RATIONAL HOMOLOGY AND WHITEHEAD PRODUCTS

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D. W. Kahn defined a spectral sequence \( E(X; R) \) for the Postnikov system \( P(X) \) of a 1-connected CW-complex which converges to \( H_*(X; R) \), the singular homology of \( X \) with coefficients in \( R \). We study \( E(X; R) \) in two settings: (a) to give a generalization of the classical theorem of Eilenberg and MacLane concerning the dependence of \( H_i(X; Z) \) on the first nonzero homotopy group of \( X \) (2.1) and (b) to give a complete computation of \( H_t(X; Q) \) \((Q = \text{rationals})\) for \( i \leq 3 \cdot c(X) \) \((c(X) = \text{connectivity of } X)\) in terms of the graded homotopy group \( H \otimes Q = \{\pi_i(X) \otimes Q \mid 0 < i \leq 3 \cdot c(X)\} \) and the Whitehead product on this group (0.1 and 0.2).

In § 1 we give a quick description of \( E(X; R) \) for later use and in § 2 we generalize the Eilenberg-MacLane theorem by giving an exact sequence involving the first two nonzero homotopy groups. \( E(X, Q) \) is studied in § 3, with the result that we are able to identify \( E'(X; Q) \) somewhat above the diagonal (Kahn identified it below the diagonal in [7]) (3.3) and to show that the Whitehead product is the only non-zero differential operator, provided the total degree is less than \( 3 \cdot c(X) \) (3.10). Section 4 gives the computations of \( H_t(X; Q) \) and various other applications.

1. Description of the Spectral Sequence of \( P(X) \). In this note \( X \) is a \((n - 1)\)-connected space, \( n > 1 \), having the homotopy type of a CW-complex. All maps and spaces are “pointed”.

Let \( \{X_i, r_i, \pi_i\} = P(X) \) be a Postnikov system for \( X \) (see [6] for definition). Choose \( m > n \) and convert the map \( r_m: X \rightarrow X_m \) into a fiber map. Use the same notation for the new map. In the tower of spaces

\[
X \rightarrow X_m \xrightarrow{r_m} X_{m-1} \xrightarrow{r_{m-1}} \cdots \xrightarrow{r_{m-i}} X_i = K(\pi_i(X), n)
\]

\( \pi_0 \circ \cdots \circ \pi_{m-i} r_m \simeq r_{n-1} (n + 1 \leq \alpha \leq m) \). Let \( r_{n-1} \) denote this composition, \( \alpha = n + 1, \cdots, m \). Since all these maps are Hurewicz fibrations, \( r_{n-1}(\alpha - 1 < m) \) is a fiber map. Let \( F_{i+1} = r_i^{-1} \) (base point) denote the fiber of \( r_i: X \rightarrow X_i, i \leq m \). The following is proved in [7].

**Lemma 1.1.** (a) \( F_{i+1} \) is \( i \)-connected.
(b) \( F_{i+1} \) is fibered over \( K(\pi_{i+1}(X), i + 1) \), with fiber \( F_{i+2} \), via the map \( r_i \mid F_{i+1} \).
(c) \( X = F_n \supset F_{n+1} \supset \cdots \supset F_m \supset F_{m+1} \) is a finite de-
creasing filtration of $X$.

For each $m$, the exact couple ([7]) $\mathcal{E}(\mathcal{P}(X), m; G)$ is defined by

$$D_{r,s} = \begin{cases} H_{r+s}(F_r; G), & \text{if } r, s \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$E_{r,s} = \begin{cases} H_{r+s}(F_r, F_{r+1}; G), & \text{if } r, s \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $G$ is any abelian group and $H_*$ is singular homology. If $D_i = \sum_\oplus D_{r,s}, E_i = \sum_\oplus E_{r,s}$ then the couple maps $i: D^i \to D^j, j: D^i \to E^i$ and $k: E^i \to D^i$ are of bidegree (respectively) $(-1, 1), (0, 0), (1, -2)$. The bidegree of the differential operator $d_i: E^i \to E^i$ is $(i, -i - 1)$.

In [7], Kahn shows that

$$\mathcal{E}_i = \mathcal{H}_i(\mathcal{P}(X), j; G)$$

is an isomorphism, provided $s \leq j$, where

$$q_j = r_j|F_j; (F_j, F_{j+1}) \to (K(\pi_j(X), j), ^*),$$

thus indentifying the $E^i$ term below the diagonal.

2. Generalization of a theorem of Eilenberg-MacLane. In [4], Eilenberg and MacLane showed the dependence of the first few homology groups of a space $X$ upon the first nonzero homotopy group of $X$. We prove the following generalization.

