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An integration on βN, the Stone-Cech compactification of the natural numbers N, is defined such that if s is a bounded sequence and ϕ is a summation method evaluating s to σ, $\int s d\phi = \sigma$. The Fourier transform $\hat{\phi}$ of a summation method ϕ is defined as a linear functional on a space of test functions analytic in the unit disc: if

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n, \ |z| < 1, \text{ then } \hat{\phi}(f) = \int \hat{f}(n) d\phi .$$

A functional which agrees with the Fourier transform of a regular summation method must annihilate the Hardy space H_1. Our space of test functions is often the space M_p of functions $f = \sum \hat{f}(n) z^n$, analytic in the unit disc, such that

$$||f||_{M_p} = \lim\ \sup [(1-r)\int_0^{2\pi} |f(r^{1/p} e^{i\theta})|^p \ d\theta/2\pi]^{1/p}$$

is finite for some $p > 1$. A functional L which is well defined on a space M_p for some $p \geqq 2$ such that $L(1/(1-z)) = 1$ agrees with the Fourier transform of a summation method which is slightly stronger than convergence.

Let $s = \{s_n\}$ be an infinite sequence of complex numbers, that is, a continuous function on the discrete additive semigroup of natural numbers N. The sequence s has a continuous extension s^β to βN, the Stone-Cech compactification of N (s^β takes the value ∞ if s is unbounded). Throughout the paper, the symbol βZ denotes the Stone-Cech compactification of the space Z, and the continuous extension of a function f defined on Z to βZ will be denoted by f^β; for a description of the Stone-Cech compactification we refer the reader to [2, pp. 82-93]. We impose the norm

$$\begin{aligned} ||s|| &= \lim \sup |s_n| \\ &= LUB\, |s^\beta(\gamma)| , \quad \gamma \in \beta N - N \end{aligned}$$

on the space m_0 of bounded sequences. Thus m_0 is isometric to $C(\beta N - N)$, the ring of continuous complex functions on $\beta N - N$; sequences differing by a null sequence are identified in m_0.

Let ϕ denote a summation method-that is, a linear functional on a subspace of m_0. We assume that the ϕ-transform of every sequence s to which ϕ is applicable is either a continuous function on N or else a continuous function on the half open unit interval I: $[0, 1)$.

For example, if ϕ is representable by a summation matrix $A = (a_{nk})$, then the ϕ-transform of a sequence s is the sequence t given by

$$t_n = \sum_{k=0}^{\infty} a_{nk}\, s_k \qquad\qquad n = 0, 1, \cdots ,$$

which is continuous function on N; if ϕ is the Abel method $\mathscr{A}$, then the ϕ transform of s is the continuous function on I given by

$$t(r) = (1 - r) \sum_{n=0}^{\infty} s_n\, r^n \qquad\qquad 0 \leqq r < 1 .$$

If ϕ is a regular and nonnegative summation method, then $\underline{\phi}$ is a functional of norm one on a closed subspace of m_0. Moreover if we denote the $\underline{\phi}$-transform of s by t then $\lim\sup |t|$ is a semi-norm on m_0. Thus by the Hahn Banach theorem, the linear functional $\underline{\phi}$ may be extended to a nonnegative linear functional on m_0 which satisfies

$$(1) \qquad\qquad |\underline{\phi}(s)| \leqq \lim\sup |t| ,$$

for each bounded sequence s; we shall denote this extension of ϕ also by $\underline{\phi}$; *throughout the paper we will assume that* ϕ *has been extended to* m_0 *in such a way that* (1) *is fulfilled.* Such an extension is never unique, and the results to be described hold for each such extension $\underline{\phi}$:

As a linear functional on m_0, $\underline{\phi}$ gives rise to a nonnegative measure on βN which we also denote by ϕ. Since $\underline{\phi}$ is a regular summation method, the measure $\underline{\phi}$ is concentrated on $\underline{\beta}N - N$ — we have $\int_{\underline{\beta} N} d\underline{\phi} = 1$. We shall write $\int s d\underline{\phi}$ for $\int s^{[\underline{\beta}]} d\underline{\phi}$.

Using (1) we can show

REMARK. *If s is a bounded sequence and $\underline{\phi}$ is a regular nonnegative summation method which is representable by either a summation matrix or a sequence-to-function transformation, then*

$$\lim\inf t \leqq \int_{\underline{\beta} N} s d\underline{\phi} \leqq \lim\sup t ,$$

where t denotes the $\underline{\phi}$-transform of s.

