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D. W. Kahn defined a spectral sequence $\mathscr{E}(X; R)$ for the Postnikov system $\mathscr{P}(X)$ of a 1-connected CW-complex which converges to $H_*(X; R)$, the singular homology of X with coefficients in R. We study $\mathscr{E}(X; R)$ in two settings: (a) to give a generalization of the classical theorem of Eilenberg and MacLane concerning the dependence of $H_i(X; Z)$ on the first nonzero homotopy group of X (2.1) and (b) to give a complete computation of $H_i(X; Q)$ ($Q =$ rationals) for $i \leqq 3 \cdot c(X)$ ($c(X) =$ connectivity of X) in terms of the *graded* homotopy group $\Pi \otimes Q = \{\pi_i(X) \otimes Q \mid 0 < i \leqq 3 \cdot c(X)\}$ and the Whitehead product on this group (0.1 and 0.2).

In § 1 we give a quick description of $\mathscr{E}(X; R)$ for later use and in § 2 we generalize the Eilenberg-MacLane theorem by giving an exact sequence involving the first *two* nonzero homotopy groups. $\mathscr{E}(X, Q)$ is studied in § 3, with the result that we are able to identify $E^1(X; Q)$ somewhat above the diagonal (Kahn identified it below the diagonal in [7]) (3.3) and to show that the Whitehead product is the only non-zero differential operator, provided the total degree is less than $3 \cdot c(X)$ (3·10). Section 4 gives the computations of $H_i(X; Q)$ and various other applications.

1. Description of the Spectral Sequence of $\mathscr{P}(X)$.

In this note X is a $(n-1)$-connected space, $n > 1$, having the homotopy type of a CW-complex. All maps and spaces are "pointed".

Let $\{X_i, r_i, \pi_i\} = \mathscr{P}(X)$ be a Postnikov system for X (see [6] for definition). Choose $m > n$ and convert the map $r_m: X \to X_m$ into a fiber map. Use the same notation for the new map. In the tower of spaces

$$X \xrightarrow{r_m} X_m \xrightarrow{\pi_m} X_{m-1} \xrightarrow{\pi_{m-1}} \longrightarrow \cdots \xrightarrow{\pi_{n+1}} X_n = K(\pi_n(X), n)$$

$\pi_\alpha \circ \cdots \circ \pi_m \circ r_m \simeq r_{\alpha-1}$ $(n + 1 \leqq \alpha \leqq m)$. Let $r_{\alpha-1}$ denote this composition, $\alpha = n+1, \cdots, m$. Since all these maps are Hurewicz fibrations, $r_{\alpha-1} (\alpha - 1 < m)$ is a fiber map. Let $F_{i+1} = r_i^{-1}$ (base point) denote the fiber of $r_i: X \to X_i$, $i \leqq m$. The following is proved in [7].

LEMMA 1.1. (a) *F_{i+1} is i-connected.*

(b) *F_{i+1} is fibered over $K(\pi_{i+1}(X), i+1)$, with fiber F_{i+2}, via the map $r_{i+1} | F_{i+1}$.*

(c) *$X = F_n \supset F_{n+1} \supset \cdots \supset F_m \supset F_{m+1}$ is a finite de-*

creasing filtration of X.

For each m, the exact couple ([7]) $\mathscr{C}(\mathscr{P}(X), m; G)$ is defined by

$$D^1_{r,s} = \begin{cases} H_{r+s}(F_r; G), & \text{if } r, s \geqq 0 . \\ 0 , & \text{otherwise,} \end{cases}$$

$$E^1_{r,s} = \begin{cases} H_{r+s}(F_r, F_{r+1}; G), & \text{if } r, s \geqq 0 . \\ 0 , & \text{otherwise,} \end{cases}$$

where G is any abelian group and H_* is singular homology. If $D^1 = \sum_\oplus D^1_{r,s}$, $E^1 = \sum_\oplus E^1_{r,s}$ then the couple maps $i: D^1 \to D^1$, $j: D^1 \to E^1$ and $k: E^1 \to D^1$ are of bidegree (respectively) $(-1, 1)$, $(0, 0)$, $(1, -2)$. The bidegree of the differential operator $d_i: E^i \to E^i$ is $(i, -i-1)$.

