Pacific Journal of Mathematics

RATIONAL HOMOLOGY AND WHITEHEAD PRODUCTS

MICHEAL NEAL DYER

Vol. 40, No. 1 September 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 1, 1972

RATIONAL HOMOLOGY AND WHITEHEAD PRODUCTS

MICHAEL DYER

D. W. Kahn defined a spectral sequence $\mathscr{E}(X; R)$ for the Postnikov system $\mathscr{P}(X)$ of a 1-connected CW-complex which converges to $H_*(X; R)$, the singular homology of X with coefficients in R. We study $\mathscr{E}(X; R)$ in two settings: (a) to give a generalization of the classical theorem of Eilenberg and MacLane concerning the dependence of $H_i(X; Z)$ on the first nonzero homotopy group of X (2.1) and (b) to give a complete computation of $H_i(X; Q)$ (Q = rationals) for $i \leqq 3 \cdot c(X)$ ($c(X)$ = connectivity of X) in terms of the *graded* homotopy group $\Pi \otimes Q = \{\pi_i(X) \otimes Q \mid 0 < i \leqq 3 \cdot c(X)\}$ and the Whitehead product on this group (0.1 and 0.2).

In § 1 we give a quick description of $\mathscr{E}(X; R)$ for later use and in § 2 we generalize the Eilenberg-MacLane theorem by giving an exact sequence involving the first *two* nonzero homotopy groups. $\mathscr{E}(X, Q)$ is studied in § 3, with the result that we are able to identify $E^1(X; Q)$ somewhat above the diagonal (Kahn identified it below the diagonal in [7]) (3.3) and to show that the Whitehead product is the only non-zero differential operator, provided the total degree is less than $3 \cdot c(X)$ (3·10). Section 4 gives the computations of $H_i(X; Q)$ and various other applications.

1. Description of the Spectral Sequence of $\mathscr{P}(X)$. In this note X is a $(n-1)$-connected space, $n > 1$, having the homotopy type of a CW-complex. All maps and spaces are "pointed".

Let $\{X_i, r_i, \pi_i\} = \mathscr{P}(X)$ be a Postnikov system for X (see [6] for definition). Choose $m > n$ and convert the map $r_m: X \to X_m$ into a fiber map. Use the same notation for the new map. In the tower of spaces

$$X \xrightarrow{r_m} X_m \xrightarrow{\pi_m} X_{m-1} \xrightarrow{\pi_{m-1}} \longrightarrow \cdots \xrightarrow{\pi_{n+1}} X_n = K(\pi_n(X), n)$$

$\pi_\alpha \circ \cdots \circ \pi_m \circ r_m \simeq r_{\alpha-1}$ $(n+1 \leqq \alpha \leqq m)$. Let $r_{\alpha-1}$ denote this composition, $\alpha = n+1, \cdots, m$. Since all these maps are Hurewicz fibrations, $r_{\alpha-1} (\alpha - 1 < m)$ is a fiber map. Let $F_{i+1} = r_i^{-1}$ (base point) denote the fiber of $r_i: X \to X_i$, $i \leqq m$. The following is proved in [7].

LEMMA 1.1. (a) *F_{i+1} is i-connected.*

(b) *F_{i+1} is fibered over $K(\pi_{i+1}(X), i+1)$, with fiber F_{i+2}, via the map $r_{i+1} | F_{i+1}$.*

(c) *$X = F_n \supset F_{n+1} \supset \cdots \supset F_m \supset F_{m+1}$ is a finite de-*

creasing filtration of X.

For each m, the exact couple ([7]) $\mathscr{C}(\mathscr{P}(X), m; G)$ is defined by

$$D^1_{r,s} = \begin{cases} H_{r+s}(F_r; G), & \text{if } r, s \geqq 0 . \\ 0 , & \text{otherwise,} \end{cases}$$

$$E^1_{r,s} = \begin{cases} H_{r+s}(F_r, F_{r+1}; G), & \text{if } r, s \geqq 0 . \\ 0 , & \text{otherwise,} \end{cases}$$

where G is any abelian group and H_* is singular homology. If $D^1 = \sum_{\oplus} D^1_{r,s}$, $E^1 = \sum_{\oplus} E^1_{r,s}$ then the couple maps $i\colon D^1 \to D^1$, $j\colon D^1 \to E^1$ and $k\colon E^1 \to D^1$ are of bidegree (respectively) $(-1, 1)$, $(0, 0)$, $(1, -2)$. The bidegree of the differential operator $d_i\colon E^i \to E^i$ is $(i, -i-1)$.

In [7], Kahn shows that

$$E^1_{j,s} = H_{j+s}(F_j, F_{j+1}; G) \xrightarrow{q_{j*}} \tilde{H}_{j+s}(\pi_j(X), j; G) \tag{1.2}$$

is an isomorphism, provided $s \leqq j$, where

$$q_j = r_j | F_j\colon (F_j, F_{j+1}) \to (K(\pi_j(X), j), *) ,$$

thus indentifying the E^1 term below the diagonal.

2. Generalization of a theorem of Eilenberg-MacLane. In [4], Eilenberg and MacLane showed the dependence of the first few homology groups of a space X upon the first nonzero homotopy group of X. We prove the following generalization.

