EVERY GENERALIZED PETERSEN GRAPH HAS A TAIT COLORING

Frank Castagna and Geert Caleb Ernst Prins
EVERY GENERALIZED PETERSEN GRAPH HAS
A TAIT COLORING

FRANK CASTAGNA AND GEERT PRINS

Watkins has defined a family of graphs which he calls
generalized Petersen graphs. He conjectures that all but the
original Petersen graph have a Tait coloring, and proves the
conjecture for a large number of these graphs. In this paper
it is shown that the conjecture is indeed true.

DEFINITIONS. Let \( n \) and \( k \) be positive integers, \( k \leq n - 1, n \neq 2k \). The generalized Petersen graph \( G(n, k) \) has \( 2n \) vertices, denoted by \( \{0, 1, 2, \ldots, n - 1; 0', 1', 2', \ldots, (n - 1)\} \) and all edges of the
form \( (i, i + 1), (i, i'), (i', (i + k)) \) for \( 0 \leq i \leq n - 1 \), where all numbers
are read modulo \( n \). \( G(5, 2) \) is the Petersen graph. See Watkins [2].

The sets of edges \( \{(i, i + 1)\} \) and \( \{(i', (i + k))\} \) are called the
outer and inner rims respectively and the edges \( (i, i') \) are called the
spokes.

A Tait coloring of a trivalent graph is an edge-coloring in three
colors such that each color is incident to each vertex. A 2-factor
of a graph is a bivalent spanning subgraph. A 2-factor consists of dis-
joint circuits. A Tait cycle of a trivalent graph is a 2-factor all of
whose circuits have even length. A Tait cycle induces a Tait coloring
and conversely.

The method that Watkins used in proving that many generalized
Petersen graphs have a Tait coloring was to prove that certain color
patterns on the spokes induce a Tait coloring. Our method for the
remaining cases consists of the construction of 2-factors and of proof
that these 2-factors are Tait cycles under appropriate conditions.

We restrict ourselves to the generalized Petersen graphs \( G(n, k) \)
with the properties:

\[
n \text{ odd, } n \geq 7, \ (n, k) = 1, \text{ and } 2 < k < \frac{n - 1}{2}.
\]

All other cases (and some special instances of the above) were
dealt with by Watkins.

We construct three types of 2-factors. The first type is a Tait
cycle when \( k \) is odd. The second type is a Tait cycle when \( k \) is even
and \( n \equiv 3(\text{mod} \ 4) \) and also when \( k \) is even and \( n \equiv 1(\text{mod} \ 4) \) with \( k^{-1} \)
even. (As \( (n, k) = 1 \), we define \( k^{-1} \) as the unique positive integer
\( < n \), for which \( kk^{-1} \equiv 1(\text{mod} \ n) \).) The third type takes care of the
remaining graphs.
The principal tool in the proofs is the automorphism $\varphi$ (henceforth fixed) of $G(n, k)$ defined by $\varphi(i) = n - i; \varphi(i') = (n - i)$. In each case $\varphi$ induces an automorphism (also called $\varphi$) of the constructed 2-factor. To facilitate notation we write $n = 2m + 1$.

**CONSTRUCTION 1.** The subgraph $H$ of $G(n, k)$ has the following edges:

(a) On the outer rim: $(m + k, m + k + 1), (m + k + 1, m + k + 2), \cdots, (n - 1, 0), (0, 1), (1, 2), \cdots, (m - k, m - k + 1), (m - k + 2, m - k + 3), (m - k + 4, m - k + 5), \cdots, (m + k - 2, m + k - 1)$.

The last line may be written as $(m - k + 2j, m - k + 2j + 1), 1 \leq j \leq k - 1$.

(b) Spokes: $(m + k, (m + k)'), (m - k + 1, (m - k + 1)'), (m - k + 2, (m - k + 2)'), \cdots, (m + k - 1, (m + k - 1)')$.