In [7], Kahn shows that

$$E^1_{j,s} = H_{j+s}(F_j, F_{j+1}; G) \xrightarrow{q_{j*}} \tilde{H}_{j+s}(\pi_j(X), j; G) \tag{1.2}$$

is an isomorphism, provided $s \leqq j$, where

$$q_j = r_j | F_j: (F_j, F_{j+1}) \to (K(\pi_j(X), j), *) ,$$

thus indentifying the E^1 term below the diagonal.

2. Generalization of a theorem of Eilenberg-MacLane. In [4], Eilenberg and MacLane showed the dependence of the first few homology groups of a space X upon the first nonzero homotopy group of X. We prove the following generalization.

THEOREM 2.1. *Let X be an $(n-1)$-connected space having the homotopy type of a CW-complex, $n \geqq 2$. Suppose $\pi_i(X) = 0$ for $n < i < p$ and $p < i < q \leqq 2n$. Then $H_i(X; G) \approx H_i(\pi_n(X), n; G)$ for $n \leqq i < p$ and any abelian group G. Furthermore, if we abbreviate $H_j(\pi_l(X), l; G)$ by $H_j(l; G)$, we have the exact sequence*

$$H_q(n; G) \xrightarrow{\Phi_q} H_{q-1}(p; G) \xrightarrow{\psi_{q-1}} H_{q-1}(X; G) \xrightarrow{\chi_{q-1}} H_{q-1}(n; G) \xrightarrow{\Phi_{q-1}} \cdots$$
$$\cdots \longrightarrow H_i(p; G) \xrightarrow{\psi_i} H_i(X; G) \xrightarrow{\chi_i} H_i(n; G) \xrightarrow{\Phi_i} H_{i-1}(p; G) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_p(p; G) \xrightarrow{\psi_p} H_p(X; G) \xrightarrow{\chi_p} H_p(n; G) \longrightarrow 0 .$$

$\Phi_i = T_i \circ (k)_*$, where $k: K(\pi_n(X), n) \to K(\pi_p(X), p+1)$ is the first k-invariant in a Postnikov decomposition of X and $T_j: H_j(\pi_p(X), p+1; G) \to H_{j-1}(\pi_p(X), p; G)$ is the transgression, which is an isomorphism provided $0 < j \leqq 2p$. Further, ψ_p is the Hurewicz homomorphism.

Proof. We consider $\mathscr{C}(\mathscr{P}(X), m; G)$ for $m > 2n$. $\pi_i(X) = 0$ for $n < i < p$, $p < i < q$ implies by 1.1 (b) that

$$(2.2) \qquad X = F_n \supset F_{n+1} = \cdots = F_p \supset F_{p-1} = \cdots = F_q \supset \cdots .$$

Thus $E^1_{r,s} = 0$ for $0 \leqq r < n$, $n < r < p$, $p < r < q$ and all s. This gives a two-term condition (see [5], chapter VIII) on the E^1-term of $\mathscr{C}(\mathscr{P}(X), m; G)$. Using (1.2) we have that $H_i(X; G) \approx H_i(\pi_n(X), n; G)$ for $n \leqq i < p$ (a 1-term condition here) and for $p \leqq i < q$ we have the exact sequence of the theorem. Note that we did not need $q \leqq 2n$ in order to obtain the two-term condition, but only in order to use (1.2). It is clear from [7] that ψ_p (the edge homomorphism) is the Hurewicz homomorphism.

We will now show that $\Phi_i = T_i \circ (k)_*$. Since Φ_i is essentially $d^{(p-n)}: E^{p-n}_{n,i-n} \rightarrow E^{p-n}_{p,i-1-p}$ ([7]), we will show that $d^{(p-n)} = T_i \circ (k)_*$. As it has significance in its own right, we give it as a separate lemma.