**Theorem 2.1.** Let $X$ be an $(n-1)$-connected space having the homotopy type of a CW-complex, $n \geq 2$. Suppose $\pi_i(X) = 0$ for $n < i < p$ and $p < i < q \leq 2n$. Then $H_n(X; G) \approx H_n(\pi_n(X), n; G)$ for $n \leq i < p$ and any abelian group $G$. Furthermore, if we abbreviate $H_i(\pi_i(X), l; G)$ by $H_i(l; G)$, we have the exact sequence

$$H_q(n; G) \xrightarrow{\phi_q} H_{q-1}(p; G) \xrightarrow{\varphi_{q-1}} H_{q-2}(X; G) \xrightarrow{\chi_{q-2}} H_{q-1}(n; G) \xrightarrow{\phi_{q-1}} \cdots$$

$$\cdots \xrightarrow{\phi_1} H_1(p; G) \xrightarrow{\varphi_1} H_1(X; G) \xrightarrow{\chi_1} H_1(n; G) \xrightarrow{\phi_1} H_{i-1}(p; G) \xrightarrow{\varphi_1} \cdots$$

$$\cdots \xrightarrow{\phi_1} H_1(p; G) \xrightarrow{\varphi_1} H_1(X; G) \xrightarrow{\chi_1} H_1(n; G) \xrightarrow{\phi_1} H_{i-1}(p; G) \xrightarrow{\varphi_1} \cdots$$

$\Phi_i = T_i \circ (k)_*, \text{ where } k: K(\pi_p(X), n) \to K(\pi_p(X), p + 1) \text{ is the first } k\text{-invariant in a Postnikov decomposition of } X \text{ and } T_j: H_j(\pi_p(X), p + 1; G) \to H_{j-1}(\pi_p(X), p; G) \text{ is the transgression, which is an isomorphism provided } 0 < j \leq 2p$. Further, $\varphi_1$ is the Hurewicz homomorphism.

**Proof.** We consider $\mathcal{E}(\mathcal{P}(X), m; G)$ for $m > 2n$. $\pi_i(X) = 0$ for $n < i < p$, $p < i < q$ implies by 1.1 (b) that
Thus $E_{r,s}^i = 0$ for $0 \leq r < n, n < r < p, p < r < q$ and all $s$. This gives a two-term condition (see [5], chapter VIII) on the $E^i$-term of $G(\mathcal{P}(X), m; G)$. Using (1.2) we have that $H_i(X; G) \simeq H_\ast(\pi_i(X), n; G)$ for $n \leq i < p$ (a 1-term condition here) and for $p \leq i < q$ we have the exact sequence of the theorem. Note that we did not need $q \leq 2n$ in order to obtain the two-term condition, but only in order to use (1.2). It is clear from [7] that $\varphi_p$ (the edge homomorphism) is the Hurewicz homomorphism.

We will now show that $\Phi_i = T_i \circ (k)_s$. Since $\Phi_i$ is essentially $d^{p-n}: E_{p,i-s}^{p-n} \to E_{p,i-1}^{p-n}$ ([7]), we will show that $d^{p-n} = T_i \circ (k)_s$. As it has significance in its own right, we give it as a separate lemma.

**Lemma 2.3** If $\pi_i(X) = 0$ for $1 \leq i < n, n < i < p, p < i < q$, then

(a) $E_{r,s}^i = E_{r,s}^{p-n}$ for $r = n, p$ provided $s \leq q - p$.

(b) The following triangle commutes for $s \leq \min \{n, q - p\}$.

\[
\begin{array}{ccc}
E_{n+s}^{p-n} & \xrightarrow{\partial} & E_{n+s-l}^{p-n} \\
\xrightarrow{k^*} & & \xrightarrow{T} \\
\tilde{H}_{n+s}(\pi_p(X), n; G) & \xrightarrow{d^{p-n}} & \tilde{H}_{n+s-l}(\pi_p(X), p; G) = E_{p,i-s}^{p-n}
\end{array}
\]

where (i) $k: K(\pi_n(X), n) \to K(\pi_p(X), p + 1)$ is the first $k$-invariant,

(ii) $T$ is the composite $\partial \circ \omega_{s+1}$

where $K(\pi_p, p) \to PK(\pi_p, p + 1) \to K(\pi_p, p + 1)$ ($\pi_p = \pi_p(X)$) is the usual path space fibration. $T$ is an isomorphism provided $n + s \leq 2p$.

**Proof.** (a) follows because $\pi_i(X) = 0$ for $1 \leq i < n, n < i < p$

\[
E_{r,s}^i = E_{n,s}^{p-n}
\]

for all $s$, since $d^{p-n}: E_{p,i-s}^{p-n} \to E_{p,i-1}^{p-n}$ is the first nonzero differential operator. $E_{r,s}^i = E_{n,s}^{p-n}$ provided $s \leq q - p$ since $\pi_i(X) = 0$ for $n < i < p$, $p < i < q$ implies that $d^i: E_{p,i-s}^{i-1} \to E_{p,i-s}^{i}$ is zero unless $i = p - n$ and $d^i: E_{r,s}^{i} \to E_{p+i,s-i}^{i}$ is zero provided $s \leq q - p$.