THEOREM 2.1. *Let X be an $(n-1)$-connected space having the homotopy type of a CW-complex, $n \geqq 2$. Suppose $\pi_i(X) = 0$ for $n < i < p$ and $p < i < q \leqq 2n$. Then $H_i(X; G) \approx H_i(\pi_n(X), n; G)$ for $n \leqq i < p$ and any abelian group G. Furthermore, if we abbreviate $H_j(\pi_l(X), l; G)$ by $H_j(l; G)$, we have the exact sequence*

$$H_q(n; G) \xrightarrow{\Phi_q} H_{q-1}(p; G) \xrightarrow{\psi_{q-1}} H_{q-1}(X; G) \xrightarrow{\chi_{q-1}} H_{q-1}(n; G) \xrightarrow{\Phi_{q-1}} \cdots$$
$$\cdots \longrightarrow H_i(p; G) \xrightarrow{\psi_i} H_i(X; G) \xrightarrow{\chi_i} H_i(n; G) \xrightarrow{\Phi_i} H_{i-1}(p; G) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_p(p; G) \xrightarrow{\psi_p} H_p(X; G) \xrightarrow{\chi_p} H_p(n; G) \longrightarrow 0 .$$

$\Phi_i = T_i \circ (k)_*$, where $k\colon K(\pi_n(X), n) \to K(\pi_p(X), p+1)$ is the first k-invariant in a Postnikov decomposition of X and $T_j\colon H_j(\pi_p(X), p+1; G) \to H_{j-1}(\pi_p(X), p; G)$ is the transgression, which is an isomorphism provided $0 < j \leqq 2p$. Further, ψ_p is the Hurewicz homomorphism.

Proof. We consider $\mathscr{C}(\mathscr{P}(X), m; G)$ for $m > 2n$. $\pi_i(X) = 0$ for $n < i < p$, $p < i < q$ implies by 1.1 (b) that

(2.2) $$X = F_n \supset F_{n+1} = \cdots = F_p \supset F_{p-1} = \cdots = F_q \supset \cdots .$$

Thus $E^1_{r,s} = 0$ for $0 \leqq r < n$, $n < r < p$, $p < r < q$ and all s. This gives a two-term condition (see [5], chapter VIII) on the E^1-term of $\mathscr{C}(\mathscr{P}(X), m; G)$. Using (1.2) we have that $H_i(X; G) \approx H_i(\pi_n(X), n; G)$ for $n \leqq i < p$ (a 1-term condition here) and for $p \leqq i < q$ we have the exact sequence of the theorem. Note that we did not need $q \leqq 2n$ in order to obtain the two-term condition, but only in order to use (1.2). It is clear from [7] that ψ_p (the edge homomorphism) is the Hurewicz homomorphism.

We will now show that $\Phi_i = T_i \circ (k)_*$. Since Φ_i is essentially $d^{(p-n)}: E^{p-n}_{n,i-n} \to E^{p-n}_{p,i-1-p}$ ([7]), we will show that $d^{(p-n)} = T_i \circ (k)_*$. As it has significance in its own right, we give it as a separate lemma.

Lemma 2.3 *If* $\pi_i(X) = 0$ *for* $1 \leqq i < n$, $n < i < p$, $p < i < q$, *then*
(a) $E^1_{r,s} = E^{p-n}_{r,s}$ *for* $r = n, p$ *provided* $s \leqq q - p$.
(b) *The following triangle commutes for* $s \leqq \min\{n, q - p\}$.

$$\begin{array}{ccccc} E^{p-n}_{n,s} = \tilde{H}_{n+s}(\pi_n(X), n; G) & & \xrightarrow{d^{p-n}} & & \tilde{H}_{n+s-1}(\pi_p(X), p; G) = E^{p-n}_{p,-(p-n)+s-1} \\ & {}_{k_*}\searrow & & \nearrow{}_{T} & \\ & & \tilde{H}_{n+s}(\pi_p(X), p+1; G) & & \end{array}$$

where (i) $k: K(\pi_n(X), n) \to K(\pi_p(X), p+1)$ *is the first k-invariant,*
(ii) T *is the composite* $\partial \circ w_*^{-1}$
where $K(\pi_p, p) \hookrightarrow PK(\pi_p, p+1) \xrightarrow{w} K(\pi_p, p+1)$ $(\pi_p \equiv \pi_p(X))$ *is the usual path space fibration. T is an isomorphism provided* $n + s \leqq 2p$.

Proof. (a) follows because $\pi_i(X) = 0$ for $1 \leqq i < n$, $n < i < p$

$$\Longrightarrow E^1_{n,s} = E^{p-n}_{n,s}$$

for all s, since $d^{p-n}: E^1_{n,s} \to E_{p\ s-(p-n)-1}$ is the first nonzero differential operator. $E^1_{p,s} = E^{p-n}_{p,s}$ provided $s \leqq q - p$ since $\pi_i(X) = 0$ for $n < i < p$, $p < i < q$ implies that $d^i: E^i_{p-i\ s+i+1} \to E^i_{p\ s}$ is zero unless $i = p - n$ and $d^i: E^i_{p,s} \to E^i_{p+i,s-i-1}$ is zero provided $s \leqq q - p$.

(b) since d^{p-n} is given by the composition (see 2.2)

$$H_{n+s}(F_n, F_p) \xrightarrow{\partial} \tilde{H}_{n+s-1}(F_p) \xrightarrow{j_*} H_{n+s-1}(F_p, F_q)$$

we are asking that the following diagram commute:

(2.4)
$$\begin{array}{ccccccc} H_{n+s}(F_n, F_p) & \xrightarrow{\partial} & \tilde{H}_{n+s-1}(F_p) & & \xrightarrow{\quad j_* \quad} & & H_{n+s-1}(F_p, F_q) \\ \downarrow{\scriptstyle (q_n)_*} & & & & & & \downarrow{\scriptstyle (\bar{k} \circ q_p)_*} \\ \tilde{H}_{n+s}(\pi_n(X), n) & \xrightarrow{k_*} & \tilde{H}_{n+s}(\pi_p(X), p+1) & \xleftarrow{w_*} & H_{n+s}(PK, K(\pi_p(X), p)) & \xrightarrow{\partial} & \tilde{H}_{n+s-1}(\pi_p(X),),p \end{array}$$

where $\bar{k}$ is defined by (2.6) below, and $q_i = r_i|_{F_i}$. (2.4) commutes if and only if