Lemma 2.3 *If* $\pi_i(X) = 0$ *for* $1 \leqq i < n$, $n < i < p$, $p < i < q$, *then*
(a) $E^1_{r,s} = E^{p-n}_{r,s}$ *for* $r = n, p$ *provided* $s \leqq q - p$.
(b) *The following triangle commutes for* $s \leqq \min\{n, q - p\}$.

$$\begin{array}{ccc} E^{p-n}_{n,s} = \tilde{H}_{n+s}(\pi_n(X), n; G) & \xrightarrow{d^{p-n}} & \tilde{H}_{n+s-1}(\pi_p(X), p; G) = E^{p-n}_{p,-(p-n)+s-1} \\ & {\scriptstyle k_*} \searrow \quad \nearrow {\scriptstyle T} & \\ & \tilde{H}_{n+s}(\pi_p(X), p+1; G) & \end{array}$$

where (i) $k: K(\pi_n(X), n) \rightarrow K(\pi_p(X), p+1)$ *is the first k-invariant,*
(ii) T *is the composite* $\partial \circ w_*^{-1}$

where $K(\pi_p, p) \hookrightarrow PK(\pi_p, p+1) \xrightarrow{w} K(\pi_p, p+1)$ $(\pi_p \equiv \pi_p(X))$ *is the usual path space fibration.* T *is an isomorphism provided* $n + s \leqq 2p$.

Proof. (a) follows because $\pi_i(X) = 0$ for $1 \leqq i < n$, $n < i < p$

$$\Rightarrow E^1_{n,s} = E^{p-n}_{n,s}$$

for all s, since $d^{p-n}: E^1_{n,s} \rightarrow E_{p\ s-(p-n)-1}$ is the first nonzero differential operator. $E^1_{p,s} = E^{p-n}_{p,s}$ provided $s \leqq q - p$ since $\pi_i(X) = 0$ for $n < i < p$, $p < i < q$ implies that $d^i: E^i_{p-i\ s+i+1} \rightarrow E^i_{p\ s}$ is zero unless $i = p - n$ and $d^i: E^i_{p,s} \rightarrow E^i_{p+i,s-i-1}$ is zero provided $s \leqq q - p$.

(b) since d^{p-n} is given by the composition (see 2.2)

$$H_{n+s}(F_n, F_p) \xrightarrow{\partial} \tilde{H}_{n+s-1}(F_p) \xrightarrow{j_*} H_{n+s-1}(F_p, F_q)$$

we are asking that the following diagram commute:

(2.4)
$$\begin{array}{ccccccc} H_{n+s}(F_n, F_p) & \xrightarrow{\partial} & \tilde{H}_{n+s-1}(F_p) & & \xrightarrow{\qquad j_* \qquad} & & H_{n+s-1}(F_p, F_q) \\ \downarrow {\scriptstyle (q_n)_*} & & & & & & \downarrow {\scriptstyle (\bar{k} \circ q_p)_*} \\ \tilde{H}_{n+s}(\pi_n(X), n) & \xrightarrow{k_*} & \tilde{H}_{n+s}(\pi_p(X), p+1) & \xleftarrow{w_*} & H_{n+s}(PK, K(\pi_p(X), p)) & \xrightarrow{\partial} & \tilde{H}_{n+s-1}(\pi_p(X),), p \end{array}$$

where $\bar{k}$ is defined by (2.6) below, and $q_i = r_i|_{F_i}$. (2.4) commutes if and only if

$$(2.5)\qquad \begin{array}{ccc} H_{n+s}(F_n, F_p) & \xrightarrow{\quad\partial\quad} & \tilde{H}_{n+s-1}(F_p) \\ \Big\downarrow {\scriptstyle w_*^{-1}\circ k_*\circ q_{n*}} & & \Big\downarrow {\scriptstyle (\bar{k}\circ q_p\circ j)_*} \\ H_{n+s}(PK, K(\pi_p(X), p)) & \xrightarrow{\partial} & \tilde{H}_{n+s-1}(\pi_p(X), p) \end{array}$$

commutes. We have the following situation:

$$(2.6)\qquad \begin{array}{ccccc} & & X_p & \xrightarrow{\quad\bar{k}\quad} & PK \\ & \nearrow{\scriptstyle r_p} & \Big\downarrow{\scriptstyle \pi_p} & & \Big\downarrow{\scriptstyle w} \\ X = F_n & \xrightarrow{\quad r_n = q_n\quad} & K(\pi_n, n) & \xrightarrow{\quad k\quad} & K(\pi_p, p+1) \\ \cup & & & & \\ F_p & & & & \\ \cup & & & & \\ F_q & & & & \end{array}$$

where $k\circ q_n = k\circ\pi_p\circ r_p = w\circ\bar{k}\circ r_p \Rightarrow w_*^{-1}\circ k_*\circ q_{n*} = \bar{k}_*\circ r_{p*}$. But $\bar{k}\circ r_p|_{F_p} = \bar{k}\circ q_p$ is clearly the same as $\bar{k}\circ q_p\circ j$ considered as maps of the pairs $(F_p, *) \rightarrow (F_p, F_q) \rightarrow (PK, *)$. This shows that (2.5) commutes.

By an argument similar to Lemma 2.3, we may identify the d^1 operator below the diagonal. This was claimed in [7], page 176.

LEMMA 2.4. *The following commutes for $s \leqq j$.*

$$\begin{array}{ccc} \tilde{H}_{j+s}(\pi_j, j) & \xrightarrow{d^1} & \tilde{H}_{j+1}(\pi_{j+1}, j+1) \\ {\scriptstyle (k_j\circ i_j)_*}\searrow & & \nearrow{\scriptstyle T} \\ & \tilde{H}_{j+s}(\pi_{j+1}, j+2) & \end{array}$$

where (a) $k_j\colon X_j \rightarrow K(\pi_{j+1}(X), j+2)$ *is the* jth *k-invariant,*

(b) $i_j\colon K(\pi_j(Y), j) \hookrightarrow X_j$ *is the inclusion, and*

(c) T *is the transgression* (which is an isomorphism for $s \leqq j+2$).

3. Rational homology and Whitehead products. In this section we consider Kahn's spectral sequence with coefficients in Q, the rationals. For this special case we are able to identify the E^1-term considerably above the diagonal. This occurs because for Q coefficients,

$H_*(\pi, n; Q) \approx$ *a* Hopf algebra over Q on $\dim_Q(\pi \otimes_Z Q)$ generators of degree n.

In [8], J. P. Meyer demonstrated how to compute Whitehead products in $\pi_*(X)$ from a Postnikov system for X and in [7], Theorem 9.1, D. W. Kahn used Meyer's results to show that a certain higher differential operator in $\mathscr{C}(X; Q)$ *is* the Whitehead product. In the range of our identification, we show that this differential is the *only* nonzero differential operator. This allows a complete computation of $H_i(X; Q)$, $i \leqq 3 \cdot c(X)$, in terms of the homotopy groups of X and the (rational) Whitehead products, where $c(X)$ is the connectivity of X.

DEFINITION 3.1. Let G be an arbitrary Q-vector space and p be a positive integer. The skew-symmetric tensor product $S_p(G)$ is defined as

$$S_p(G) = (G \otimes_Q G)/R$$

where R is the subspace generated by $\{g_i \otimes g_j - (-1)^{p \cdot p} g_j \otimes g_i \mid g_i, g_j \in G\}$. Suppose $\nu = \dim_Q G$, and let $\Lambda(\nu, p)$ be the free commutative graded algebra over Q on generators $(t_1, \cdots, t_\nu)$ where degree $t_i = p$ (ν need not be finite).

$$\Lambda(\nu, p) \approx \begin{cases} Q[t_1, \cdots, t_\nu] \text{ if } p \text{ even}, \\ E_Q(t_1, \cdots, t_\nu) \text{ if } p \text{ odd}, \end{cases}$$

where $Q[t_1, \cdots]$ is the graded polynomial algebra over Q, $E_Q(t_1, \cdots)$ is the graded exterior algebra over Q, on generators $t_1, \cdots, t_\nu$ of degree p. Then it is easy to see that $S_p(G) \approx \Lambda(\nu, p)_{2p}$, the Q-module of $\Lambda(\nu, p)$ in degree $2p$.

LEMMA 3.2. *Let G be an abelian group. Then* $H_{2p}(G, p; Q) \approx S_p(G \otimes Q)$.