(c) On the inner rim: $(i', (i + k)'), m + 1 \leq i \leq n - 1 (i', (i' - k)'), k \leq i \leq m$.

**EXAMPLE.** $G(11, 3)$

![Figure 1](image)

Clearly $H$ is a 2-factor, and $\varphi(H) = H$. If $C_o$ is the circuit of $H$ which contains 0, then $\varphi(C_o) = C_o$. If $C_o$ has odd length, then it must contain an odd number of edges of the form $(i, -i)$ and $(i', -i')$. The only candidates are:

(A) $(m, m + 1)$

(B) $\left(\left(\frac{n - k}{2}\right)', \left(\frac{k}{2}\right)\right)$

(C) $\left(\left(\frac{n - k}{2}\right)', \left(\frac{n + k}{2}\right)\right)$.
EVERY GENERALIZED PETERSEN GRAPH HAS A TAIT COLORING

The edge (C) is not in $H$ by our construction. Either the presence of (A) in $H$ or the existence of edge (B) will imply that $k$ is even. We conclude that if $k$ is odd $C_o$ has even length.

Let $m - k + 2 \leq i \leq m + k - 1$. Then either $i', i, i + 1, (i + 1)'$ or $i', i, i - 1, (i - 1)'$ are 4 consecutive vertices on a circuit of $H$. We call such sets 4-sets. If every point of a circuit is on a 4-set, then the circuit has even length.

Now consider a vertex $i'$, $m + k < i \leq n - 1$ or $0 \leq i < m - k + 1$, which is not on $C_o$. The circuit of $H$ which contains $i'$ passes consecutively through the vertices $i', (i + k)', (i + 2k)' \cdots (i + rk)', (i + (r + 1)k)'$, where $i + rk < m - k + 1$, $i + (r + 1)k > m - k + 1$, $r \geq 0$. The vertex $(i + (r + 1)k)'$ is on a 4-set, and also $i + (r + 1)k \leq m$, hence the circuit continues through the vertices $i + (r + 1)k$, $i + (r + 1)k \pm 1, (i + (r + 1)k \pm 1)', (i + rk \pm 1)' \cdots (i \pm 1)'$. The circuit continues to $(i \pm 1 - k)'$ and by an identical argument eventually returns and hits $i'$ or $(i + 2)'$ or $(i - 2)'$. In the first case the circuit is complete and it is easily seen that it has even length. The other two cases lead to a contradiction; for assume (w.l.o.g) that the circuit is on $(i', (i + 1)', (i + 2)')$. Then by the above argument the circuit will eventually hit either $(i + 1)'$ again or else $(i + 3)'$. But the first case is impossible, because $H$ is bivalent. Hence the circuit contains $(i + 3)'$ and further $(i + 4)' \cdots (m - k + 1)'$, but this contradicts our assumption, as $(m - k + 1)'$ is on $C_o$.

CONSTRUCTION 2. $H$ has the following edges:
(a) On the outer rim: $(n - 1, 0), (0, 1), (2, 3), \cdots, (2j, 2j + 1) \cdots (n - 3, n - 2)$.
(b) Spokes: all, except $(0,0')$.
(c) On the inner rim: $(0', k'), (2k', 3k'), \cdots (2jk', (2j + 1)k'), \cdots, ((n - 1)k', 0')$.

(For the sake of clarity we have written $ck'$ instead of the formally more correct $(ck)'$)

EXAMPLE. $G(15, 4)$. See Figure 2.

Again, one checks easily that $H$ is a 2-factor and that $\varphi(H) = H$. Looking at the edges (A), (B), and (C) of Construction 1, we note that (C) is not an edge if $k$ is even. If edge (A) occurs, then $m = (n - 1)/2$ is even and $n \equiv 1(mod 4)$. If edge (B) occurs, and we write $k/2 \equiv jk(mod n)$, $j < n$, then $j$ is odd by our construction. But then $k \equiv 2jk(mod n) \equiv (2j - 1) \equiv 0 (mod n) \Rightarrow n = 2j - 1 \equiv 1 (mod 4)$.