$$\begin{array}{ccc} H_{n+s}(F_n, F_p) & \xrightarrow{\quad\partial\quad} & \tilde{H}_{n+s-1}(F_p) \\ \big\downarrow{\scriptstyle w_*^{-1}\circ k_*\circ q_{n*}} & & \big\downarrow{\scriptstyle (\bar{k}\circ q_p\circ j)_*} \\ H_{n+s}(PK, K(\pi_p(X), p)) & \xrightarrow{\partial} & \tilde{H}_{n+s-1}(\pi_p(X), p) \end{array} \tag{2.5}$$

commutes. We have the following situation:

$$\begin{array}{ccccc} & {\scriptstyle r_p}\nearrow & X_p & \xrightarrow{\qquad\bar{k}\qquad} & PK \\ & & \big\downarrow{\scriptstyle \pi_p} & & \big\downarrow{\scriptstyle w} \\ X = F_n & \xrightarrow{\quad r_n = q_n\quad} & K(\pi_n, n) & \xrightarrow{\qquad k\qquad} & K(\pi_p, p+1) \\ \cup & & & & \\ F_p & & & & \\ \cup & & & & \\ F_q & & & & \end{array} \tag{2.6}$$

where $k\circ q_n = k\circ\pi_p\circ r_p = w\circ\bar{k}\circ r_p \Rightarrow w_*^{-1}\circ k_*\circ q_{n*} = \bar{k}_*\circ r_{p*}$. But $\bar{k}\circ r_p|_{F_p} = \bar{k}\circ q_p$ is clearly the same as $\bar{k}\circ q_p\circ j$ considered as maps of the pairs $(F_p, *) \to (F_p, F_q) \to (PK, *)$. This shows that (2.5) commutes.

By an argument similar to Lemma 2.3, we may identify the d^1 operator below the diagonal. This was claimed in [7], page 176.

LEMMA 2.4. *The following commutes for* $s \leqq j$.

$$\begin{array}{ccc} \tilde{H}_{j+s}(\pi_j, j) & \xrightarrow{d^1} & \tilde{H}_{j+1}(\pi_{j+1}, j+1) \\ & {\scriptstyle (k_j\circ i_j)_*}\searrow \qquad \nearrow{\scriptstyle T} & \\ & \tilde{H}_{j+s}(\pi_{j+1}, j+2) & \end{array}$$

where (a) $k_j\colon X_j \to K(\pi_{j+1}(X), j+2)$ *is the* jth *k-invariant,*
(b) $i_j\colon K(\pi_j(Y), j) \hookrightarrow X_j$ *is the inclusion, and*
(c) T *is the transgression* (which is an isomorphism for $s \leqq j+2$).

3. **Rational homology and Whitehead products.** In this section we consider Kahn's spectral sequence with coefficients in Q, the rationals. For this special case we are able to identify the E^1-term considerably above the diagonal. This occurs because for Q coefficients,

$H_*(\pi, n; Q) \approx a$ Hopf algebra over Q on $\dim_Q(\pi \otimes_Z Q)$ generators of degree n.

In [8], J. P. Meyer demonstrated how to compute Whitehead products in $\pi_*(X)$ from a Postnikov system for X and in [7], Theorem 9.1, D. W. Kahn used Meyer's results to show that a certain higher differential operator in $\mathscr{C}(X; Q)$ *is* the Whitehead product. In the range of our identification, we show that this differential is the *only* nonzero differential operator. This allows a complete computation of $H_i(X; Q)$, $i \leqq 3 \cdot c(X)$, in terms of the homotopy groups of X and the (rational) Whitehead products, where $c(X)$ is the connectivity of X.

DEFINITION 3.1. Let G be an arbitrary Q-vector space and p be a positive integer. The skew-symmetric tensor product $S_p(G)$ is defined as

$$S_p(G) = (G \otimes_Q G)/R$$

where R is the subspace generated by $\{g_i \otimes g_j - (-1)^{p \cdot p} g_j \otimes g_i \mid g_i, g_j \in G\}$. Suppose $\nu = \dim_Q G$, and let $\Lambda(\nu, p)$ be the free commutative graded algebra over Q on generators $(t_1, \cdots, t_\nu)$ where degree $t_i = p$ (ν need not be finite).

$$\Lambda(\nu, p) \approx \begin{cases} Q[t_1, \cdots, t_\nu] \text{ if } p \text{ even ,} \\ E_Q(t_1, \cdots, t_\nu) \text{ if } p \text{ odd ,} \end{cases}$$

where $Q[t_1, \cdots]$ is the graded polynomial algebra over Q, $E_Q(t_1, \cdots)$ is the graded exterior algebra over Q, on generators $t_1, \cdots, t_\nu$ of degree p. Then it is easy to see that $S_p(G) \approx \Lambda(\nu, p)_{2p}$, the Q-module of $\Lambda(\nu, p)$ in degree $2p$.

LEMMA 3.2. *Let G be an abelian group. Then* $H_{2p}(G, p; Q) \approx S_p(G \otimes Q)$.

Proof. This follows because $H_*(G, p; Q) = \Lambda(\dim_Q (G \otimes Q), p)$.

THEOREM 3.3. *Let $c(X) = n - 1$, for $n \geqq 2$. In $\mathscr{C}(\mathscr{P}(X), \infty; Q)$, the E^1-term is given as follows* ($\otimes$ means $\otimes_Z$): *For all $p > 0$,*

$$E^1_{p,q}(X; Q) \approx \begin{cases} \pi_p \otimes Q, \text{ if } q = 0 \\ 0, \text{ if } 0 < q < p , \\ S_p(\pi_p \otimes Q), \text{ if } q = p \\ \pi_p \otimes \pi_q \otimes Q, \text{ if } p + 1 \leqq q \leqq 2p - 2 , \end{cases}$$

where $\pi_i \equiv \pi_i(X)$ (see Figure 3.1).