(b) since $d^{p-n}$ is given by the composition (see 2.2)

\[
\begin{array}{ccc}
H_{n+s}(F_n, F_p) & \xrightarrow{\partial} & \tilde{H}_{n+s-1}(F_p) \\
\xrightarrow{\partial} & & \xrightarrow{T} \\
\tilde{H}_{n+s}(\pi_p(X), n) & \xrightarrow{d} & \tilde{H}_{n+s}(\pi_p(X), p + 1)
\end{array}
\]

we are asking that the following diagram commute:

\[
\begin{array}{ccc}
H_{n+s}(F_n, F_p) & \xrightarrow{\partial} & \tilde{H}_{n+s-1}(F_p) \\
\xrightarrow{j^*} & & \xrightarrow{T} \\
\tilde{H}_{n+s}(\pi_p(X), n) & \xrightarrow{\partial} & \tilde{H}_{n+s}(\pi_p(X), p + 1)
\end{array}
\]

\[
\begin{array}{ccc}
H_{n+s}(F_n, F_p) & \xrightarrow{\partial} & \tilde{H}_{n+s-1}(F_p) \\
\xrightarrow{\partial} & & \xrightarrow{T} \\
\tilde{H}_{n+s}(\pi_p(X), n) & \xrightarrow{\partial} & \tilde{H}_{n+s}(\pi_p(X), p + 1)
\end{array}
\]
where \( \bar{k} \) is defined by (2.6) below, and \( q_i = r_i|_{r^*} \). (2.4) commutes if and only if

\[
\begin{array}{c}
H_{n+s}(F^*, F^p) \xrightarrow{\delta} \tilde{H}_{n+s-1}(F^p) \\
\downarrow w_* \circ k_* \circ q_* \\
H_{n+s}(PK, K(\pi_p(X), p)) \xrightarrow{\delta} \tilde{H}_{n+s-1}(\pi_p(X), p)
\end{array}
\] (2.5)

commutes. We have the following situation:

\[
\begin{array}{c}
X = F^*_n \\
\cup \ F^*_p \\
\cup \ F^*_q
\end{array} \xrightarrow{r_p} \begin{array}{c}
X_p \\
\pi_p \\
k \\
K(\pi_p, n) \rightarrow \begin{array}{c}
PK \\
w
\end{array}
\end{array}
\]

\[
(2.6) \ x = F^*_n \rightarrow K(\pi_p, n) \rightarrow K(\pi_p, p+1)
\]

where \( k \circ q_n = k \circ \pi_n \circ r_p = w \circ \bar{k} \circ r_p = w \circ \bar{k} \circ \pi_n \circ q_n = \bar{k} \circ r_p. \) But \( k \circ r_p|_{r_p} = \bar{k} \circ q_p \) is clearly the same as \( \bar{k} \circ q_p \circ j \) considered as maps of the pairs \((F^*_p, *) \rightarrow (F^*_p, F^*_q) \rightarrow (PK, \ast)\). This shows that (2.5) commutes.

By an argument similar to Lemma 2.3, we may identify the \( d^i \) operator below the diagonal. This was claimed in [7], page 176.

**Lemma 2.4.** The following commutes for \( s \leq j \).

\[
\begin{array}{c}
\tilde{H}_{j+s}(\pi_j, j) \xrightarrow{d^i} \tilde{H}_{j+1}(\pi_{j+1}, j+1) \\
\downarrow (k_j \circ \pi_j)_* \\
\tilde{H}_{j+s}(\pi_{j+1}, j+2)
\end{array}
\]

where

(a) \( k_j: X_j \rightarrow K(\pi_{j+1}(X), j+2) \) is the \( j \)th \( k \)-invariant,
(b) \( i_j: K(\pi_j(Y), j) \hookrightarrow X_j \) is the inclusion, and
(c) \( T \) is the transgression (which is an isomorphism for \( s \leq j + 2 \)).

3. Rational homology and Whitehead products. In this section we consider Kahn's spectral sequence with coefficients in \( \mathbb{Q} \), the rationals. For this special case we are able to identify the \( E^1 \)-term considerably above the diagonal. This occurs because for \( \mathbb{Q} \) coefficients,
$H_*(\pi, n; Q) \approx$ a Hopf algebra over $Q$ on $\dim_0(\pi \otimes \mathbb{Q})$ generators of degree $n$.

In [8], J. P. Meyer demonstrated how to compute Whitehead products in $\pi_*(X)$ from a Postnikov system for $X$ and in [7], Theorem 9.1, D. W. Kahn used Meyer's results to show that a certain higher differential operator in $\mathcal{E}(X; Q)$ is the Whitehead product. In the range of our identification, we show that this differential is the only nonzero differential operator. This allows a complete computation of $H_*(X; Q), i \leq 3 \cdot c(X)$, in terms of the homotopy groups of $X$ and the (rational) Whitehead products, where $c(X)$ is the connectivity of $X$.