Hence if $n \equiv 3(mod 4)$ and $k$ is even none of the lines (A), (B), and (C) occur, and we may conclude by the argument used in Construction 1 that the circuits through 0 and 0' have even length. All
the points of every other circuit belong to a 4-set, and hence also have even length. Therefore $H$ if a Tait cycle if $n \equiv 3 \pmod{4}$ and $k$ is even.

If $n \equiv 1 \pmod{4}$ and $k$ and $k^{-1}$ are both even, then the edge $((k + 1)', 1') = (1', (k + 1)) = (k^{-1}k', (k^{-1} + 1)k')$ exists in $H$, and so does the edge $(-1', -(k + 1))$. We then obtain the circuit:

$$0', k', k, k + 1, (k + 1)', 1', 1, 0, -1, -1',$$
$$-(k + 1)', -(k + 1), -k, -k', 0'$$

which has length 14 and contains both 0 and 0'.

We conclude that in this case $H$ is again a Tait cycle.

CONSTRUCTION 3. For this construction we assume $n \equiv 1 \pmod{4}$, $k$ even, $k^{-1}$ odd and $> n/2$. This last assumption is no real restriction, because if $k^{-1}$ is odd and $< n/2$, then Construction 1 gives a Tait cycle for $G(n, k^{-1})$ and Watkins has shown that $G(n, k)$ and $G(n, k^{-1})$ are isomorphic. Finally we need to assume $k > 2$; this restriction was not needed in Constructions 1 and 2.

$H$ has the following edges:

On the outer rim: $(-1, 0), (0, 1), (2, 3), \cdots, (k - 4, k - 3), (k - 2, k - 1), (k - 1, k), (k + 1, k + 2), \cdots, (n - k - 2, n - k - 1), (n - k, n - k + 1), (n - k + 1, n - k + 2), (n - k + 3, n - k + 4), \cdots, (n - 3, n - 2)$.

Spokes: all except $(0'0'), (k - 1, (k - 1)'), (n - k + 1, (n - k + 1))$.

On the inner rim: $(0', k'), (2k', 3k'), \cdots, ((n - k^{-1})k', (n - k^{-1} + 1)k'), ((n - k^{-1} + 1)k', (n - k^{-1} + 2)k'), ((n - k^{-1} + 3)k', (n - k^{-1} + 4)k'), \cdots, ((k^{-1} - 2)k', (k^{-1} - 1)k'), ((k^{-1} - 1)k', k^{-1}k'), ((k^{-1} + 1)k', (k^{-1} + 2)k'), \cdots, ((n - 1)k', 0')$.
EVERY GENERALIZED PETERSEN GRAPH HAS A TAIT COLORING

EXAMPLE. $G(17, 4)$

$H$ is a 2-factor, as long as $n - k^{-1} + 1 < k^{-1} - 1$, which assures that the constructed edges on the inner rim cover all vertices of the inner rim. But this condition holds whenever $k^{-1} > (n + 1/2)$ or alternatively when $k^{-1} > (n/2)$, and $k > 2$. It is clear that $\varphi(H) = H$.

Since $n = 1 \pmod 4$, $m$ is even and $(m, m + 1)$ is not an edge of $H$. As $(n - k)/2$ is not an integer $H$ does not have an edge $((n - k)/2', (n + k)/2')$. Finally, since $n - k^{-1} + 1 \leq (n - 1)/2 = m < m + 1 = (n + 1)/2 \leq k^{-1} - 1$, and $m$ is even, $H$ does not contain the edge $(mk', (m + 1)k') = (-k'/2, k'/2)$. As before we conclude that the circuits containing $0$ and $0'$ have even length. The circuit containing $0$ also contains $n - 1, (n - 1)', (k - 1)'$ and $1, 1', (n - k + 1)'$, while the circuit containing $0'$ also contains $k', k, k - 1, k - 2, (k - 2)'$ and $(n - k'), n - k, n - k + 1, n - k + 2, (n - k + 2)'$. Hence the other circuits only contain vertices of 4-sets and every circuit of $H$ has even length.