Proof. This follows because $H_*(G, p; Q) = \Lambda(\dim_Q (G \otimes Q), p)$.

THEOREM 3.3. *Let $c(X) = n - 1$, for $n \geqq 2$. In $\mathscr{C}(\mathscr{P}(X), \infty; Q)$, the E^1-term is given as follows* ($\otimes$ means $\otimes_Z$): *For all $p > 0$,*

$$E^1_{p,q}(X; Q) \approx \begin{cases} \pi_p \otimes Q, \ \textit{if } q = 0 \\ 0, \ \textit{if } 0 < q < p, \\ S_p(\pi_p \otimes Q), \ \textit{if } q = p \\ \pi_p \otimes \pi_q \otimes Q, \ \textit{if } p + 1 \leqq q \leqq 2p - 2, \end{cases}$$

where $\pi_i \equiv \pi_i(X)$ (see Figure 3.1).

12

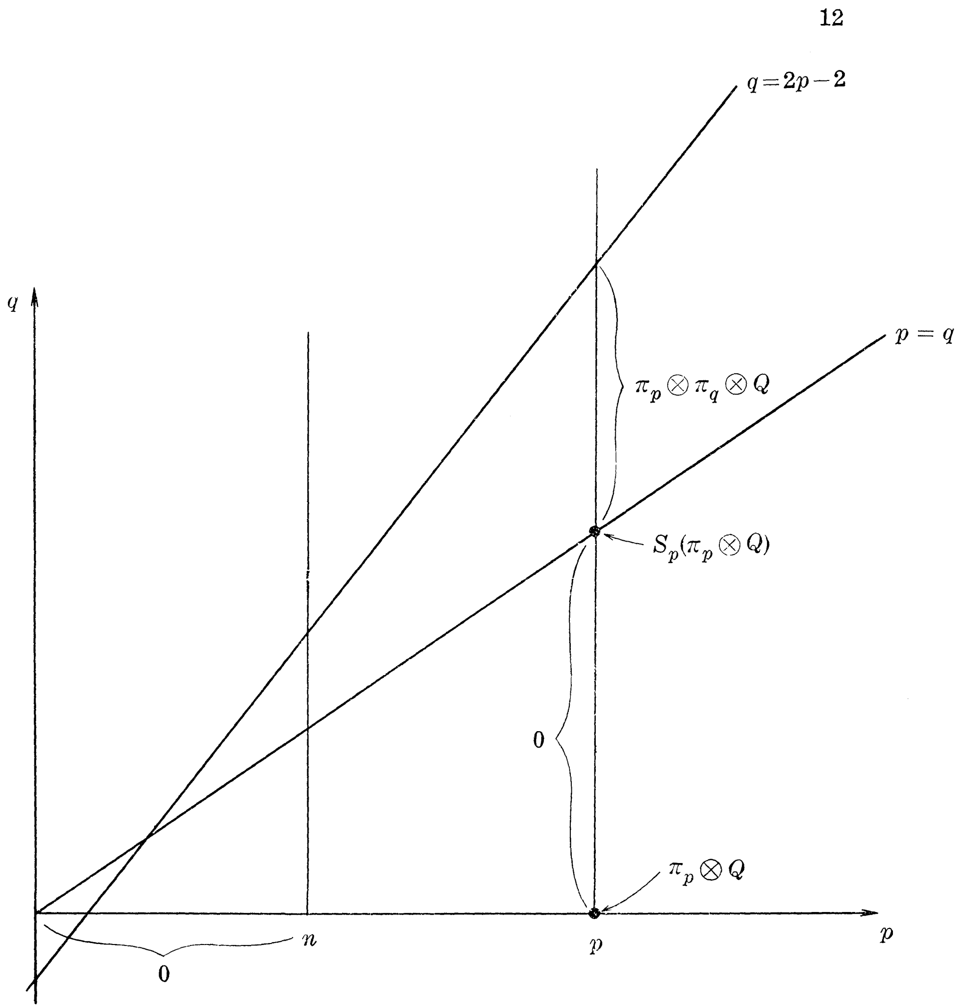


FIG. 3.1. $E^1(X; Q)$.