**Definition 3.1.** Let $G$ be an arbitrary $Q$-vector space and $p$ be a positive integer. The skew-symmetric tensor product $S_p(G)$ is defined as

$$S_p(G) = (G \otimes \mathbb{Q} G)/R$$

where $R$ is the subspace generated by $\{g_1 \otimes g_j - (-1)^{p_1} g_j \otimes g_1 | g_i, g_j \in G\}$. Suppose $\nu = \dim_0 G$, and let $A(\nu, p)$ be the free commutative graded algebra over $Q$ on generators $(t_1, \cdots, t_\nu)$ where degree $t_i = p$ ($\nu$ need not be finite).

$$A(\nu, p) \approx \begin{cases} Q[t_1, \cdots, t_\nu] & \text{if } p \text{ even} \\ E_0(t_1, \cdots, t_\nu) & \text{if } p \text{ odd} \end{cases}$$

where $Q[t_1, \cdots]$ is the graded polynomial algebra over $Q$, $E_0(t_1, \cdots)$ is the graded exterior algebra over $Q$, on generators $t_1, \cdots, t_\nu$ of degree $p$. Then it is easy to see that $S_p(G) \approx A(\nu, p)_{2p}$, the $Q$-module of $A(\nu, p)$ in degree $2p$.

**Lemma 3.2.** Let $G$ be an abelian group. Then $H_{2p}(G, p; Q) \approx S_p(G \otimes \mathbb{Q})$.

**Proof.** This follows because $H_*(G, p; Q) = A(\dim_0(G \otimes \mathbb{Q}), p)$.

**Theorem 3.3.** Let $c(X) = n - 1$, for $n \geq 2$. In $\mathcal{E}(\mathcal{P}(X), \infty; Q)$, the $E^1$-term is given as follows ($\otimes$ means $\otimes_2$): For all $p > 0$,

$$E^1_{n,s}(X; Q) \approx \begin{cases} \pi_p \otimes \mathbb{Q}, & \text{if } q = 0 \\ 0, & \text{if } 0 < q < p \\ S_p(\pi_p \otimes \mathbb{Q}), & \text{if } q = p \\ \pi_p \otimes \pi_q \otimes \mathbb{Q}, & \text{if } p + 1 \leq q \leq 2p - 2 \end{cases}$$

where $\pi_i = \pi_i(X)$ (see Figure 3.1).
Proof. Let $p > 1$ and consider the homology Serre spectral sequence \[5\] for the fibration $F_{p+1} \hookrightarrow (F_p, F_{p+1}) \twoheadrightarrow (K(\pi_p, p), *)$. The $E^2$-term, with coefficients in $\mathbb{Q}$, is

$$E^2_{r,s} = H_r(\pi_p, p), *; H_s(F_{p+1}; Q) \approx \tilde{H}_s(\pi_p, p; Q) \otimes Q H_s(F_{p+1}; Q).$$

Note that if $r < 2p$, then $E^2_{r,s} = 0$ unless $r = p$ and

$$E^2_{r,s} = \pi_p \otimes H_s(F_{p+1}; Q).$$

It is easy to see from this, 1.1 (a), and the fact that

$$H_s(\pi_p, p; Q) \approx \wedge (\dim_q (\pi_p \otimes Q), p)$$

that
\[ E_{p,q}(X; Q) \cong H_{p+q}(F_p, F_{p+1}; Q) \]
\[ \cong \begin{cases} \pi_p \otimes_{\mathbb{Z}} H_q(F_{p+1}; Q), & \text{if } 0 \leq q \leq 2p - 2, \quad q \neq p ; \\ H_{2p}(\pi_p, p; Q) = S_p(\pi_p \otimes Q), & \text{if } q = p . \end{cases} \]

Now we show that if \( p \leq q \leq 2p - 2 \), then \( H_q(F_p; Q) \cong H_q(F_q; Q) \). Consider the homology Serre spectral sequence with coefficients in \( Q \) of the fibration \( F_{p+1} \to F_p \to K(\pi_p, p) \) given by 1.1 (a) and the Hurewicz theorem. If \( q = p \), then \( H_q(F_p; Q) \cong \pi_q \otimes Q \) by 1.1 (a) and the Hurewicz theorem. If \( p < q \leq 2p - 2 \), then the exact sequence of [5], page 284, implies that \( i_*: H_q(F_{p+1}) \cong H_q(F_p) \). Similar arguments on the homology Serre spectral sequences for \( F_{i+1} \to F_i \to K(\pi_i, i) \), \( i = p + 1, \cdots, q \) show that

\[ H_q(F_p; Q) \cong H_q(F_{p+1}; Q) \cong \cdots \cong H_q(F_{q-1}; Q) \cong H_q(F_q; Q) \cong \pi_q \otimes Q \]

provided \( p \leq q \leq 2p - 2 \).