12

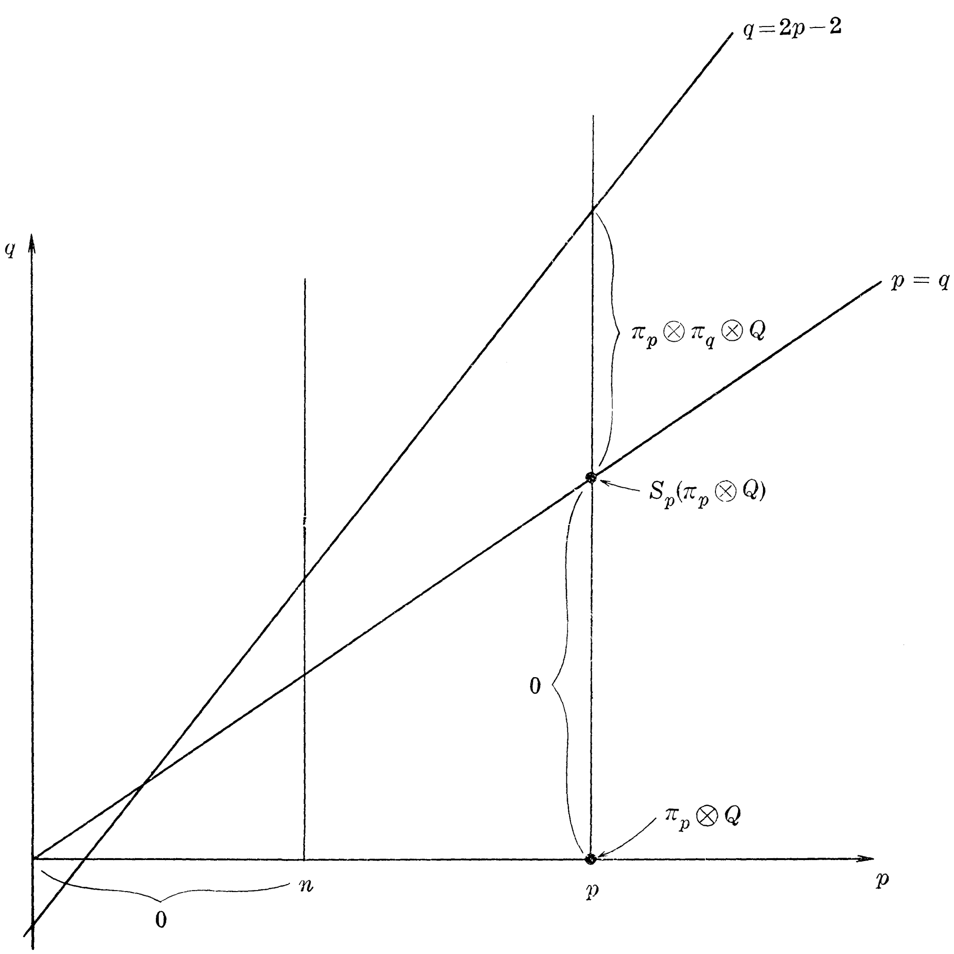


FIG. 3.1. $E^1(X; Q)$.

Proof. Let $p > 1$ and consider the homology Serre spectral sequence [5] for the fibration $F_{p+1} \hookrightarrow (F_p, F_{p+1}) \rightarrow (K(\pi_p, p), *)$. The E^2-term, with coefficients in Q, is

$$E^2_{r,s} \approx H_r(K(\pi_p, p), *; H_s(F_{p+1}; Q) \approx \tilde{H}_r(\pi_p, p; Q) \otimes_Q H_s(F_{p+1}; Q) .$$

Note that if $r < 2p$, then $E^2_{r,s} = 0$ unless $r = p$ and

$$E^2_{p,s} \approx \pi_p \otimes_Z H_s(F_{p+1}; Q) .$$

It is easy to see from this, 1.1 (a), and the fact that

$$H_*(\pi_p, p; Q) \approx \wedge (\dim_Q (\pi_p \otimes Q), p)$$

that

$$E_{p\,q}(X;\,Q) \approx H_{p+q}(F_p,\,F_{p+1};\,Q)$$
$$\approx \begin{cases} \pi_p \otimes_Z H_q(F_{p+1};\,Q), \text{ if } 0 \leqq q \leqq 2p-2,\, q \neq p\,. \\ H_{2p}(\pi_p,\,p;\,Q) = S_p(\pi_p \otimes Q), \text{ if } q = p\,. \end{cases}$$

Now we show that *if* $p \leqq q \leqq 2p-2$, *then* $H_q(F_p;\,Q) \approx H_q(F_q;\,Q) \approx \pi_q \otimes_Z Q$. If $q = p$, then $H_p(F_p;\,Q) \approx \pi_p \otimes Q$ by 1.1 (a) and the Hurewicz theorem. Consider the homology Serre spectral sequence with coefficients in Q of the fibration $F_{p+1} \subset F_p \rightarrow K(\pi_p,\,p)$ given by 1.1 (b). If $p < q \leqq 2p-2$, then the exact sequence of [5], page 284, implies that $i_*\colon H_q(F_{p+1}) \approx H_q(F_p)$. Similar arguments on the homology Serre spectral sequences for $F_{i+1} \hookrightarrow F_i \rightarrow K(\pi_i,\,i)$, $i = p+1, \cdots, q$ show that

$$H_q(F_p;\,Q) \approx H_q(F_{p+1};\,Q) \approx \cdots \approx H_q(F_{q-1};\,Q) \approx H_q(F_q;\,Q) \approx \pi_q \otimes Q$$

provided $p \leqq q \leqq 2p-2$.

COROLLARY 3.4. (*Rational Hurewicz Theorem*) *If* $i \leqq 2c(X)$ *then* $h_i \otimes 1\colon \pi_i(X) \otimes Q \rightarrow H_i(X;\,Q)$ *is an isomorphism.*

Proof. This is follows from 3.3 because the only non-zero term $E^1_{p,q}$ of total degree i (for $i \leqq 2c(X)$) is $E^1_{i,0} = \pi_i(X) \otimes Q = E^\infty_{i,0}$. Thus $\pi_i(X) \otimes Q \rightarrow H_i(X;\,Q)$ is an isomorphism. Kahn's theorem 4.1 [7] identifies this map (the edge homomorphism) as $h_i \otimes 1$.