Proof. Let $p > 1$ and consider the homology Serre spectral sequence [5] for the fibration $F_{p+1} \hookrightarrow (F_p, F_{p+1}) \to (K(\pi_p, p), *)$. The E^2-term, with coefficients in Q, is

$$E^2_{r,s} \approx H_r(K(\pi_p, p), *; H_s(F_{p+1}; Q) \approx \tilde{H}_r(\pi_p, p; Q) \otimes_Q H_s(F_{p+1}; Q) .$$

Note that if $r < 2p$, then $E^2_{r,s} = 0$ unless $r = p$ and

$$E^2_{p,s} \approx \pi_p \otimes_Z H_s(F_{p+1}; Q) .$$

It is easy to see from this, 1.1 (a), and the fact that

$$H_*(\pi_p, p; Q) \approx \wedge (\dim_Q (\pi_p \otimes Q), p)$$

that

$$E_{p\,q}(X; Q) \approx H_{p+q}(F_p, F_{p+1}; Q)$$
$$\approx \begin{cases} \pi_p \otimes_Z H_q(F_{p+1}; Q), \text{ if } 0 \leqq q \leqq 2p-2, q \neq p\,. \\ H_{2p}(\pi_p, p; Q) = S_p(\pi_p \otimes Q), \text{ if } q = p\,. \end{cases}$$

Now we show that *if* $p \leqq q \leqq 2p-2$, *then* $H_q(F_p; Q) \approx H_q(F_q; Q) \approx \pi_q \otimes_Z Q$. If $q = p$, then $H_p(F_p; Q) \approx \pi_p \otimes Q$ by 1.1 (a) and the Hurewicz theorem. Consider the homology Serre spectral sequence with coefficients in Q of the fibration $F_{p+1} \subset F_p \to K(\pi_p, p)$ given by 1.1 (b). If $p < q \leqq 2p-2$, then the exact sequence of [5], page 284, implies that $i_*: H_q(F_{p+1}) \approx H_q(F_p)$. Similar arguments on the homology Serre spectral sequences for $F_{i+1} \hookrightarrow F_i \to K(\pi_i, i)$, $i = p+1, \cdots, q$ show that

$$H_q(F_p; Q) \approx H_q(F_{p+1}; Q) \approx \cdots \approx H_q(F_{q-1}; Q) \approx H_q(F_q; Q) \approx \pi_q \otimes Q$$

provided $p \leqq q \leqq 2p-2$.

COROLLARY 3.4. (*Rational Hurewicz Theorem*) *If* $i \leqq 2c(X)$ *then* $h_i \otimes 1: \pi_i(X) \otimes Q \to H_i(X; Q)$ *is an isomorphism.*

Proof. This is follows from 3.3 because the only non-zero term $E^1_{p,q}$ of total degree i (for $i \leqq 2c(X)$) is $E^1_{i,0} = \pi_i(X) \otimes Q = E^\infty_{i,0}$. Thus $\pi_i(X) \otimes Q \to H_i(X; Q)$ is an isomorphism. Kahn's theorem 4.1 [7] identifies this map (the edge homomorphism) as $h_i \otimes 1$.

This result was known to Cartan and Serre in [2].

We will now study the differentials in $\mathscr{C}(X; \infty; Q)$. According to Theorem 2.2 of [3] (see also [9], Chapter 2), given X, $\exists$ aCW-complex $X \otimes Q$ and a map $f: X \to X \otimes Q$

(a) $\pi_i(X \otimes Q) \approx \pi_i(X) \otimes Q$

(b) f is a homotopy equivalence modulo the class $\mathscr{T}$ of torsion groups.

(c) $\exists$ an isomorphism ν such that the following commutes:

$$\begin{array}{ccc} & \overset{f_\sharp}{\nearrow} & \pi_i(X \otimes Q) \\ \pi_i(X) & & \downarrow \nu \\ & \underset{t}{\searrow} & \pi_i(X) \otimes Q \end{array}$$

where $t(\alpha) = \alpha \otimes 1$, for $\alpha \in \pi_i(X)$.

