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**ON REMOVING COINCIDENCES OF TWO MAPS WHEN ONLY
ONE, RATHER THAN BOTH, OF THEM MAY BE DEFORMED
BY A HOMOTOPY**

ROBIN B. S. BROOKS

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OF THEM MAY BE DEFORMED BY
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It is known that if $f, g: X \rightarrow Y$ are maps of a topological space X into a topological manifold Y , and that f and g can be deformed by homotopies to maps f' and g' which are coincidence-free, then f may be deformed by a homotopy to a map f'' such that f'' and g are coincidence-free. This result is generalized as follows:

If $f, g: X \rightarrow Y$ are maps of a topological space X into a topological manifold Y and f' and g' are homotopic to f and g respectively, then for any homotopy $\{g_t\}$ from g to g' , there is a homotopy $\{f_t\}$ from f' such that the set of coincidences of f_t and g_{1-t} is the same for all $t \in [0, 1]$.

Some applications of this result to fixed point theory and root theory are indicated.

A coincidence of two maps $f, g: X \rightarrow Y$ of topological spaces is a point $x \in X$ such that $f(x) = g(x)$. Denote the set of all such points by $\Gamma(f, g)$. It may be possible to deform both f and g by homotopies to maps f' and g' such that $\Gamma(f', g')$ is empty, but still impossible to deform just f to a map f'' such that $\Gamma(f'', g)$ is empty. This is the case, for example, when $X = Y$ is the unit interval, g is the identity, and f is arbitrary. However, Fuller [4] has shown that when Y is a manifold, the ability to free f and g of coincidences by deforming both f and g is equivalent to being able to free f and g by deforming just f . The purpose of this paper is to generalize Fuller's result by showing that (in that case where Y is a manifold) any change in the coincidence set $\Gamma(f, g)$ that may be effected by deforming both f and g can also be effected by deforming just f . Precisely, we establish the following theorem.

THEOREM 1. *Suppose $f, g: X \rightarrow Y$ maps of a topological space X into a topological manifold Y , and let f' and g' be homotopic to f and g respectively. Then there is a map f'' homotopic to f' , and therefore to f , such that $\Gamma(f'', g) = \Gamma(f', g')$. In fact, given any homotopy $\{g_t | t \in I\}$ from g to g' , there is a homotopy $\{f_t | t \in I\}$ beginning at f' such that $\Gamma(f_t, g_{1-t}) = \Gamma(f', g')$ for all $t \in I$.*

Here I denotes the closed unit interval.

This theorem (with a weaker hypothesis on Y) is proved in § 2. The main tool used in its proof is the theory of numerable bundle pairs. This theory is reviewed in the first section of the paper. The last section of the paper is devoted to tracing out some of the consequences of the theorem. In particular, we show that when Y is a manifold the local fixed point theory as well as the global theory can be regarded as a special case of the local coincidence theory, and similarly for the local study of roots (study of equations of the form $f(x) = a$, where a is a given point in Y).

1. **Bundle theory.** The following treatment is largely an adaptation to the relative case of terminology and results that may be found in [5].

A *bundle pair* $(p: E \rightarrow B, p_0: E_0 \rightarrow B)$ over a space B is a pair of maps such that $E_0 \subset E$ and $p_0 = p|_{E_0}$. Two such pairs $(p: E \rightarrow B, p_0: E_0 \rightarrow B)$ and $(p': E' \rightarrow B, p'_0: E'_0 \rightarrow B)$ are *isomorphic over B* if there is an homeomorphism $h: E' \rightarrow E$ such that $p' = p \circ h$, and $h(E'_0) = E_0$.

The bundle pair (p, p_0) is *locally trivial* iff there is an open cover $\{U_\alpha | \alpha \in A\}$ of B and a family

$$\{(h_\alpha: p^{-1}(U_\alpha) \rightarrow U_\alpha \times F_\alpha, h_{0\alpha}: p_0^{-1}(U_\alpha) \rightarrow U_\alpha \times F_{0\alpha}) | \alpha \in A\}$$

of pairs of homomorphisms such that $h_{0\alpha} = h_\alpha|_{p_0^{-1}(U_\alpha)}$ and $\pi_\alpha \circ h_\alpha(e) = p(e)$ for every $\alpha \in A$ and $e \in p^{-1}(U_\alpha)$. Here $\pi_\alpha: U_\alpha \times F_\alpha \rightarrow U_\alpha$ is the indicated projection. The cover $\{U_\alpha | \alpha \in A\}$ is called a *trivializing cover*.