This result was known to Cartan and Serre in [2].

We will now study the differentials in $\mathscr{C}(X;\,\infty;\,Q)$. According to Theorem 2.2 of [3] (see also [9], Chapter 2), given X, $\exists$ a CW-complex $X \otimes Q$ and a map $f\colon X \rightarrow X \otimes Q$

(a) $\pi_i(X \otimes Q) \approx \pi_i(X) \otimes Q$

(b) f is a homotopy equivalence modulo the class $\mathscr{T}$ of torsion groups.

(c) $\exists$ an isomorphism ν such that the following commutes:

$$\begin{array}{ccc} & \overset{f_\#}{\nearrow} & \pi_i(X \otimes Q) \\ \pi_i(X) & & \downarrow \nu \\ & \underset{t}{\searrow} & \pi_i(X) \otimes Q \end{array}$$

where $t(\alpha) = \alpha \otimes 1$, for $\alpha \in \pi_i(X)$.

Let $\widehat{X \otimes Q}$ be the space obtained from $X \otimes Q$ by killing off all the homotopy groups of $X \otimes Q$ in dimensions $\geqq 2 \cdot c(X) + 1$; $i\colon X \otimes Q \rightarrow \widehat{X \otimes Q}$ the inclusion map. Consider the composite map $i \circ f\colon X \rightarrow \widehat{X \otimes Q}$. This induces an exact couple map from

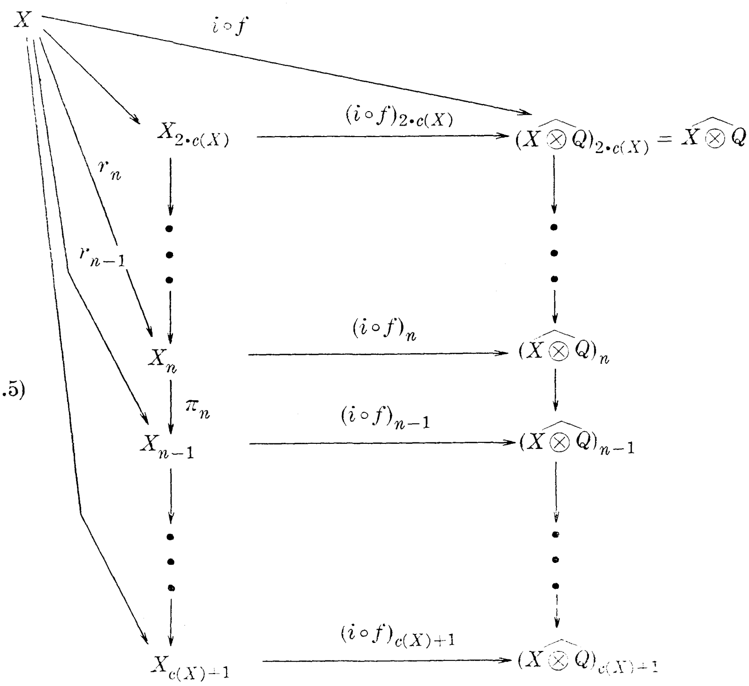
$$\mathscr{C}(\mathscr{P}(X);\,Q) \xrightarrow{\mathscr{C}(i \circ f)} \mathscr{C}(\mathscr{P}(\widehat{X \otimes Q});\,Q)$$

which we shall see is an isomorphism in a certain range of dimensions on the E^1-term. Theorem 4.4 of [3] implies that all the k-invariants of $X \otimes Q$ are trivial, i.e.,

$$\widehat{X \otimes Q} \cong \prod_{i=c(X)+1}^{2\cdot c(X)} K(\pi_i(X) \otimes Q, i) .$$

This implies that the spectral sequence $\{E^i(\widehat{X \otimes Q}; Q); \hat{d}^i\}$ collapses; i.e., all the $\hat{d}^i$ are zero. It follows from a theorem of Kahn [6], that $i \circ f$ induces maps $\mathscr{P}(i \circ f): \mathscr{P}(X) \to \mathscr{P}(\widehat{X \otimes Q})$ such that the following diagram commutes.

(3.5)



and $\pi_i(X_n) \xrightarrow{(i \circ f)_\#} \pi_i((\widehat{X \otimes Q})_n)$ $(i > 0)$ is an isomorphism mod $\mathscr{T}$. The commutativity of (3.5) $\Rightarrow (i \circ f)(F_n(X)) \subset F_n(\widehat{X \otimes Q})$ for $n \leqq 2\cdot c(X)$. An easy induction using the mod $\mathscr{T}$ 5-Lemma [5], and the homotopy ladder induced by

$$\begin{array}{ccc}
F_{n+1}(X) & \xrightarrow{(i\circ f)|F_{n+1}} & F_{n+1}(\widehat{X \otimes Q}) \\
\downarrow & & \downarrow \\
F_n(X) & \xrightarrow{(i\circ f)|F_n} & F_n(\widehat{X \otimes Q}) \\
\downarrow & & \downarrow \\
K(\pi_n(X), n) & \xrightarrow{(i\circ f)_n|K(\pi_n, n)} & K(\pi_n(X) \otimes Q, n)
\end{array}$$

shows that $(i\circ f|_{F_n(X)})_*\colon H_j(F_n(X); Q) \to H_j(F_n(\widehat{X\otimes Q}); Q)$ is a $\mathscr{T}$-isomorphism for $j \leqq 2\cdot c(X)$ (and an epimorphism for $j > 2\cdot c(X)$). By the Whitehead theorem mod $\mathscr{T}$ [5], page 512, we then have that

$$(3.6) \qquad (i\circ f|_{F_n(X)})_*\colon H_j(F_n(X); Q) \to H_j(F_n(\widehat{X\otimes Q}); Q)$$

is an isomorphism for $j \leqq 2\cdot c(X)$ and an epimorphism for $j = 2\cdot c(X) + 1$.