Let $\widehat{X \otimes Q}$ be the space obtained from $X \otimes Q$ by killing off all the homotopy groups of $X \otimes Q$ in dimensions $\geqq 2 \cdot c(X) + 1$; $i: X \otimes Q \to \widehat{X \otimes Q}$ the inclusion map. Consider the composite map $i \circ f: X \to \widehat{X \otimes Q}$. This induces an exact couple map from

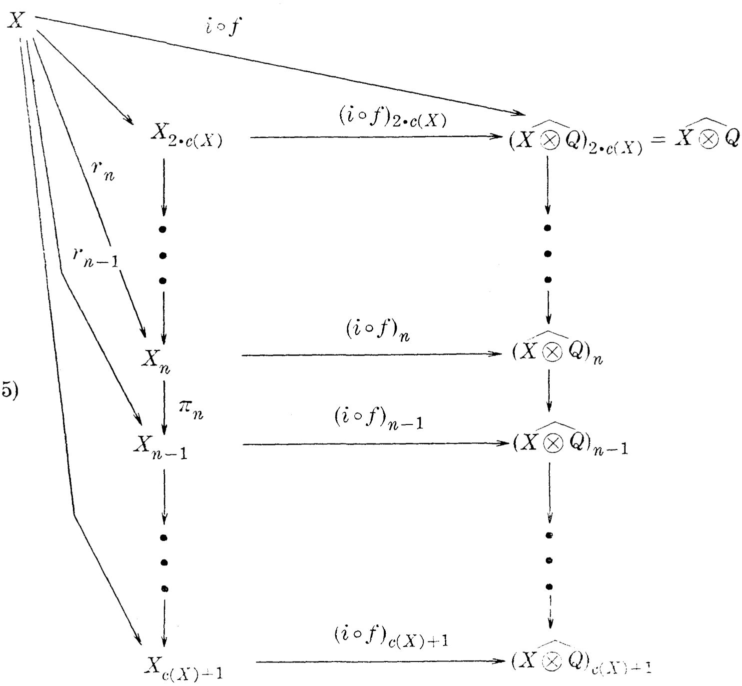
$$\mathscr{C}(\mathscr{P}(X); Q) \xrightarrow{\mathscr{C}(i \circ f)} \mathscr{C}(\mathscr{P}(\widehat{X \otimes Q}); Q)$$

which we shall see is an isomorphism in a certain range of dimensions on the E^1-term. Theorem 4.4 of [3] implies that all the k-invariants of $X \otimes Q$ are trivial, i.e.,

$$\widehat{X \otimes Q} \cong \prod_{i=c(X)+1}^{2 \cdot c(X)} K(\pi_i(X) \otimes Q, i) .$$

This implies that the spectral sequence $\{E^i(\widehat{X \otimes Q}; Q); \hat{d}^i\}$ collapses; i.e., all the $\hat{d}^i$ are zero. It follows from a theorem of Kahn [6], that $i \circ f$ induces maps $\mathscr{P}(i \circ f)\colon \mathscr{P}(X) \to \mathscr{P}(\widehat{X \otimes Q})$ such that the following diagram commutes.

(3.5)



and $\pi_i(X_n) \xrightarrow{(i \circ f)_\sharp} \pi_i((\widehat{X \otimes Q})_n)$ $(i > 0)$ is an isomorphism mod $\mathscr{T}$. The commutativity of (3.5) $\Rightarrow (i \circ f)(F_n(X)) \subset F_n(X \otimes Q)$ for $n \leqq 2 \cdot c(X)$. An easy induction using the mod $\mathscr{T}$ 5-Lemma [5], and the homotopy ladder induced by

$$\begin{array}{ccc}
F_{n+1}(X) & \xrightarrow{(i \circ f)|F_{n+1}} & F_{n+1}(\widehat{X \otimes Q}) \\
\downarrow & & \downarrow \\
F_n(X) & \xrightarrow{(i \circ f)|F_n} & F_n(\widehat{X \otimes Q}) \\
\downarrow & & \downarrow \\
K(\pi_n(X), n) & \xrightarrow{(i \circ f)_n|K(\pi_n, n)} & K(\pi_n(X) \otimes Q, n)
\end{array}$$