The Abel summation method $\underline{\mathscr{A}}$ induces translation-invariant measures on $\underline{\beta}N$. This summation method will play a vital role in our discussion of Fourier transforms of sequences.

1. L^p Spaces. If $p \geqq 1$ and ϕ is a regular summation method which is representable either by a summation matrix or by a sequence-

to-function transformation, we define $L^p(\phi)$ as the space of sequences s with the property that for each $\varepsilon > 0$ *there is a bounded sequence* $s^{(\varepsilon)}$ *such that the sequence* $|s - s^{(\varepsilon)}|^p$ *has a* ϕ *transform whose limit superior is bounded in absolute value by* ε; this definition is more restrictive than the usual definition of L^p spaces. If s is a sequence in an L^p space we define

$$\int_{\beta N} s d\phi = \lim_{\varepsilon \to 0} \int_{\beta N} s^{(\varepsilon)} d\phi ,$$

where $\{s^{(\varepsilon)}\}$ is a set of bounded sequences which approximate s in the sense that for each $\varepsilon > 0$, there is a bounded sequence $s^{(\varepsilon)}$ such that the limit superior of the ϕ-transform of $|s - s^{(\varepsilon)}|^p$ is less than ε in absolute value. We norm L^p by:

$$||s||_p = \left(\int |s|^p \, d\phi\right)^{1/p} = \lim_{\varepsilon \to 0} \left[\int |s|^{(\varepsilon)} |^p d\phi\right]^{1/p} .$$

(Clearly the limit is independent of the choice of $\{s^{(\varepsilon)}\}$).

By Holder's inequality we have that for $1 \leqq q \leqq p$, $L^p(\phi) \subseteqq L^q(\phi)$, and moreover $||s||_q \leqq ||s||_p$.

As usual we identify two sequences s and t in $L^p(\phi)$ if

$$||s - t||_p = 0 .$$

THEOREM. *Let* ϕ *be a regular nonnegative summation method and let* s *be a sequence in* $L^p(\phi)$, $p \geqq 1$. *Let* t *denote the* ϕ*-transform of* $|s|^p$. Then

$$\liminf t \leqq \int |s|^p d\phi \leqq \limsup t < \infty .$$

In particular if ϕ evaluates the sequence $|s_n|^p$ to σ, then

$$\int |s|^p d\phi = \sigma .$$

Proof. We deal only with the case where ϕ is represented by a summation matrix $A = (a_{nk})$ — the case where ϕ is representable by a sequence-to-function may be dealt with in a similar fashion. Let $s^{(\varepsilon)}$ be a set of bounded sequences approximating s, that is, for each $\varepsilon > 0$ there is a bounded sequence $s^{(\varepsilon)}$ such that

$$\limsup \sum_{k=0}^{\infty} a_{nk} |s_k - s_k^{(\varepsilon)}|^p \leqq \varepsilon .$$

If we take $\varepsilon = 1$,

$$\begin{aligned}
&\limsup \sum_{k=0}^{\infty} a_{nk} |s_k|^p \\
&\leqq 2^p \Big[\limsup \sum_{k=0}^{\infty} a_{nk} |s_k^{(\varepsilon)}|^p \\
&\quad + \limsup \sum_{k=0}^{\infty} a_{nk} |s_k - s_k^{(\varepsilon)}|^p \Big] \\
&\leqq 2^p [\limsup \sum a_{nk} |s_k^{(\varepsilon)}|^p + 1] .
\end{aligned}$$

Hence $\limsup |t|$ is finite.

Also

$$\int |s|^p dA = \lim_{\varepsilon \to 0} \int |s^{(\varepsilon)}|^p dA .$$

Since each $s^{(\varepsilon)}$ is a bounded sequence,

$$\begin{aligned}
\liminf t_n &\leqq \liminf \sum a_{nk} |s_k^{(\varepsilon)}|^p + C_1 \varepsilon^{1/p} \\
&\leqq \int |s^{(\varepsilon)}|^p dA + C_1 \varepsilon^{1/p} \\
&\leqq \limsup \sum_{k=0}^{\infty} a_{nk} |s^{(\varepsilon)}|^p + C_1 \varepsilon^{1/p} \\
&\leqq \limsup t_p + C_2 \varepsilon^{1/p} ,
\end{aligned}$$

where C_1 and C_2 are numbers not depending on ε. If we let ε tend to zero we have the theorem.