By the naturality of the universal coefficient theorem and the Serre spectral sequence, we have the following commutative diagram for $p \leqq 2\cdot c(X)$ and $p < q \leqq 2p - 2$.

(3.7)

$$\begin{array}{ccc}
E^1_{p,q}(X; Q) & \xrightarrow{E(i\circ f)} & E^1_{p,q}(\widehat{X\otimes Q}; Q) \\
\| & & \| \\
H_{p+q}(F_p(X), F_{p+1}(X); Q) & \xrightarrow{(i\circ f|_{F_p(X)})_*} & H_{p+q}(F_p(\widehat{X\otimes Q}), F_{p+1}(\widehat{X\otimes Q}); Q) \\
s(X) \downarrow \approx & & \approx \downarrow s(\widehat{X\otimes Q}) \\
H_p(K(\pi_p(X), p); H_q[F_{p+1}(X); Q]) & \xrightarrow{H_p(i\circ f_n|_{K(\pi_p,p)}; (i\circ f|_{F_{p+1}})_*)} & H_p(K(\pi_p(\widehat{X\otimes Q}), p); H_q[F_{p+1}(\widehat{X\otimes Q}); Q]) \\
UCT \downarrow \approx & & UCT \downarrow \approx \\
H_p(\pi_p, p) \otimes H_q(F_{p+1}(X)) \otimes Q & \xrightarrow{if_{n*} \otimes (i\circ f|_{F_{p+1}})_* \otimes 1} & H_p(\pi_p \otimes Q, p) \otimes H_q(F_{p+1}(\widehat{X\otimes Q})) \otimes Q
\end{array}$$

where $s(\cdot)$ in the above is the isomorphism defined from the Serre spectral sequence for $F_{p+1}(\cdot) \hookrightarrow F_p(\cdot) \to K(\pi_p(\cdot), p)$. In this range of dimensions ($p \leqq 2\cdot c(X)$, $p < q \leqq 2p - 2$) the vertical arrows are isomorphisms. 3.6 implies that the bottom *row* is an isomorphism, provided $q \leqq 2\cdot c(X)$. A similar argument gives the case $q = p$.

From this we deduce that

$$(3.8) \qquad E^1(i\circ f)\colon E^1_{p,q}(X; Q) \to E^1_{p,q}(\widehat{X\otimes Q}; Q)$$

is an isomorphism provided $0 \leqq p \leqq 2\cdot c(X)$, $0 \leqq q \leqq 2\cdot c(X)$. See Figure 3.2. (3.8) implies

$$(3.9) \qquad E^1_{p,q}(X; Q) \xrightarrow{E^1(i\circ f)} E^1_{p,q}(\widehat{X\otimes Q}; Q)$$

is an isomorphism for $p + q \leqq 3c(X) + 1$, $p \leqq 2c(X)$. (see Figure 3.2.)

Assume now that $c(X) \geqq 2$. *We will show that*

$$E^i_{p,q} = E^1_{p,q} \text{ for } 2 \leqq i \leqq q - 2$$

whenever $c(X) + 1 \leqq p \leqq 2\cdot c(X)$, $p \leqq q \leqq 3c(X) - p$. (These are the only nonzero terms of total degree $\leqq 3c(X)$ such that $q > 0$. See shaded area in Figure 3.2.) *Furthermore, all differential operators coming into* $E^i_{p,q}$ $(i > 0)$ *are zero and all differential operators issuing* *forth from* $E^i_{p,q}$ *are zero except for* $i = q - 1$.

We show this by arguing on the total degree j $(2c(X) + 2 \leqq j \leqq 3cX)$.

(a) $p + q = 2c(X) + 2 \Rightarrow p = c(X) + 1$. All differential operators with range $E^i_{cX+1\ cX+1}$ are zero for $i > 0$ since $E^1_{cX+1-i\ cX+1+i+1} = 0$ for all $i > 0$. Similarly all $d^i\colon E^i_{cX+1\ cX+1} \to E^i_{cX+1+i, cX+1-i-1}$ are zero for $i \leqq c(X) - 1$ since the latter group is zero in that range.

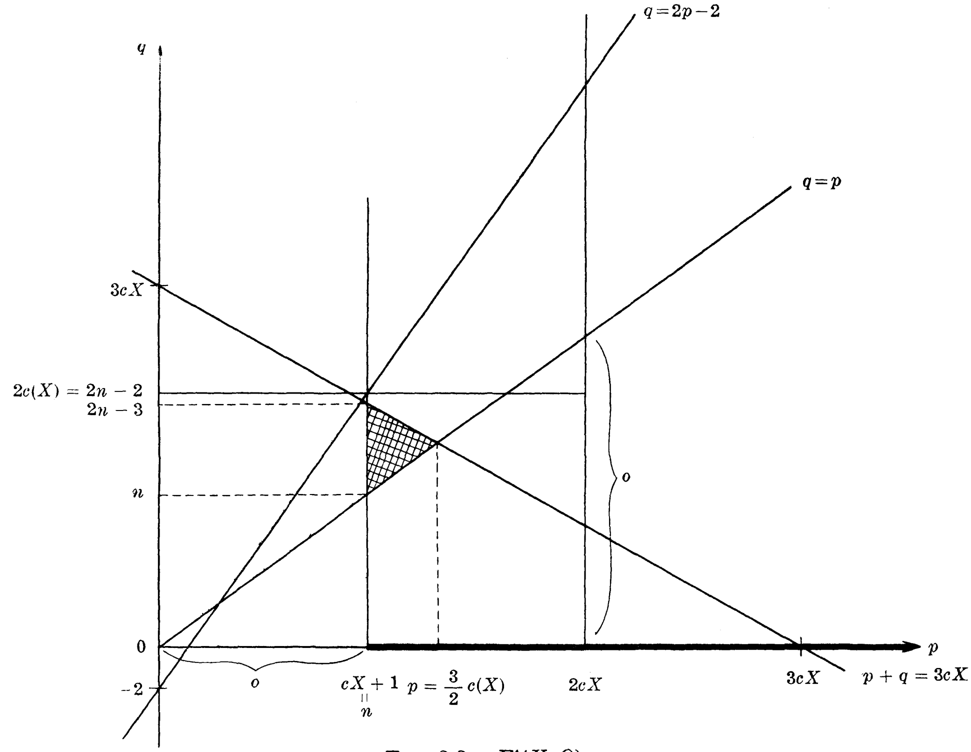


FIG. 3.2. $E^1(X; Q)$.