**Corollary 3.4.** (Rational Hurewicz Theorem) If \( i \leq 2c(X) \) then
\[ h_i \otimes 1: \pi_i(X) \otimes Q \to H_i(X; Q) \]

is an isomorphism.

**Proof.** This is follows from 3.3 because the only non-zero term \( E^i_{p,q} \) of total degree \( i \) (for \( i \leq 2c(X) \)) is \( E^i_{i,0} = \pi_i(X) \otimes Q = E^\infty_{i,0} \). Thus \( \pi_i(X) \otimes Q \to H_i(X; Q) \) is an isomorphism. Kahn's theorem 4.1 [7] identifies this map (the edge homomorphism) as \( h_i \otimes 1 \).

This result was known to Cartan and Serre in [2].

We will now study the differentials in \( \mathcal{G}(X; \infty; Q) \). According to Theorem 2.2 of [3] (see also [9], Chapter 2), given \( X, \exists \text{ aCW-complex } \]
\[ X \otimes Q \text{ and a map } f: X \to X \otimes Q \]
\[ \text{(a) } \pi_i(X \otimes Q) \cong \pi_i(X) \otimes Q \]
\[ \text{(b) } f \text{ is a homotopy equivalence modulo the class } \mathcal{T} \text{ of torsion groups.} \]
\[ \text{(c) } \exists \text{ an isomorphism } \nu \text{ such that the following commutes:} \]
\[ \pi_i(X) \otimes Q \]
\[ \pi_i(X) \]
\[ f_* \]
\[ \nu \]
\[ t(\alpha) = \alpha \otimes 1, \text{ for } \alpha \in \pi_i(X). \]

Let \( X \otimes Q \) be the space obtained from \( X \otimes Q \) by killing off all the homotopy groups of \( X \otimes Q \) in dimensions \( \geq 2 \cdot c(X) + 1 \); \( i: X \otimes Q \to \hat{X} \otimes Q \)
the inclusion map. Consider the composite map \( i \circ f: X \to X \otimes Q \). This induces an exact couple map from

\[ \mathcal{G}(\mathcal{P}(X); Q) \to \mathcal{G}(\mathcal{P}(X \otimes Q); Q) \]
which we shall see is an isomorphism in a certain range of dimensions on the $E^0$-term. Theorem 4.4 of [3] implies that all the $k$-invariants of $X \otimes Q$ are trivial, i.e.,

$$X \otimes Q \cong \mathop{\bigoplus}_{i=1}^{2g'(X)} K(\pi_i(X) \otimes Q, \hat{i}).$$

This implies that the spectral sequence $\{E^i(X \otimes Q; Q); \hat{d}^i\}$ collapses; i.e., all the $\hat{d}^i$ are zero. It follows from a theorem of Kahn [6], that $i \circ f$ induces maps $\mathcal{P}(i \circ f): \mathcal{P}(X) \to \mathcal{P}(X \otimes Q)$ such that the following diagram commutes.

$$
\begin{array}{cccc}
X & \xrightarrow{i \circ f} & (X \otimes Q)_{2 \cdot c(X)} = X \otimes Q \\
\downarrow r_n & & \downarrow (i \circ f)_{2 \cdot c(X)} & \\
X_{2 \cdot c(X)} & \xrightarrow{(i \circ f)_{2 \cdot c(X)}} & (X \otimes Q)_{2 \cdot c(X)} \\
\downarrow r_n & & \downarrow (i \circ f)_{2 \cdot c(X)} & \\
X_{n-1} & \xrightarrow{(i \circ f)_{n}} & (X \otimes Q)_{n-1} \\
\downarrow (3.5) & & \downarrow (i \circ f)_{n-1} & \\
X_{n-1} & \xrightarrow{(i \circ f)_{n-1}} & (X \otimes Q)_{n-1} \\
\downarrow \pi_n & & \downarrow (i \circ f)_{n-1} & \\
\cdots & & \cdots & \\
\downarrow \pi_{n-1} & & \cdots & \\
X_{c(X)-1} & \xrightarrow{(i \circ f)_{c(X)-1}} & (X \otimes Q)_{c(X)-1} \\
\downarrow \pi_{c(X)-1} & & \downarrow (i \circ f)_{c(X)-1} & \\
\cdots & & \cdots & \\
\downarrow \pi_{c(X)+1} & & \cdots & \\
\end{array}
$$

and $\pi_i(X) \xrightarrow{(i \circ f)_i} \pi_i((X \otimes Q)_n)$ ($i > 0$) is an isomorphism mod $\mathcal{I}$. The commutativity of (3.5) $\Rightarrow (i \circ f)(F_n(X)) \subset F_n(X \otimes Q)$ for $n \leq 2 \cdot c(X)$. An easy induction using the mod $\mathcal{I}$ 5-Lemma [5], and the homotopy ladder induced by