Holder's inequality together with the technique of the above proof may be used to yield:

THEOREM. *Let ϕ be a regular nonnegative summation method and let s be a sequence in $L^p(\phi)$ $p \geqq 1$. If t denotes the ϕ-transform of s, then*

$$\liminf t \leqq \int s d\phi \leqq \limsup t .$$

In particular if ϕ evaluates s to σ, then $\int_{\beta N} s d\phi = \sigma$.

2. **Fourier transforms.** The Fourier transform $\hat{\underline{\phi}}$ of a summation method $\underline{\phi}$ is defined as a functional on a space M of test functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ analytic in the unit disc D: $|z| < 1$, given by

$$\begin{aligned}
\hat{\underline{\phi}}(f) &= \int_{\beta N - N} (\hat{f}(n))^{\beta} d\underline{\phi} \\
&= \int_{\beta N} \hat{f}(n) d\underline{\phi} ;
\end{aligned}$$

the Fourier transform $\hat{s}$ of a sequence $s = \{s_n\}$ is defined as the linear functional on M given by

$$\hat{s}(f) = \int_{\beta N} s^{\beta}(\hat{f}(n))^{\beta} d\mathscr{A} ,$$
$$= \int s_n \hat{f}(n)\, d\mathscr{A} , \quad f \in M ,$$

where $\mathscr{A}$ is any measure on $\beta N - N$ induced by the Abel method.

The more customary definition of the Fourier transform, namely as the function of $[0, 2\pi]$ given by

$$\int_N \exp(-i\ n\alpha) s_n\, d\mathscr{A} , \qquad 0 \leqq \alpha < 2\pi ,$$

is insufficient; S. P. Lloyd has given examples of sequences s such that $|s_n| = 1$ for all α and such that $\int_N \exp(-i\ n\alpha) s_n\, d\mathscr{A}$ vanishes for all α cf [6]. Later we shall make some remarks about sequences s which may be written

$$s_k = \sum_n a_n \exp(i\ \alpha_n\ k) ,$$

where the Fourier coefficients a_n are given by the formulas

$$a_n = \int_{\beta N} s_k \exp(-i\ \alpha_n k) d\mathscr{A} ,$$

(that is, the sequence $s_k \exp(i\alpha k)$ is Abel summable for all α), where each α_n is a number in $[0, 2\pi)$.

By H_p, $p \geqq 1$ we understand the Hardy space of functions f analytic in D: $|z| < 1$ such that $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ is bounded for $0 \leqq r < 1$ [cf. 5 pp. 39].

THEOREM. *If L is a linear functional on a space of functions analytic in D which agrees with the Fourier transform $\hat{\phi}$ of a regular summation method ϕ, then*

$$(1) \qquad L(f) = 0$$

for each $f \in M$ which is also in H_1; also

$$(3) \qquad L(1/(1 - z)) = 1 .$$

Proof. If $f \in H_1$ then $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, $|z| < 1$, and $\{\hat{f}(n)\}$ is a null sequence [cf. 5 pp. 70]. Since ϕ is a regular method, ϕ must evaluate $\{\hat{f}(n)\}$ to zero. Hence $\hat{\phi}(f) = 0$ for each $f \in H_1 \cap M$. To establish (3) we simply note that since ϕ is regular, it must evaluate the sequence $\{1, 1, \cdots\}$ to one, that is $\hat{\phi}(1/(1 - z)) = 1$.