(b) Suppose $j > 2c(X) + 2$. Consider $p + q = j \leqq 3c(X)$, where $c(X) + 1 \leqq p \leqq [j/2]$, and the following commutative diagram

$$\begin{array}{ccc} E^1_{p-1,q+2} & \xrightarrow{E^1(i\circ f)_{p-1}} & \hat{E}^1_{p-1,q+2} \\ \downarrow d_p & & \downarrow \hat{d}_p \\ E^1_{p,q} & \xrightarrow{E^1(i\circ f)_p} & \hat{E}^1_{p,q} \\ \downarrow d_{p+1} & & \downarrow \hat{d}_{p+1} \\ E^1_{p+1,q-2} & \xrightarrow{E^1(i\circ f)_{p+1}} & \hat{E}^1_{p+1,q-2} \end{array}$$

where $E^1 \equiv E^1(X; Q)$, $\hat{E}^1 \equiv E^1(\widehat{X \otimes Q}; Q)$. $E^1(i\circ f)_k$ $(k = p - 1, p, p + 1)$ is an isomorphism by 3.9 since the total degree in each case is $\leqq 3c(X) + 1$.

Since $\hat{d}_i = 0$, we have $d_i = 0$ for $i = p, p+1$. Thus $E^1_{p,q} = E^2_{p,q}$ for (p, q) satisfying the above. Similar arguments imply $E^i_{p,q} = E^1_{p,q}$ for $i = 3, 4, \cdots, q-2$.

(c) $d^i: E^i_{p,q} \to E^i_{p+1,q-i-1}$ is zero for $i > q-1$ since $q - i - 1 < 0 \Rightarrow E_{p+i,q-i-1} = 0$. $d^i: E^i_{p-i,q+i-1} \to E^i_{p,q}$ is zero for $i \geqq q-1$ since $i \geqq q-1$, $q \geqq p \Rightarrow p - i \leqq p - q + 1 \Rightarrow E^1_{p-i,q+i-1} = 0$.

Thus the only (possibly) nonzero differential operator for each (p, q) satisfying $c(X) + 1 \leqq p \leqq 2\cdot c(X)$, $p \leqq q \leqq 3c(X) - p$ is

$$d^{q-1}: E^{q-1}_{p,q} \to E^{q-1}_{p+q-1,0} .$$

But this has been identified by Kahn in [7], Theorem 9.1, as the (rational) Whitehead product: If $q > p$

$$\begin{array}{ccc} \pi_p \otimes \pi_q \otimes Q & \xrightarrow{[\,,\,]\otimes id} & \pi_{p+q-1} \otimes Q \\ \uparrow \approx & & \uparrow \approx \\ E^{q-1}_{p,q} & \xrightarrow{d^{p-1}} & E^{q-1}_{p+q-1,0} \end{array} \qquad (q > p)$$

or, if $q = p$

$$\begin{array}{ccc} S_p(\pi_p \otimes Q) & \xrightarrow{[\,,\,]\otimes id} & \pi_{2p-1} \otimes Q \\ \uparrow \approx & & \uparrow \approx \\ E^{q-1}_{q,q} & \xrightarrow{d^{q-1}} & E^{q-1}_{2q-1,0} \end{array}$$

where [,] is the Whitehead product.

We have thus proved the following.

THEOREM 3.10. *Let* $c(X) \geqq 2$. *If* $p + q \leqq 3\cdot(X)$ *and* $q \geqq p$, *then*

(a) $d^i: E^i_{p-i,q+i+1} \to E^i_{p,q}$ *is zero for all* $i > 0$.

(b) $d^i: E^i_{p,q} \to E^i_{p+i,q-i-1}$ *is zero for* $i = 1, 2, \cdots, q-2, q, q+1, \cdots$

(c) $d^{q-1}: E^{q-1}_{p,q} \to E^{q-1}_{p+q-1,0}$ *is the rational Whitehead product.*

4. Applications. We are now in a position to compute $H_i(X; Q)$ $(i \leqq 3\cdot c(X))$ completely in terms of the graded homotopy group $\Pi = \{\pi_i \otimes Q \mid 1 \leqq i \leqq 3\cdot c(X)\}$ and the rational Whitehead product on this group. For $i \leqq 2\cdot c(X)$ this is given by the rational Hurewicz theorem (3.4). Let

$$\mathrm{Ker}_{ij} = \begin{cases} \mathrm{Ker}\,\{\pi_j \otimes \pi_{i-j} \otimes Q \xrightarrow{[\,,\,]\otimes id} \pi_{i-1} \otimes Q\}\ , & c(X) < j \leqq \left[\dfrac{i-1}{2}\right] \\ \mathrm{Ker}\,\{S(\pi_{i/2} \otimes Q) \xrightarrow{[\,,\,]\otimes id} \pi_{i-1} \otimes Q\}\ , & \text{if } i \text{ even, } j = \left[\dfrac{i}{2}\right] \\ 0\ , & \text{if } i \text{ odd, } j = \left[\dfrac{i}{2}\right] . \end{cases}$$

and

$$\mathrm{Ker}_i = \bigoplus_{c(X)<j\leqq[i/2]} \mathrm{Ker}_{ij} \qquad (\oplus \text{ denotes direct sum}),$$

where [,] is the Whitehead product.