An open covering $\{U_\alpha | \alpha \in A\}$ of a space B is *numerable* if it is locally finite and has a partition of unity subordinate to it. A bundle pair is *numerable* if it has a numerable trivializing cover. Every locally trivial bundle pair over a paracompact space is numerable.

A bundle pair $(p': E' \rightarrow B', p'_0: E'_0 \rightarrow B')$ is a *pullback* under the map $f: B' \rightarrow B$ of a bundle pair $(p: E \rightarrow B, p_0: E_0 \rightarrow B)$ if there is a map $f^*: (E', E'_0) \rightarrow (E, E_0)$ such that $p(f^*(e)) = f(p'(e))$ for all $e \in E'$, and if $(p'': E'' \rightarrow B', p''_0: E''_0 \rightarrow B')$ is any other bundle pair over B' , and $f^{**}: (E'', E''_0) \rightarrow (E, E_0)$ a map such that $p(f^{**}(e)) = f(p'(e))$ for all $e \in E''$, then there is a unique map $u: (E'', E''_0) \rightarrow (E', E'_0)$ such that $f^{**} = f^* \circ u$, and $p'(u(e)) = p''(e)$ for all for $e \in E''$. Any two pullbacks of (p, p_0) under f are isomorphic over B' . Given $f: B' \rightarrow B$ one may always construct a pullback $(p': E' \rightarrow B', p'_0: E'_0 \rightarrow B')$ and a map $f^*: (E', E'_0) \rightarrow (E, E_0)$ by letting $E' = \{(b', e) \in B' \times E | p(e) = f(b')\}$, $p'(b', e) = b'$ and $f^*(b', e) = e$ for all $(b', e) \in E'$, $E'_0 = f^{*-1}(E_0)$, and $p'_0 = p'|_{E'_0}$.

A pullback of a locally trivial (numerable) bundle pair is locally trivial (numerable).

The main result we need from bundle theory is Theorem 2 below.

The corresponding result for principle G bundles is proved in [5, p. 50]. The same proof carries over to the case of numerable bundles, if one substitutes Exercise 5 on page 22 of [5] for Lemma 4.1 of [5]. The proof for numerable bundle pairs is essentially the same as the proof for numerable bundles. Therefore, we state the result without further indication of proof.

THEOREM 2. *Suppose (p, p_0) is a numerable bundle pair over the cartesian product $B \times I$ of a space B with the unit interval I . Let $r: B \times I \rightarrow B \times I$ be the map defined by $r(b, t) = (b, 0)$ for all $(b, t) \in B \times I$. Then any pullback under r of the bundle pair (p, p_0) is isomorphic over $B \times I$ to (p, p_0) .*

We will be concerned with pullbacks of the bundle pair $(q: Y \times Y \rightarrow Y, q_0: D(Y) \rightarrow Y)$ where Y is a topological space, $D(Y) = \{y, y\} | y \in Y\}$ is the diagonal in $Y \times Y$, and $q: Y \times Y \rightarrow Y$ is the projection onto the first factor. The following proposition gives a necessary and sufficient condition for (q, q_0) to be locally trivial (numerable). The condition may be thought of as a sort of local homogeneity condition.

PROPOSITION 1. *The bundle pair (q, q_0) is locally trivial (numerable) iff there is a (numerable) open cover $\{U_\alpha | \alpha \in A\}$ of Y and for each $\alpha \in A$ a continuous family $\{\phi_{\alpha xy} | (x, y) \in U_\alpha \times U_\alpha\}$ of homeomorphisms $\phi_{\alpha xy}: Y \rightarrow Y$ such that $\phi_{\alpha xy}(x) = y$ for every $\alpha \in A$ and $(x, y) \in U_\alpha \times U_\alpha$.*

(The family $\{\phi_{\alpha xy} | (x, y) \in U_\alpha \times U_\alpha\}$ is continuous if $\phi_{\alpha xy}(z)$ is a continuous function of the triple (x, y, z) .)