$$
\begin{array}{cccc}
F_{n+1}(X) & \xrightarrow{(i \circ f)_{F_{n+1}}} & F_{n+1}(X \otimes Q) \\
\downarrow & & \downarrow & \\
F_n(X) & \xrightarrow{(i \circ f)_{F_n}} & F_n(X \otimes Q) \\
\downarrow & & \downarrow & \\
K(\pi_n(X), n) & \xrightarrow{(i \circ f)_n} & K(\pi_n(X) \otimes Q, n) \\
\end{array}
$$
shows that \((\iota \circ f \mid_{F_n(X)})_*: H_j(F_n(X); Q) \rightarrow H_j(F_n(X \otimes Q); Q)\) is a \(\mathcal{R}\)-isomorphism for \(j \leq 2 \cdot c(X)\) (and an epimorphism for \(j > 2 \cdot c(X)\)). By the Whitehead theorem mod \(\mathcal{R}\) [5], page 512, we then have that

\[(3.6) \quad (\iota \circ f \mid_{F_n(X)})_*: H_j(F_n(X); Q) \rightarrow H_j(F_n(X \otimes Q); Q)\]

is an isomorphism for \(j \leq 2 \cdot c(X)\) and an epimorphism for \(j = 2 \cdot c(X) + 1\).

By the naturality of the universal coefficient theorem and the Serre spectral sequence, we have the following commutative diagram for \(p \leq 2 \cdot c(X)\) and \(p < q \leq 2p - 2\).

\[\begin{array}{ccc}
E_\ast F_p(X; Q) & \xrightarrow[]{E(\iota \circ f)} & E_\ast F_p(X \otimes Q; Q) \\
H_{p-q}(F_p(X), F_{p-q}(X); Q) & \xrightarrow{\cong} & H_{p-q}(F_p(X \otimes Q), F_{p-q}(X \otimes Q); Q) \\
H_p(K(\pi_p(X), \pi_p(X); Q)) & \xrightarrow{\cong} & H_p(K(\pi_p(X \otimes Q), \pi_p(X \otimes Q); Q)) \\
\end{array}\]

where \(s(\cdot)\) in the above is the isomorphism defined from the Serre spectral sequence for \(F_{p+1}(\pi_p(X), \pi_p(X); Q)\). In this range of dimensions \((p \leq 2 \cdot c(X), p < q \leq 2p - 2)\) the vertical arrows are isomorphisms. 3.6 implies that the bottom row is an isomorphism, provided \(q \leq 2 \cdot c(X)\). A similar argument gives the case \(q = p\).

From this we deduce that

\[(3.8) \quad E_\ast(i \circ f): E_\ast F_p(X; Q) \rightarrow E_\ast F_p(X \otimes Q; Q)\]

is an isomorphism provided \(0 \leq p \leq 2 \cdot c(X), 0 \leq q \leq 2 \cdot c(X)\). See Figure 3.2. (3.8) implies

\[(3.9) \quad E_\ast F_p(X; Q) \xrightarrow{E_\ast(i \circ f)} E_\ast F_p(X \otimes Q; Q)\]

is an isomorphism for \(p + q \leq 3c(X) + 1, p \leq 2c(X)\). (see Figure 3.2.)

Assume now that \(c(X) \geq 2\). We will show that

\[E_{p,q}^i = E_{p,q}^i \text{ for } 2 \leq i \leq q - 2\]

whenever \(c(X) + 1 \leq p \leq 2 \cdot c(X), p \leq q \leq 3c(X) - p\). (These are the only nonzero terms of total degree \(\leq 3c(X)\) such that \(q > 0\). See shaded area in Figure 3.2.) Furthermore, all differential operators coming into \(E_{p,q}^i (i > 0)\) are zero and all differential operators issuing
forth from $E_{p,q}^i$ are zero except for $i = q - 1$.

We show this by arguing on the total degree $j (2c(X) + 2 \leq j \leq 3c(X))$.

(a) $p + q = 2c(X) + 2 \Rightarrow p = c(X) + 1$. All differential operators with range $E_{i+1}^{c+1}$ are zero for $i > 0$ since $E_{i+1}^{c+1} = 0$ for all $i > 0$. Similarly all $d'_{i}: E_{i+1}^{c+1} \rightarrow E_{i+1}^{c+1}$ are zero for $i \leq c(X) - 1$ since the latter group is zero in that range.