Our spaces of test functions will be
(a) the space M_p, $p > 1$, of functions

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

analytic in D, such that

$$\| f \|_{M_p} = \limsup_{r \to 1-} (1 - r)^{1/p'} \left[\int_0^{2\pi} | f(r^{1/p'} \exp i\theta) |^p d\theta / 2\pi \right]^{1/p}$$

is finite-throut the paper the symbol p' denotes the number $p/(p-1)$: Two functions f, g are identified in M_p in case

$$(1 - r)^{p/p'} \int_0^{2\pi} | f(r^{1/p'} \exp i\theta) - g(^{1/}{}_p{}' \exp i\theta) |^p d\theta$$

tends to zero as r tends to one. We norm each space M_p by $\| \ \|_{M_p}$,
(b) the space of functions

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

such that

$$\| f \|_{M_\infty} = \limsup_{r \to 1-} (1 - r) | f(r \exp i\theta) |$$

is finite. We identify two functions f and g in M_∞ in case

$$(1 - r) | f(r \exp i\theta) - g(r \exp i\theta) |$$

tends to zero as r tends to 1. We norm M_∞ by $\| \ \|_{M_\infty}$. For $1 < p < q < \infty$ we have $M_p \subseteqq M_q$ of [3 pp. 623-625].

A linear functional L on a normed space M will be said to be welldefined if $L(f) = L(g)$ whenever $\| f - g \| = 0$, f, $g \in M$.

For $p > 0$ a sequence s will be said to be strongly Abel-p-summable to σ if

$$\lim_{r \to 1} (1 - r) \sum_{n=0} | s_n - \sigma |^p r^n = 0 .$$

The method of strong Abel-p-summability is regular for $p > 0$.

THEOREM. *If $2 \leqq p < \infty$, and L is a well-defined linear functional on M_p such that*

$$L(1/1 - z) = 1 , \tag{4}$$

then there is a summation method ϕ which includes strong Abel-p'-summability such that

$$\hat{\phi}(f) = L(f) \qquad f \in M_p \,.$$

Proof. We define a summation method ϕ by $\int_{\beta N} s d\phi = L(S)$, where $S(z) = \sum_{n=0}^{\infty} s_n z^n$, whenever the right hand is defined. If $f \in M_p$, then $L(f)$ is defined and $\hat{\phi}(f) = \int_N \hat{f}(n) d\phi = L(f)$. Now let $\{s_n\}$ be strongly Abel-p'-summable to σ. Then $(1 - r) \sum |s_n - \sigma|^{p'} r^n \to 0$. Since $\sum (s_n - \sigma) z^n = S(z) - \sigma/(1 - z)$ we have, by the Hausdorff-Young theorem cf [7, pp. 145], $(1 - r) \int_0^{2\pi} |S(r^{1/p'} e^{i\theta}) - \sigma/(1 - r^{1/p'} e^{i\theta})|^p d\theta \to 0$; thus $\|S - \sigma/(1 - z)\|_{M_p} = 0$. Since L is well defined,

$$L(S) = \sigma L(1/(1 - z)) = \sigma$$

by (4). Hence $\int_N s d\phi = \sigma$, that is, the method ϕ includes strong-Abel-p'-summability.

Similarly

THEOREM. *If L is a well defined linear functional on M_∞ which satisfies* (4), *then there is a summation ϕ which includes strong-Abel-1-summability such that $\hat{\phi}(f) = L(f)$, $f \in M_\infty$.*

If a summation matrix $A = (a_{nk})$ has a sizable convergence field, then $\lim_{n\to\infty} \max_k |a_{n,k}| = 0$; for example this condition must be satisfied if A has the Borel property (cf [3]).

We denote by $\hat{A}$ the the Fourier transform of the summation method represented by the matrix A.

THEOREM. *If $A = (a_{nk})$ is a non-negative regular row-finite summation matrix such that $\lim_{n\to\infty} \text{l.u.b.}_k |a_{nk}| = 0$, $a_{n0} \geqq a_{n1} \geqq a_{n2} \cdots$, then $\hat{A}(1/(1 - ze^{i\alpha}) = 1$ or 0 according as α is or is not congruent to zero modulo 2π.*

Proof. We have $1/(1 - ze^{i\alpha}) = \sum_{n=0}^{\infty} e^{in\alpha} z^n$. If $\alpha \equiv 0 \pmod{2\pi}$, then $\hat{A}(1/(1 - ze^{i\alpha})) = 1$ by the regularity of A. If $\alpha \neq 0 \pmod{2\pi}$, then since the sequence $\{a_{n\,k}\}$ is nonincreasing in k,

$$\left| \sum_{k=0}^{\infty} a_{nk} e^{ik\alpha} \right| \leqq 8 a_{n0}/\eta$$

where η is the distance of the point α from the multiples of 2π. Thus A evaluates to zero each sequence $\{e^{in\alpha}\}$ such that α is not a multiple of 2π, that is, $\hat{A}(1/(1 - ze^{i\alpha}) = 0$ if $\alpha \not\equiv 0 \pmod{2\pi}$.