Proof. Suppose first that such a (numerable) covering and indexed family of homeomorphisms exists. For each $\alpha \in A$ choose an element $u_\alpha \in U_\alpha$ and let $F_\alpha = Y$ and $F_{0\alpha} = \{u_\alpha\}$. Define $h_\alpha: q^{-1}(U_\alpha) \rightarrow U_\alpha \times F_\alpha$ by $h_\alpha(x, y) = (x, \phi_{\alpha xu_\alpha}(y))$. Then h_α is a homeomorphism, h_α followed by projection onto U_α is simply $q|q^{-1}(U_\alpha)$, and $h_\alpha(x, y) = u_\alpha$ when $x = y$, so $h_\alpha(q_0^{-1}(U_\alpha)) = F_{0\alpha}$. Thus $\{U_\alpha | \alpha \in A\}$ is a (numerable) trivializing cover for (q, q_0) . Conversely, suppose $\{U_\alpha | \alpha \in A\}$ a (numerable) open cover of Y and $\{(h_\alpha: q^{-1}(U_\alpha) \rightarrow U_\alpha \times F_\alpha, h_{0\alpha}: q_0^{-1}(U_\alpha) \rightarrow U_\alpha \times F_{0\alpha}) | \alpha \in A\}$ a family of pairs of homeomorphisms such that $h_{0\alpha} = h_\alpha|U_\alpha \times F_{0\alpha}$ and $\pi_\alpha \circ h_\alpha(x, y) = q(x)$ for each $\alpha \in A$ and $(x, y) \in q^{-1}(U_\alpha)$, where $\pi_\alpha: U_\alpha \times F_\alpha \rightarrow U_\alpha$ is the projection. For each $\alpha \in A$ and $(x, y) \in U_\alpha \times U_\alpha$ define $\phi_{\alpha xy}: Y \rightarrow Y$ by $\phi_{\alpha xy}(z) = \pi'_\alpha \circ h_\alpha^{-1}(y, \pi''_\alpha \circ h_\alpha(x, z))$ where $\pi'_\alpha: U_\alpha \times Y \rightarrow Y$ and $\pi''_\alpha: U_\alpha \times F_\alpha \rightarrow F_\alpha$ are the indicated projections. The map $\phi_{\alpha xy}$ is illustrated below. Then $\phi_{\alpha xy}$ is a homeomorphism, in fact $\phi_{\alpha xy}^{-1} = \phi_{\alpha yx}$, for each $\alpha \in A$ and $(x, y) \in U_\alpha \times U_\alpha$. Moreover $\phi_{\alpha xy}(z)$ is continuous in x, y, z simultaneously. Finally, suppose $z = x$. Then $(x, z) \in D(Y)$. This

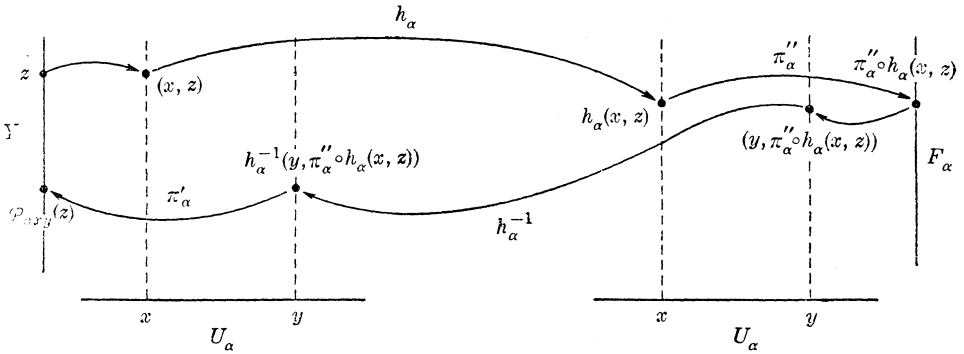


FIG. 1

implies that $h_\alpha(x, z) \in \{x\} \times F_0$, so $\pi''_\alpha \circ h_\alpha(x, z) \in F_0$, and therefore

$$(y, \pi''_\alpha \circ h_\alpha(x, z)) \in \{y\} \times F_0 .$$

Hence $h_\alpha^{-1}(y, \pi''_\alpha \circ h_\alpha(x, z)) \in D(Y)$, and therefore $\pi'_\alpha(h_\alpha^{-1}(y, \pi''_\alpha \circ h_\alpha(x, z))) = y$. Thus $\phi_{\alpha xy}(x) = y$ for all $\alpha \in A$ and $(x, y) \in U_\alpha \times U_\alpha$.