(b) Suppose $j > 2c(X) + 2$. Consider $p + q = j \leq 3c(X)$, where $c(X) + 1 \leq p \leq \lfloor j/2 \rfloor$, and the following commutative diagram

$$
\begin{array}{ccc}
E_{p-1,q+2}^{i} & \xrightarrow{E^i(i\circ f)_{p-1}} & \tilde{E}_{p-1,q+2}^{i} \\
| & d_p & | \\
E_{p,q}^{i} & \xrightarrow{E^i(i\circ f)_{p}} & \tilde{E}_{p,q}^{i} \\
| & d_{p+1} & | \\
E_{p+1,q-2}^{i} & \xrightarrow{E^i(i\circ f)_{p+1}} & \tilde{E}_{p+1,q-2}^{i}
\end{array}
$$

where $E^i = E^i(X; Q)$, $\tilde{E}^i = E^i(X \otimes Q; Q)$. $E^i(i\circ f)_{k} (k = p - 1, p, p + 1)$ is an isomorphism by 3.9 since the total degree in each case is $\leq 3c(X) + 1$. 

![Fig. 3.2. $E^i(X; Q)$](image-url)
Since \( \hat{d}_i = 0 \), we have \( d_i = 0 \) for \( i = p, p + 1 \). Thus \( E_{p,q}^i = E_{p,q}^i \) for \( (p, q) \) satisfying the above. Similar arguments imply \( E_{p,q}^i = E_{p,q}^i \) for \( i = 3, 4, \ldots, q - 2 \).

(c) \( d^i: E_{p,q}^i \to E_{p+1,q-i-1}^i \) is zero for \( i > q - 1 \) since \( q - i - 1 < 0 \Rightarrow E_{p+1,q-i-1}^i = \). \( d^i: E_{p-i,q+i-1}^i \to E_{p,q}^i \) is zero for \( i \geq q - 1 \) since \( i \geq q - 1 \), \( q \geq p = p - i \leq p - q + 1 = E_{p-i,q+i-1}^i = 0 \).

Thus the only (possibly) nonzero differential operator for each \( (p, q) \) satisfying \( c(X) + 1 \leq p \leq 2c(X), p \leq q \leq 3c(X) - p \) is

\[
d^{q-1}: E_{p,q}^{q-1} \to E_{p+q-1,0}^{q-1}.
\]

But this has been identified by Kahn in [7], Theorem 9.1, as the (rational) Whitehead product: If \( q > p \)

\[
\pi_p \otimes \pi_q \otimes Q \xrightarrow{[\ , \ ] \otimes \text{id}} \pi_{p+q-1} \otimes Q
\]

\[
E_{p,q}^{q-1} \xrightarrow{d^{q-1}} E_{p+q-1,0}^{q-1} (q > p)
\]

or, if \( q = p \)

\[
S_p(\pi_p \otimes Q) \xrightarrow{[\ , \ ] \otimes \text{id}} \pi_{2p-1} \otimes Q
\]

\[
E_{p,q}^{p-1} \xrightarrow{d^{p-1}} E_{2p-1,0}^{p-1}
\]

where \([\ , \ ]\) is the Whitehead product.

We have thus proved the following.

**Theorem 3.10.** Let \( c(X) \geq 2 \). If \( p + q \leq 3 \cdot c(X) \) and \( q \geq p \), then

(a) \( d^i: E_{p-i,q+i-1}^i \to E_{p,q}^i \) is zero for all \( i > 0 \).

(b) \( d^i: E_{p,q}^i \to E_{p+i,q-i-1}^i \) is zero for \( i = 1, 2, \ldots, q - 2, q, q + 1, \ldots \)

(c) \( d^{q-1}: E_{p,q}^{q-1} \to E_{p+q-1,0}^{q-1} \) is the rational Whitehead product.

4. Applications. We are now in a position to compute \( H_i(X; Q) \) \( (i \leq 3 \cdot c(X)) \) completely in terms of the graded homotopy group \( H = \{\pi_i \otimes Q|1 \leq i \leq 3 \cdot c(X)\} \) and the rational Whitehead product on this group. For \( i \leq 2 \cdot c(X) \) this is given by the rational Hurewicz theorem (3.4). Let

\[
\text{Ker}_{ij} = \begin{cases} 
\text{Ker} \{\pi_j \otimes \pi_{i-j} \otimes Q \xrightarrow{[\ , \ ] \otimes \text{id}} \pi_{i-1} \otimes Q\}, & c(X) < j \leq \left[ \frac{i-1}{2} \right] \\
\text{Ker} \{S(\pi_{i/2} \otimes Q) \xrightarrow{[\ , \ ] \otimes \text{id}} \pi_{i-1} \otimes Q\}, & \text{if } i \text{ even}, \ j = \left[ \frac{i}{2} \right] \\
0, & \text{if } i \text{ odd}, \ j = \left[ \frac{i}{2} \right]. 
\end{cases}
\]
and
\[
\text{Ker}_i = \bigoplus_{e(X) < j \in [i/2]} \text{Ker}_{ij} \quad (\oplus \text{denotes direct sum}),
\]

where \([ , \) is the Whitehead product.