We note that our constructions are not mutually exclusive. For example, Construction 1 also produces a Tait cycle, when $k$ is even, and the largest positive integer $q$ such that $qk < n$ is an odd number.

We conclude with a new conjecture. G.N. Robertson [1] has shown that $G(n, 2)$ is Hamiltonian unless $n \equiv 5 \pmod 6$. As $G(n, 2) \cong G(n, (n + 1)/2) \cong G(n, (n - 1)/2) \cong G(n, n - 2)$ (see [2]), none of these graphs has a Hamiltonian if $n \equiv 5 \pmod 6$. We conjecture that all other generalized Petersen graphs are Hamiltonian. In all examples that we have worked out $G(n, k)$ possesses a Hamiltonian $H$ with $\varphi(H) = H$, but our three constructions are Hamiltonians only in a minority of cases.
REFERENCES


Received June 16, 1970 and in revised form August 20, 1970.

WAYNE STATE UNIVERSITY
Alex Bacopoulos and Athanassios G. Kartsatos, *On polynomials approximating the solutions of nonlinear differential equations* .......... 1
Monte Boisen and Max Dean Larsen, *Prüfer and valuation rings with zero divisors* .................................................. 7
James J. Bowe, *Neat homomorphisms* .................................................. 13
David W. Boyd and Hershy Kisilevsky, *The Diophantine equation*
\[ u(u+1)(u+2)(u+3) = v(v+1)(v+2) \] .................................................. 23
George Ulrich Brauer, *Summability and Fourier analysis* .................. 33
Robin B. S. Brooks, *On removing coincidences of two maps when only one, rather than both, of them may be deformed by a homotopy* .................. 45
Frank Castagna and Geert Caleb Ernst Prins, *Every generalized Petersen graph has a Tait coloring* .................................................. 53
Micheal Neal Dyer, *Rational homology and Whitehead products* ........ 59
John Fuelberth and Mark Lawrence Teply, *The singular submodule of a finitely generated module splits off* .................................................. 73
Robert Gold, \(\Gamma\)-extensions of imaginary quadratic fields ................ 83
Myron Goldberg and John W. Moon, *Cycles in k-strong tournaments* ........ 89
Darald Joe Hartfiel and J. W. Spellman, *Diagonal similarity of irreducible matrices to row stochastic matrices* .................................................. 97
Wayland M. Hubbart, *Some results on blocks over local fields* ............ 101
Alan Loeb Kostinsky, *Projective lattices and bounded homomorphisms* .... 111
Kenneth O. Leland, *Maximum modulus theorems for algebras of operator valued functions* .................................................. 121
Jerome Irving Malitz and William Nelson Reinhardt, *Maximal models in the language with quantifier “there exist uncountably many”* ............. 139
John Douglas Moore, *Isometric immersions of space forms in space forms* .................................................. 157
Ronald C. Mullin and Ralph Gordon Stanton, *A map-theoretic approach to Davenport-Schinzel sequences* .................................................. 167
Chull Park, *On Fredholm transformations in Yeh-Wiener space* .......... 173
Stanley Poreda, *Complex Chebyshev alterations* ................................ 197
Ray C. Shiflett, *Extreme Markov operators and the orbits of Ryff* ........ 201
Robert L. Snider, *Lattices of radicals* .................................................. 207
Ralph Richard Summerhill, *Unknotting cones in the topological category* .................................................. 221
Charles Irvin Vinsonhaler, *A note on two generalizations of QF – 3* .... 229
William Patterson Wardlaw, *Defining relations for certain integrally parameterized Chevalley groups* .................................................. 235
William Jennings Wickless, *Abelian groups which admit only nilpotent multiplications* .................................................. 251