Note that if Y is a topological group then we may define $\phi_{xy}: Y \rightarrow Y$ by $\phi_{xy}(z) = yx^{-1}z$ for all $(x, y, z) \in Y \times Y \times Y$, so we obtain a trivializing cover for (q, q_0) consisting of Y alone. More important is the case where Y is a manifold.

PROPOSITION 2. *If Y is a topological manifold, then $(q: Y \times Y \rightarrow Y, q_0: D(Y) \rightarrow Y)$ is numerable.*

Proof. Let Y be a topological manifold. Then there is a family $\{\psi_\alpha: U_\alpha \rightarrow \mathbf{R}^n | \alpha \in A\}$ of charts for Y such that $\{U_\alpha | \alpha \in A\}$ is numerable. For each $\alpha \in A$ and $(x, y) \in U_\alpha \times U_\alpha$ define $\phi_{\alpha xy}: Y \rightarrow Y$ by

$$\phi_{\alpha xy}(z) = \begin{cases} z, & \text{if } z \in Y - U_\alpha \\ \psi_\alpha^{-1}\left(\psi_\alpha(z) + \frac{\psi_\alpha(y) - \psi_\alpha(x)}{1 + |\psi_\alpha(z) - \psi_\alpha(x)|}\right), & \text{if } z \in U_\alpha \end{cases}$$

and then apply the preceding proposition.

2. Proof of Theorem 1. Throughout this section we will assume that either the bundle pair $(q: Y \times Y \rightarrow Y, q_0: D(Y) \rightarrow Y)$ is numerable, or that it is locally trivial and X is a paracompact space. A necessary and sufficient condition for the local triviality (numeration) of (q, q_0) has been given in Proposition 1. In particular, (q, q_0) will be numerable if Y is a topological manifold.

LEMMA. *Suppose that $\{g_t: X \rightarrow Y | t \in I\}$ is a homotopy. Then there*

is an isotopy $\{h_t: X \times Y \rightarrow X \times Y | t \in I\}$ such that

$$h_t(x, g_0(x)) = (x, g_t(x))$$

for all $x \in X$ and $t \in I$.

Proof. Define $G: X \times I \rightarrow Y$ by $G(x, t) = g_t(x)$ for all $(x, t) \in X \times I$. Let $p: X \times I \times Y \rightarrow X \times I$ be the projection and $p_0: \text{Graph}(G) \rightarrow X \times I$ its restriction to the graph of G , i.e., the set of all triples $(x, t, G(x, t))$ with $(x, t) \in X \times I$. Then it is easy to verify that (p, p_0) is a pullback under G of (q, q_0) . (To see this, define $G^*: (X \times I \times Y, \text{Graph}(G)) \rightarrow (Y \times Y, D(Y))$ by $G^*(x, t, y) = (G(x, t), y)$. Therefore, under the assumptions at the beginning of this section, (p, p_0) is numerable. Define $r: X \times I \rightarrow X \times I$ by $r(x, t) = (x, 0)$, and let p'_0 be the restriction of p to the space $\{(x, t, y) \in X \times I \times Y | y = g_0(x)\}$. Then (p, p'_0) is a pullback under r of (p, p_0) . (Let $r^*(x, t, y) = (x, 0, y)$.) Hence, by Theorem 2, (p, p'_0) is isomorphic to (p, p_0) . There is therefore a homeomorphism $H: X \times I \times Y \rightarrow X \times I \times Y$ such that $H(x, t, g_0(x)) = (x, t, G(x, t))$ for every $(x, t) \in X \times I$, and $p \circ H = p$. Thus, if we write $h_t(x, y)$ for the projection of $H(x, t, y)$ in $X \times Y$ we obtain the desired isotopy $\{h_t: X \times Y \rightarrow X \times Y | t \in I\}$.

We turn now to the proof of Theorem 1. We are given maps $f, g: X \rightarrow Y$ and maps $f', g': X \rightarrow Y$ homotopic to f and g respectively. We are also given a specific homotopy $\{g_t | t \in I\}$ from g to g' . We must find a homotopy $\{f_t: X \rightarrow Y | t \in I\}$ beginning at f' such that $\Gamma(f_t, g_{1-t}) = \Gamma(f', g')$ for all $t \in I$.