Furthermore, let
\[
\text{Im}_{ij} = \begin{cases} 
\text{Im}\{\pi_j \otimes \pi_{i+1-j} \otimes Q \overset{[ , ] \otimes id}{\longrightarrow} \pi_i \otimes Q\}, & \text{if } e(X) < j \leq \left\lfloor \frac{i}{2} \right\rfloor \\
\text{Im}\{S(\pi_{(i+1)/2} \otimes Q) \overset{1,1 \otimes id}{\longrightarrow} \pi_i \otimes Q\}, & \text{if } i + 1 \text{ even}, j = \left\lfloor \frac{i + 1}{2} \right\rfloor \\
0, & \text{if } i + 1 \text{ odd}, j = \left\lfloor \frac{i + 1}{2} \right\rfloor 
\end{cases}
\]

and (since \text{Im}_{ij} \subset \pi_i \otimes Q \text{ for each } j)
\[
\text{Im}_i = \sum_{c(X) < j \in [i/2]} \text{Im}_{ij} \subset \pi_i \otimes Q. \text{ (+ denotes sum, not necessarily direct)}
\]

\textbf{Theorem 4.1.} If \(2c(X) < i \leq 3 \cdot c(X)\), then
\[
H_i(X; Q) \approx \text{Ker}_i \oplus (\pi_i \otimes Q/\text{Im}_i)
\]

\textbf{Proof.} 3.4, 3.10 \(= E^{\pi_i}_{c,0} \approx (\pi_i \otimes Q/\text{Im}_i)\) and \(E^{\pi_i}_{p,q}(c(X) < p \leq [i/2], p + q = i) \approx \text{Ker}_{ip}\). These are the only nonzero terms of total degree \(i\). Since all extensions split we have
\[
H_i(X; Q) \approx E^{\pi_i}_{c,0} \oplus \bigoplus_{c(X) < p \in [i/2]} E^{\pi_i}_{p,i-p} \\
\approx (\pi_i \otimes Q/\text{Im}_i) \oplus \text{Ker}_i.
\]

Since Kahn [7] has identified the edge homomorphism with the Hurewicz homomorphism we see

\textbf{Theorem 4.2.} If \(i \leq 3 \cdot c(X)\) and \(h_i \otimes 1: \pi_i(X) \otimes Q \rightarrow H_i(X; Q)\) is the Hurewicz homomorphism, then
\begin{enumerate}
\item[(a)] \text{Ker} \(h_i \otimes 1 = \text{Im}_i\)
\item[(b)] \text{coker} \(h_i \otimes 1 = \text{Ker}_i\)
\end{enumerate}

\textbf{Proof.} This follows because \(h_i \otimes 1\) is the natural map
\[
\pi_i \otimes Q \rightarrow \text{Ker}_i \oplus (\pi_i \otimes Q/\text{Im}_i).
\]

\textbf{Corollary 4.3.} If \(i \leq 3 \cdot c(X)\), then
\begin{enumerate}
\item[(a)] \(h_i \otimes 1\) is a monomorphism \(\Rightarrow\) \text{Im}_i = 0
\item[(b)] \(h_i \otimes 1\) is an epimorphism \(\Rightarrow\) \text{Ker}_i = 0.
\end{enumerate}
Note. By Proposition 2.1 (respectively, 4.1) of [1], $h_i \otimes 1$ is epic (respectively, monic) $\iff$ the $i^{th}$ $k$'-invariant ($k$-invariant) of any homology (Postnikov) decomposition is of finite order. 4.3 gives another such characterization. This gives, for instance, the following theorem.

**Theorem 4.4** If $\pi_i(X; \mathbb{Q}) = 0$ for $i > 3 \cdot c(X)$, then all $k$-invariants are of finite order $\iff$ all rational Whitehead products vanish.

Finally, since it is usually easier to compute $H_i(X; \mathbb{Q})$ than it is the Whitehead product, we will use these relations (4.1 and 4.2) to give information about the Whitehead products themselves.

**Theorem 4.5.** Let $i \leq 3 \cdot c(X)$ and consider the following statements:

(a) $\pi_i \otimes \mathbb{Q}$ is generated by Whitehead products.

(b) For all $r$ such that $c(X) < r \leq [(i - 1)/2]$, $\pi_r \otimes \pi_{i-r} \otimes \mathbb{Q} \to \pi_{i-1} \otimes \mathbb{Q}$ is injective.

(c) If $i$ even, $S(\pi_i \otimes \mathbb{Q}) \to \pi_{i-1} \otimes \mathbb{Q}$ is injective. The following are true.

(d) $h_i \otimes 1 = 0 \iff$ (a)

(e) $\text{coker } h_i \otimes 1 = 0 \iff$ (b) and (c)

(f) $H_i(X; \mathbb{Q}) = 0 \iff$ (a), (b) and (c).

**References**


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