THEOREM. *Let P denote the Norlund summation method, so that the P-transform of a sequence s is the sequence $\{\sum_{k=0}^{\infty} p_{n-k}s_k/P_n\}$, where the numbers p_n, P_n satisfy the conditions*

$$P_n = \sum_{k=0}^{n} p_k \ , \quad p_k = 0(1) \ , \quad P_n \longrightarrow \infty \ .$$

Then for almost all α in $[0, 2\pi)$

$$\hat{P}(1/1 - z \exp i\alpha) = 0 \ .$$

This result is proved in [1, pp. 325-326].

THEOREM. *If s is a sequence in $L^p(\mathscr{A})$, $1 < p \leqq 2$, then $\hat{s}$ is a bounded functional on M_p, and*

$$||\hat{s}||^p \leqq \limsup(1 - r) \sum_{n=0}^{\infty} |s_n|^p r^n \ ,$$

Proof. If $p \leqq 2$, then by the Hausdorff-Young theorem

$$\left(\sum_{n=0}^{\infty} |\hat{f}(n)|^{p'} r^n\right)^{1/p'} \leqq \left[\int_0^{2\pi} |f(r^{1/p'} \exp (i\theta)|^p \, d\theta/2\pi\right]^{1/p} \ , \quad f \in M_p \ .$$

Hence, if $s \in L^p(\mathscr{A})$, we have by Holder's inequality

$$\begin{aligned} |\hat{s}(f)| &\leqq \left| \int_{\beta N} \{s_n \hat{f}(n)\} d\mathscr{A} \right. \\ &\leqq \lim_{r \to 1-} \sup \, (1 - r)\left(\sum_{n=0}^{\infty} |s_n|^p r^n\right)^{1/p} \left(\sum_{n=0}^{\infty} |\hat{f}(n)|^{p'} r^n\right)^{1/p'} \\ &\leqq ||f||_{M_p} \lim \mathrm{sum}\left[(1 - r)\left(\sum_{n=0}^{\infty} |s_n|^p r^n\right)\right]^{1/p} \ . \end{aligned}$$

Since the last member is bounded, $\hat{s}$ is a bounded functional on M_p. If s is a bounded sequence such that the sequence $\{|s_n|^p\}$ is Abel summable, then $||\hat{s}|| \leqq ||s||_p$ — when $\hat{s}$ is considered a linear functional on M_p.

THEOREM. *If s is a sequence in $L^p(\mathscr{A})$ $2 \leqq p < \infty$, then*

$$||\hat{s}|| \geqq ||s||/\limsup(1 - r) \sum |s_n|^p r^n \ ,$$

when $\hat{s}$ is considered a functional on M_p, provided that the sequence $\{|s_n|^p\}$ is not Abel summable to zero. If the sequence $\{|s_n|^p\}$ is Abel summable, then $||\hat{s}|| \geqq ||s||$. If $\hat{s}(f) = 0$ for all $f \in M_p$, then $||s||_p = 0$.

Proof. We let

$$\begin{aligned}\hat{f}(n) &= |s_n|^{p-2}\overline{s_n} && \text{if } s_n \neq 0 ,\\ &= 0 && \text{if } s_n = 0 .\end{aligned}$$

If follows from the Hausdorff Young theorem that $f(z) = \sum \hat{f}(n)z^n \in M_p$, and

$$\begin{aligned}\|f\|_{M_p} &\leqq \limsup[(1-r)\sum |\hat{f}(n)|^{p'}r^n]^{1/p'}\\ &= \limsup\left[(1-r)\sum_{n=0}^{\infty}|s_n|^p r^n\right]^{1/p'} .\end{aligned}$$

Hence if $\|f\|_{M_p} \neq 0$,

$$\begin{aligned}\|\hat{s}\| &\geqq |\hat{s}(f)|/\|f\|_{M_p}\\ &\geqq \|s\|_p{}^p/\limsup[(1-r)\sum |s_n|^p r^n]^{1/p'} .\end{aligned}$$

If the sequence $\{|s_n|^p\}$ is Abel summable to a nonzero value,

$$\|\hat{s}\| \geqq \|s\|_p{}^p/\|s\|_p{}^{p/p'} = \|s\|_p .$$

If $\hat{s}$ annihilates M_p it must annihilate the function f defined above, and thus $\|s\|_p = 0$.

