = \sum_n a_n \exp(i \alpha_n k) ,$$

where the Fourier coefficients a_n are given by the formulas

$$a_n = \int_{\beta N} s_k \exp(-i \alpha_n k) d.\underline{\mathcal{A}} ,$$

(that is, the sequence $s_k \exp(i \alpha k)$ is Abel summable for all α), where each α_n is a number in $[0, 2\pi)$.

By H_p , $p \geq 1$ we understand the Hardy space of functions f analytic in $D: |z| < 1$ such that $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ is bounded for $0 \leq r < 1$ [cf. 5 pp. 39].

THEOREM. *If L is a linear functional on a space of functions analytic in D which agrees with the Fourier transform $\hat{\phi}$ of a regular summation method ϕ , then*

$$(1) \quad L(f) = 0$$

for each $f \in M$ which is also in H_1 ; also

$$(3) \quad L(1/(1 - z)) = 1 .$$

Proof. If $f \in H_1$ then $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, $|z| < 1$, and $\{\hat{f}(n)\}$ is a null sequence [cf. 5 pp. 70]. Since ϕ is a regular method, ϕ must evaluate $\{\hat{f}(n)\}$ to zero. Hence $\hat{\phi}(f) = 0$ for each $f \in H_1 \cap M$. To establish (3) we simply note that since ϕ is regular, it must evaluate the sequence $\{1, 1, \dots\}$ to one, that is $\hat{\phi}(1/(1 - z)) = 1$.

Our spaces of test functions will be

(a) the space M_p , $p > 1$, of functions

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

analytic in D , such that

$$\|f\|_{M_p} = \limsup_{r \rightarrow 1^-} (1-r)^{1/p'} \left[\int_0^{2\pi} |f(r^{1/p'} \exp i\theta)|^p d\theta / 2\pi \right]^{1/p}$$

is finite through the paper the symbol p' denotes the number $p/(p-1)$:
Two functions f, g are identified in M_p in case

$$(1-r)^{p/p'} \int_0^{2\pi} |f(r^{1/p'} \exp i\theta) - g(r^{1/p'} \exp i\theta)|^p d\theta$$

tends to zero as r tends to one. We norm each space M_p by $\|\cdot\|_{M_p}$,

(b) the space of functions

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

such that

$$\|f\|_{M_\infty} = \limsup_{r \rightarrow 1^-} (1-r) |f(r \exp i\theta)|$$

is finite. We identify two functions f and g in M_∞ in case

$$(1-r) |f(r \exp i\theta) - g(r \exp i\theta)|$$

tends to zero as r tends to 1. We norm M_∞ by $\|\cdot\|_{M_\infty}$. For $1 < p < q < \infty$ we have $M_p \subseteq M_q$ of [3 pp. 623-625].

A linear functional L on a normed space M will be said to be well-defined if $L(f) = L(g)$ whenever $\|f - g\| = 0$, $f, g \in M$.

For $p > 0$ a sequence s will be said to be strongly Abel- p -summable to σ if

$$\lim_{r \rightarrow 1} (1-r) \sum_{n=0}^{\infty} |s_n - \sigma|^p r^n = 0.$$

The method of strong Abel- p -summability is regular for $p > 0$.

THEOREM. *If $2 \leq p < \infty$, and L is a well-defined linear functional on M_p such that*

$$(4) \quad L(1/1-z) = 1,$$

then there is a summation method ϕ which includes strong Abel- p -summability such that

$$\hat{\phi}(f) = L(f) \quad f \in M_p.$$

Proof. We define a summation method ϕ by $\int_{\beta_N} sd\phi = L(S)$, where $S(z) = \sum_{n=0}^{\infty} s_n z^n$, whenever the right hand is defined. If $f \in M_p$, then $L(f)$ is defined and $\hat{\phi}(f) = \int_N \hat{f}(n)d\phi = L(f)$. Now let $\{s_n\}$ be strongly Abel- p' -summable to σ . Then $(1-r) \sum |s_n - \sigma|^{p'} r^n \rightarrow 0$. Since $\sum (s_n - \sigma)z^n = S(z) - \sigma/(1-z)$ we have, by the Hausdorff-Young theorem cf [7, pp. 145], $(1-r) \int_0^{2\pi} |S(r^{1/p'} e^{i\theta}) - \sigma/(1-r^{1/p'} e^{i\theta})|^{p'} d\theta \rightarrow 0$; thus $\|S - \sigma/(1-z)\|_{M_p} = 0$. Since L is well defined,

$$L(S) = \sigma L(1/(1-z)) = \sigma$$

by (4). Hence $\int_N sd\phi = \sigma$, that is, the method ϕ includes strong-Abel- p' -summability.

Similarly

THEOREM. *If L is a well defined linear functional on M_{∞} which satisfies (4), then there is a summation ϕ which includes strong-Abel-1-summability such that $\hat{\phi}(f) = L(f)$, $f \in M_{\infty}$.*

If a summation matrix $A = (a_{n,k})$ has a sizable convergence field, then $\lim_{n \rightarrow \infty} \max_k |a_{n,k}| = 0$; for example this condition must be satisfied if A has the Borel property (cf [3]).