Let $\{h_t: X \times Y \rightarrow X \times Y | t \in I\}$ be an isotopy with the properties given in the lemma. Let $\pi: X \times Y \rightarrow Y$ be the projection. Then we define f_t by

$$f_t = \pi \circ h_{1-t} \circ h_1^{-1}(x, f'(x))$$

for every $x \in X$ and $t \in I$. To see that $\{f_t | t \in I\}$ is the desired homotopy note first that $f_0(x) = \pi \circ h_1 \circ h_1^{-1}(x, f'(x)) = f'(x)$ for every $x \in X$, so $\{f_t | t \in I\}$ begins at f' . It remains to show that $\Gamma(f_t, g_{1-t}) = \Gamma(f_0, g_1)$, for all $t \in I$. Let $t \in I$, and suppose first that $x \in \Gamma(f_t, g_{1-t})$, so $f_t(x) = g_{1-t}(x)$. Then

$$\begin{aligned} f_0(x) &= \pi \circ h_1 \circ h_{1-t}^{-1} \circ h_{1-t} \circ h_1^{-1}(x, f_0(x)) \\ &= \pi \circ h_1 \circ h_{1-t}^{-1}(x, f_t(x)) \\ &= \pi \circ h_1 \circ h_{1-t}^{-1}(x, g_{1-t}(x)) \\ &= \pi \circ h_1(x, g_0(x)) \\ &= \pi(x, g_1(x)) \\ &= g_1(x) , \end{aligned}$$

so $x \in \Gamma(f_0, g_1)$. Conversely, suppose $x \in \Gamma(f_0, g_1)$ so $f_0(x) = g_1(x)$. Then

$$\begin{aligned} f_t(x) &= \pi \circ h_{1-t} \circ h_1^{-1}(x, f'(x)) \\ &= \pi \circ h_{1-t} \circ h_1^{-1}(x, f_0(x)) \\ &= \pi \circ h_{1-t} \circ h_1^{-1}(x, g_1(x)) \\ &= \pi \circ h_{1-t}(x, g_0(x)) \\ &= \pi(x, g_{1-t}(x)) \\ &= g_{1-t}(x), \end{aligned}$$

so $x \in \Gamma(f_t, g_{1-t})$.

3. Applications to coincidence theory. In this section we use some of the notation, definitions, and elementary results that are summarized in the second section of [3]. In particular, we will regard a homotopy F of maps from X into Y as a function from the unit interval into $\text{Map}(X, Y)$, with the proviso that $F(t)(x)$ be continuous in both its arguments. Throughout this section Y will be a topological manifold (or, more generally, the bundle pair $(p: Y \times Y \rightarrow p_0: D(Y) \rightarrow Y)$ will be assumed to be numerable.) We will let $f, g: X \rightarrow Y$ be maps. The class of all pairs (F, G) of homotopies of maps from X into Y will be denoted, as in [3], by Δ_1 ; the subset of Δ_1 consisting of those pairs for which G is the constant homotopy at g will be denoted by Δ_1^* . Let C be the minimum number of coincidence of f' and g' when f' and g' are allowed to vary over all maps homotopic to f and g respectively; let C^* be the minimum number when just f' is allowed to vary but g' must remain fixed at g . Then Theorem 1 has the following immediate corollary.

COROLLARY 1. $C = C^*$.

We turn now to essentiality of coincidences and the Nielsen numbers.

COROLLARY 2. $x \in \Gamma(f, g)$ is Δ_1 essential iff it is Δ_1^* essential.

Proof. If f is Δ_1 essential it is certainly Δ_1^* essential. Conversely, suppose $x \in \Gamma(f, g)$ is Δ_1^* essential, and let F and G be homotopies beginning at f and g respectively. According to Theorem 1, there is a homotopy F' beginning at $F(1)$ such that for any $x \in X$ and any $t \in I$ we have $F'(t)(x) = G(1-t)(x)$ iff $F(1)(x) = G(1)(x)$. Since x is Δ_1^* essential it is $F'F', g$ related to a point $x' \in \Gamma(F'(1), g)$. Since $F'(1)(x) = g(x) = G(0)(x)$, we have $F'(1-t)(x') = G(t)(x')$ for all $t \in I$. Thus $x' \in \Gamma(F(1), G(1))$ and is F'^{-1}, G related to itself. (The constant path at

x' is a path for which $[\langle F'^{-1}, x' \rangle] = [\langle G, x' \rangle]$. It follows that x is $(FF')F'^{-1}, gG$ related to x' . But $[(FF')F'^{-1}] = [F]$ and $[gG] = [G]$, so x is F, G related to x' . It follows that x is Δ_1 -essential.