We make a few remarks about the sequence s which may be written as exponential series

$$s_k = \sum_{n=0}^{\infty} a_n \exp(i\alpha_n k) \qquad k = 0, 1, \cdots,$$

where the numbers α_n lie in the interval $[0, 2\pi)$ and the numbers a_n are given by the formulas

$$\begin{aligned}a_n &= \int_{\beta N} s_k \exp(-i\alpha_n k)d\mathscr{A}\\ &= \lim_{r=1-}(1-r)\sum_{n=0}^{\infty} s_k \exp(-i\alpha_n k)r^k \qquad n = 0, 1, \cdots,\end{aligned}$$

(we assume that the sequence $\{s_k \exp(i\alpha k)\}$ is Abel summable for each α in $[0, 2\pi)$). We also have

$$a_n = \hat{s}(1/1 - z\exp(-i\alpha_n)) .$$

We have the following version of the Riesz Fisher theorem:

THEOREM. *If* $\sum |a_p|^2 < \infty$, *then the Fourier transforms of the exponential polynomials*

$$s_k^{(j)} = \sum_{n=i}^{j} a_n \exp(i\alpha_n k) , \qquad j = 1, 2, \cdots,$$

converge to a bounded linear functional σ *on* M_2, *in the sense that*

$$\lim_{j\to\infty} ||\sigma - \hat{s}^{(j)}|| = 0 ,$$

and

$$||\sigma||^2 = \sum_{n=1}^{\infty} |a_n|^2 = \lim_{j=\infty} ||\hat{s}^{(j)}||_2^2 ,$$

when each $\hat{s}^{(j)}$ is considered a functional on M_2.

Proof. Let $f(z) = \sum \hat{f}(n)z^n$ be a function in M_2. Then

$$\begin{aligned} &|\hat{s}^{(j')}(f) - \hat{s}^{(j'')}(f)| \\ &= \int_{\beta N} \Big(\sum_{j'}^{j''} a_n \exp(i\alpha_n k)\Big) \hat{f}(k) d\underline{\mathscr{A}} \\ &\leqq \Big(\int_{\beta N} \Big|\sum_{j'}^{j''} a_n \exp(i\alpha_n k)\Big|^2 d\underline{\mathscr{A}}\Big)^{1/2} ||f||_{M_2} \\ &\leqq \Big(\sum_{n=j'}^{j''} |a_n|^2\Big)^{1/2} ||f||_{M_2} , \end{aligned}$$

which tends to zero as j' and j'' tend to infinity, where the above integration is carried out with respect to k. Therefore, for each $f \in M_2$ the sequence $\{\hat{s}^{(j)}(f)\}$ is a Cauchy sequence of numbers and hence converges. Let $\sigma(f) = \lim \hat{s}^{(j)}(f)$. It is readily verified that $\sigma(f)$ depends linearly on f. Also

$$\begin{aligned} |\sigma(f)| &= |\lim \hat{s}^{(j)}(f)| \\ &\leqq \Big(\sum_{n=0}^{j} |a_n|^2\Big)^{1/2} ||f||_{M_2} ; \end{aligned}$$

hence if we regard σ as a functional on M_2, $||\sigma|| < (\sum |a_j|^2)^{1/2}$. If we take

$$f(z) = \sum \hat{f}(k)z^k ,$$

where

$$\hat{f}(k) = \sum_{n=0}^{j} a_n \exp(-i\alpha_n k) ,$$

then the sequence $\{|\hat{f}(k)|\}^2$ is Abel summable to $\sum_{n=1}^{j} |a_n|^2$; thus

$$\int_{\beta N} |\hat{f}(k)^2 d\underline{\mathscr{A}} = ||f||_{M_2}^2 = \sum_{n=1}^{j} |a_n|^2 .$$

Since $s^{(j)}(f) = \sum |a_n|^2$, $||\hat{s}^{(j)}||^2 = \sum_{n=1}^{j} |a_n|^2$. Since $||\sigma|| = \lim_{j\to\infty} ||\hat{s}^{(j)}||$, $||\sigma||^2 = \sum_{n=1}^{\infty} |a_n|^2$.

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