We denote by \hat{A} the the Fourier transform of the summation method represented by the matrix A .

THEOREM. *If $A = (a_{n,k})$ is a non-negative regular row-finite summation matrix such that $\lim_{n \rightarrow \infty} \text{l.u.b.}_k |a_{n,k}| = 0$, $a_{n0} \geq a_{n1} \geq a_{n2} \dots$, then $\hat{A}(1/(1 - ze^{i\alpha})) = 1$ or 0 according as α is or is not congruent to zero modulo 2π .*

Proof. We have $1/(1 - ze^{i\alpha}) = \sum_{n=0}^{\infty} e^{in\alpha} z^n$. If $\alpha \equiv 0 \pmod{2\pi}$, then $\hat{A}(1/(1 - ze^{i\alpha})) = 1$ by the regularity of A . If $\alpha \not\equiv 0 \pmod{2\pi}$, then since the sequence $\{a_{n,k}\}$ is nonincreasing in k ,

$$\left| \sum_{k=0}^{\infty} a_{n,k} e^{ik\alpha} \right| \leq 8a_{n0}/\eta$$

where η is the distance of the point α from the multiples of 2π . Thus A evaluates to zero each sequence $\{e^{in\alpha}\}$ such that α is not a multiple of 2π , that is, $\hat{A}(1/(1 - ze^{i\alpha})) = 0$ if $\alpha \not\equiv 0 \pmod{2\pi}$.

THEOREM. Let P denote the Norlund summation method, so that the P -transform of a sequence s is the sequence $\{\sum_{k=0}^{\infty} p_{n-k}s_k/P_n\}$, where the numbers p_n, P_n satisfy the conditions

$$P_n = \sum_{k=0}^n p_k, \quad p_k = o(1), \quad P_n \rightarrow \infty.$$

Then for almost all α in $[0, 2\pi)$

$$\hat{P}(1/1 - z \exp i\alpha) = 0.$$

This result is proved in [1, pp. 325-326].

THEOREM. If s is a sequence in $L^p(\mathcal{A})$, $1 < p \leq 2$, then \hat{s} is a bounded functional on M_p , and

$$\|\hat{s}\|^p \leq \limsup \sum_{n=0}^{\infty} (1-r) |s_n|^p r^n,$$

Proof. If $p \leq 2$, then by the Hausdorff-Young theorem

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} |\hat{f}(n)|^{p'} r^n \right)^{1/p'} \\ & \leq \left[\int_0^{2\pi} |f(r^{1/p'} \exp(i\theta))|^p d\theta/2\pi \right]^{1/p}, \quad f \in M_p. \end{aligned}$$

Hence, if $s \in L^p(\mathcal{A})$, we have by Holder's inequality

$$\begin{aligned} |\hat{s}(f)| & \leq \left| \int_{\beta_N} \{s_n \hat{f}(n)\} d\mathcal{A} \right| \\ & \leq \limsup_{r \rightarrow 1-} (1-r) \left(\sum_{n=0}^{\infty} |s_n|^p r^n \right)^{1/p} \left(\sum_{n=0}^{\infty} |\hat{f}(n)|^{p'} r^n \right)^{1/p'} \\ & \leq \|f\|_{M_p} \limsup \left[(1-r) \sum_{n=0}^{\infty} |s_n|^p r^n \right]^{1/p}. \end{aligned}$$

Since the last member is bounded, \hat{s} is a bounded functional on M_p . If s is a bounded sequence such that the sequence $\{|s_n|^p\}$ is Abel summable, then $\|\hat{s}\| \leq \|s\|_p$ — when \hat{s} is considered a linear functional on M_p .

THEOREM. If s is a sequence in $L^p(\mathcal{A})$, $2 \leq p < \infty$, then

$$\|\hat{s}\| \geq \|s\| / \limsup \sum |s_n|^p r^n,$$

when \hat{s} is considered a functional on M_p , provided that the sequence $\{|s_n|^p\}$ is not Abel summable to zero. If the sequence $\{|s_n|^p\}$ is Abel summable, then $\|\hat{s}\| \geq \|s\|$. If $\hat{s}(f) = 0$ for all $f \in M_p$, then $\|s\|_p = 0$.