COROLLARY 3. $N(f, g, \Delta_1) = N(f, g, \Delta_1^*)$, i.e., the Δ_1 and Δ_1^* Nielsen numbers of f and g are equal.

Proof. By Corollary 2, each Δ_1^* -essential coincidence class of f and g is also Δ_1 -essential and conversely.

In the study of fixed points one is interested in finding a lower bound for the number of solutions to equations of the form $f(x) = x$, where $f: Y \rightarrow Y$ is allowed to vary throughout a homotopy class. In the study of roots one wants a lower bound for the number of solutions to equations of the form $f(x) = a$, where $f: X \rightarrow Y$ is again allowed to vary throughout a homotopy class, and a is a given point in Y . In general coincidence theory, one wants a lower bound to the number of solutions to an equation of the form $f(x) = g(x)$, where $f: X \rightarrow Y$ is allowed to vary throughout one homotopy class and $g: X \rightarrow Y$ throughout another. The first two equations are special cases of the third (let $g = 1_Y$ in the first and g be the constant map in the second). But the first two theories are not, in general, special cases of the third—since in the third g as well as f is allowed to vary by a homotopy. However our results show that when Y is a manifold the first two theories are indeed special cases of the third. As an example we give an alternative proof of the following result announced in [2].

PROPOSITION 3. *Suppose $f: X \rightarrow Y$ a map of a topological space X into a topological manifold Y , and suppose $a \in Y$. Then if the equation $f(x) = a$ has at least one Δ_2 -essential solution (Δ_2 is the class of all pairs (F, G) of homotopies of maps from X into Y in which G is the constant homotopy at the constant map into a), then it has at least $R(f)$ Δ_2 -essential solutions, where $R(f)$ is the order of the cokernel of the fundamental group homomorphism $f_*: \pi(X) \rightarrow \pi(Y)$ induced by f .*

Proof. Since $N(f, g, \Delta_2) > 0$, there is an $x_0 \in X$ with $f(x_0) = a$. Base the fundamental groups of X and Y at x_0 and a respectively. Let $g: X \rightarrow Y$ be the constant map into $a \in Y$. Then Δ_1^* is Δ_2 , so Corollary 3 implies $N(f, g, \Delta_1) = N(f, g, \Delta_2) > 0$. Thus, according to Theorem 1 of [3], $N(f, g, \Delta_1) \geq J(f, g, \Delta_1)$, where, since $g_*: \pi(X, x_0) \rightarrow \pi(Y, a)$ is trivial, $J(f, g, \Delta_1)$ is the number of elements of the cokernel of f_* that have representatives of the form $[\langle F, x_0 \rangle][\langle G, x_0 \rangle]$, where F is a homotopy from f to itself and G is a homotopy from g to it-

self. But since g is the constant map, every loop C in Y at a is of the form $[\langle G, x_0 \rangle]$ where G is a homotopy from g to itself (set $G(t)(x) = C(t)$ for all $x \in X$). Thus $J(f, g, \Delta_1) = \text{order cokernel } f_* = R(f)$, so $N(f, g, \Delta_2) \geq R(f)$. On the other hand [3, p. 557], we always have $R(f) \geq N(f, g, \Delta_2)$, so $R(f) = N(f, g, \Delta_2)$.

We may also combine our results with one due to Schirmer [6] to provide a partial converse to Proposition 3.

PROPOSITION 4. *Suppose X and Y are triangulable manifolds of the same dimension ≥ 3 . Then there is a map f'' homotopic to f such that $f''(x) = a$ has exactly $N(f, g, \Delta_2)$ solutions, so, in particular, if $N(f, g, \Delta_2) > 0$ then $f''(x) = a$ has exactly $R(f)$ solutions.*

Proof. In [1, p. 109] a result of Schirmer's is translated into our terminology to give the following result: there is a map f' homotopic to f and a map g' homotopic to the constant map g such that $f'(x) = g'(x)$ has exactly $N(f, g, \Delta_1)$ solutions. Theorem 1 and Corollary 3 therefore imply that there is a map f'' homotopic to f' and therefore to f such that $f''(x) = a$ has exactly $N(f, g, \Delta_2) = N(f, g, \Delta_1)$ solutions.

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