Furthermore, let

$$\mathrm{Im}_{ij} = \begin{cases} \mathrm{im}\{\pi_j \otimes \pi_{i+1-j} \otimes Q \xrightarrow{[\,,\,]\otimes id} \pi_i \otimes Q\}\ , & \text{if } c(X) < j \leqq \left[\dfrac{i}{2}\right] \\ \mathrm{im}\,\{S(\pi_{(i+1)/2} \otimes Q) \xrightarrow{[\,,\,]\otimes id} \pi_i \otimes Q\}\ , & \text{if } i+1 \text{ even, } j = \left[\dfrac{i+1}{2}\right] \\ 0\ , & \text{if } i+1 \text{ odd, } j = \left[\dfrac{i+1}{2}\right] \end{cases}$$

and (since $\mathrm{Im}_{ij} \subset \pi_i \otimes Q$ for each j)

$$\mathrm{Im}_i = \sum_{c(X)<j\leqq[(i+1)/2]} \mathrm{Im}_{ij} \subset \pi_i \otimes Q. \quad (+ \text{ denotes sum, not necessarily direct})$$

THEOREM 4.1. *If* $2c(X) < i \leqq 3\cdot c(X)$, *then*

$$H_i(X; Q) \approx \mathrm{Ker}_i \oplus (\pi_i \otimes Q/\mathrm{Im}_i)$$

Proof. 3.4, 3.10 $\Rightarrow E^\infty_{i,0} \approx (\pi_i \otimes Q/\mathrm{Im}_i)$ and $E^\infty_{p,q}(c(X) < p \leqq [i/2], p + q = i) \approx \mathrm{Ker}_{ip}$. These are the only nonzero terms of total degree i. Since all extensions split we have

$$\begin{aligned} H_i(X; Q) &\approx E^\infty_{i,0} \oplus \bigoplus_{c(X)<p\leqq[i/2]} E^\infty_{p,i-p} \\ &\approx (\pi_i \otimes Q/\mathrm{Im}_i) \oplus \mathrm{Ker}_i\ . \end{aligned}$$

Since Kahn [7] has identified the edge homomorphism with the Hurewicz homomorphism we see

THEOREM 4.2. *If* $i \leqq 3\cdot c(X)$ *and* $h_i \otimes 1\colon \pi_i(X) \otimes Q \to H_i(X; Q)$ *is the Hurewicz homomorphism, then*

(a) $\mathrm{Ker}\, h_i \otimes 1 = \mathrm{Im}_i$

(b) $\mathrm{coker}\, h_i \otimes 1 = \mathrm{Ker}_i$

Proof. This follows because $h_i \otimes 1$ is the natural map

$$\pi_i \otimes Q \to \mathrm{Ker}_i \oplus (\pi_i \otimes Q/\mathrm{Im}_i)\ .$$

COROLLARY 4.3. *If* $i \leqq 3\cdot c(X)$, *then*

(a) $h_i \otimes 1$ *is a monomorphism* $\Leftrightarrow \mathrm{Im}_i = 0$

(b) $h_i \otimes 1$ *is an epimorphism* $\Leftrightarrow \mathrm{Ker}_i = 0$.

Note. By Proposition 2.1 (respectively, 4.1) of [1], $h_i \otimes 1$ is epic (respectively, monic) $\Leftrightarrow$ the i^{th} k'-invariant (k-invariant) of any homology (Postnikov) decomposition is of finite order. 4.3 gives another such characterization. This gives, for instance, the following theorem.

THEOREM 4.4 *If $\pi_i(X; Q) = 0$ for $i > 3 \cdot c(X)$, then all k-invariants are of finite order $\Leftrightarrow$ all rational Whitehead products vanish.*

Finally, since it is usually easier to compute $H_i(X; Q)$ than it is the Whitehead product, we will use these relations (4.1 and 4.2) to give information about the Whitehead products themselves.

THEOREM 4.5. *Let $i \leqq 3 \cdot c(X)$ and consider the following statements:*

(a) *$\pi_i \otimes Q$ is generated by Whitehead products.*

(b) *For all r such that $c(X) < r \leqq [(i-1)/2]$, $\pi_r \otimes \pi_{i-r} \otimes Q \to \pi_{i-1} \otimes Q$ is injective.*

(c) *If i even, $S(\pi_{i/2} \otimes Q) \to \pi_{i-1} \otimes Q$ is injective. The following are true.*

(d) $h_i \otimes 1 = 0 \Leftrightarrow$ (a)

(e) $\operatorname{coker} h_i \otimes 1 = 0 \Leftrightarrow$ (b) *and* (c)

(f) $H_i(X; Q) = 0 \Leftrightarrow$ (a), (b) *and* (*c*).

REFERENCES

1. M. Arkowitz, and C. R. Curjel, *The Hurewicz homomorphism and finite homotopy invariants*, Trans. Amer. Math. Soc., **110** (1964), 538–551.
2. H. Cartan, and J.-P. Serre, *Espaces fibres et groupes d'homotopie, II, Applications*, C. R. Acad. Sci. Paris, **234** (1952), 393–395.
3. M. Dyer, *Replacing Postnikov systems by simpler ones*, (unpublished).
4. S. Eilenberg, and S. MacLane, *Relations between homology and homotopy groups oj spaces*, Ann. of Math., (2) **46** (1945), 480–509.
5. S.-T. Hu, *Homotopy Theory*, Academic Press, New York, 1959.
6. D. W. Kahn, *Induced maps for Postnikov systems*, Trans. Amer. Math. Soc., **107** (1963), 432–450.
7. ———, *The spectral sequence of a Postnikov system*, Comm. Math. Helv., **40** (1966), 196–198.
8. J.-P. Meyer, *Whitehead products and Postnikov systems*, Amer. J. Math., **82** (1960).
9. D. Sullivan, *Geometric Topology, Part I*, Mass. Inst. of Tech. Notes, 1970.

Received October 5, 1970 and in revised form June 7, 1971.

UNIVERSITY OF OREGON

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 40, No. 1 September, 1972