Proof. We let

$$\begin{aligned} \widehat{f}(n) &= |s_n|^{p-2} \overline{s_n} & \text{if } s_n \neq 0, \\ &= 0 & \text{if } s_n = 0. \end{aligned}$$

If follows from the Hausdorff Young theorem that $f(z) = \sum \widehat{f}(n)z^n \in M_p$, and

$$\begin{aligned} \|f\|_{M_p} &\leq \limsup[(1-r) \sum |\widehat{f}(n)|^{p'} r^n]^{1/p'} \\ &= \limsup \left[(1-r) \sum_{n=0}^{\infty} |s_n|^{p'} r^n \right]^{1/p'}. \end{aligned}$$

Hence if $\|f\|_{M_p} \neq 0$,

$$\begin{aligned} \|\widehat{s}\| &\geq |\widehat{s}(f)| / \|f\|_{M_p} \\ &\geq \|s\|_p^p / \limsup[(1-r) \sum |s_n|^{p'} r^n]^{1/p'}. \end{aligned}$$

If the sequence $\{|s_n|^p\}$ is Abel summable to a nonzero value,

$$\|\widehat{s}\| \geq \|s\|_p^p / \|s\|_p^{p/p'} = \|s\|_p.$$

If \widehat{s} annihilates M_p it must annihilate the function f defined above, and thus $\|s\|_p = 0$.

We make a few remarks about the sequence s which may be written as exponential series

$$s_k = \sum_{n=0}^{\infty} a_n \exp(i\alpha_n k) \quad k = 0, 1, \dots,$$

where the numbers α_n lie in the interval $[0, 2\pi)$ and the numbers a_n are given by the formulas

$$\begin{aligned} a_n &= \int_{\beta_N} s_k \exp(-i\alpha_n k) d\underline{\mathcal{N}} \\ &= \lim_{r \rightarrow 1-} (1-r) \sum_{k=0}^{\infty} s_k \exp(-i\alpha_n k) r^k \quad n = 0, 1, \dots, \end{aligned}$$

(we assume that the sequence $\{s_k \exp(i\alpha k)\}$ is Abel summable for each α in $[0, 2\pi)$). We also have

$$a_n = \widehat{s}(1/1 - z \exp(-i\alpha_n)).$$

We have the following version of the Riesz Fisher theorem:

THEOREM. *If $\sum |a_p|^2 < \infty$, then the Fourier transforms of the exponential polynomials*

$$s_k^{(j)} = \sum_{n=0}^j a_n \exp(i\alpha_n k), \quad j = 1, 2, \dots,$$

converge to a bounded linear functional σ on M_2 , in the sense that

$$\lim_{j \rightarrow \infty} \|\sigma - \hat{s}^{(j)}\| = 0,$$

and

$$\|\sigma\|^2 = \sum_{n=1}^{\infty} |a_n|^2 = \lim_{j \rightarrow \infty} \|\hat{s}^{(j)}\|_2^2,$$

when each $\hat{s}^{(j)}$ is considered a functional on M_2 .

Proof. Let $f(z) = \sum \hat{f}(n)z^n$ be a function in M_2 . Then

$$\begin{aligned} & |\hat{s}^{(j')} (f) - \hat{s}^{(j'')} (f)| \\ &= \int_{\beta_N} \left(\sum_j^{j''} a_n \exp(i\alpha_n k) \right) \hat{f}(k) d_{\mathcal{A}} \\ &\leq \left(\int_{\beta_N} \left| \sum_j^{j''} a_n \exp(i\alpha_n k) \right|^2 d_{\mathcal{A}} \right)^{1/2} \|f\|_{M_2} \\ &\leq \left(\sum_{n=j'}^{j''} |a_n|^2 \right)^{1/2} \|f\|_{M_2}, \end{aligned}$$

which tends to zero as j' and j'' tend to infinity, where the above integration is carried out with respect to k . Therefore, for each $f \in M_2$ the sequence $\{\hat{s}^{(j)}(f)\}$ is a Cauchy sequence of numbers and hence converges. Let $\sigma(f) = \lim \hat{s}^{(j)}(f)$. It is readily verified that $\sigma(f)$ depends linearly on f . Also

$$\begin{aligned} |\sigma(f)| &= |\lim \hat{s}^{(j)}(f)| \\ &\leq \left(\sum_{n=0}^j |a_n|^2 \right)^{1/2} \|f\|_{M_2}; \end{aligned}$$

hence if we regard σ as a functional on M_2 , $\|\sigma\| < (\sum |a_j|^2)^{1/2}$. If we take

$$f(z) = \sum \hat{f}(k)z^k,$$

where

$$\hat{f}(k) = \sum_{n=0}^j a_n \exp(-i\alpha_n k),$$

then the sequence $\{|\hat{f}(k)|\}^2$ is Abel summable to $\sum_{n=1}^j |a_n|^2$; thus

$$\int_{\beta_N} |\hat{f}(k)|^2 d_{\mathcal{A}} = \|f\|_{M_2}^2 = \sum_{n=1}^j |a_n|^2.$$

Since $s^{(j)}(f) = \sum |a_n|^2$, $\|\hat{s}^{(j)}\|^2 = \sum_{n=1}^j |a_n|^2$. Since $\|\sigma\| = \lim_{j \rightarrow \infty} \|\hat{s}^{(j)}\|$, $\|\sigma\|^2 = \sum_{n=1}^{\infty} |a_n|^2$.

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