shows that $(i\circ f|_{F_n(X)})_*: H_j(F_n(X); Q) \to H_j(F_n(\widehat{X \otimes Q}); Q)$ is a $\mathscr{T}$-isomorphism for $j \leqq 2\cdot c(X)$ (and an epimorphism for $j > 2\cdot c(X)$). By the Whitehead theorem mod $\mathscr{T}$ [5], page 512, we then have that

$$(3.6) \qquad (i\circ f|_{F_n(X)})_*: H_j(F_n(X); Q) \to H_j(F_n(\widehat{X \otimes Q}); Q)$$

is an isomorphism for $j \leqq 2\cdot c(X)$ and an epimorphism for $j = 2\cdot c(X) + 1$.

By the naturality of the universal coefficient theorem and the Serre spectral sequence, we have the following commutative diagram for $p \leqq 2\cdot c(X)$ and $p < q \leqq 2p - 2$.

(3.7)

$$\begin{array}{ccc}
E^1_{p,q}(X; Q) & \xrightarrow{E(i\circ f)} & E^1_{p,q}(\widehat{X \otimes Q}; Q) \\
\Vert & & \Vert \\
H_{p+q}(F_p(X), F_{p+1}(X); Q) & \xrightarrow{(i\circ f|_{F_p(X)})_*} & H_{p+q}(F_p(\widehat{X \otimes Q}), F_{p+1}(\widehat{X \otimes Q}); Q) \\
s(X) \downarrow \approx & & \approx \downarrow s(\widehat{X \otimes Q}) \\
H_p(K(\pi_p(X), p); H_q[F_{p+1}(X); Q]) & \xrightarrow{H_p(i\circ f_n|_{K(\pi_p, p)}; (i\circ f|_{F_{p+1}})_*)} & H_p(K(\pi_p(\widehat{X \otimes Q}), p); H_q[F_{p+1}(\widehat{X \otimes Q}); Q]) \\
UCT \downarrow \approx & & UCT \downarrow \approx \\
H_p(\pi_p, p) \otimes H_q(F_{p+1}(X)) \otimes Q & \xrightarrow{if_{n*} \otimes (i\circ f|_{F_{p+1}})_* \otimes 1} & H_p(\pi_p \otimes Q, p) \otimes H_q(F_{p+1}(\widehat{X \otimes Q})) \otimes Q
\end{array}$$

where $s(\cdot)$ in the above is the isomorphism defined from the Serre spectral sequence for $F_{p+1}(\cdot) \hookrightarrow F_p(\cdot) \to K(\pi_p(\cdot), p)$. In this range of dimensions ($p \leqq 2\cdot c(X)$, $p < q \leqq 2p - 2$) the vertical arrows are isomorphisms. 3.6 implies that the bottom *row* is an isomorphism, provided $q \leqq 2\cdot c(X)$. A similar argument gives the case $q = p$.

From this we deduce that

$$(3.8) \qquad E^1(i\circ f): E^1_{p,q}(X; Q) \to E^1_{p,q}(\widehat{X \otimes Q}; Q)$$

is an isomorphism provided $0 \leqq p \leqq 2\cdot c(X)$, $0 \leqq q \leqq 2\cdot c(X)$. See Figure 3.2. (3.8) implies

$$(3.9) \qquad E^1_{p,q}(X; Q) \xrightarrow{E^1(i\circ f)} E^1_{p,q}(\widehat{X \otimes Q}; Q)$$

is an isomorphism for $p + q \leqq 3c(X) + 1$, $p \leqq 2c(X)$. (see Figure 3.2.)

Assume now that $c(X) \geqq 2$. *We will show that*

$$E^i_{p,q} = E^1_{p,q} \ \textit{for} \ 2 \leqq i \leqq q - 2$$

whenever $c(X) + 1 \leqq p \leqq 2\cdot c(X)$, $p \leqq q \leqq 3c(X) - p$. (These are the only nonzero terms of total degree $\leqq 3c(X)$ such that $q > 0$. See shaded area in Figure 3.2.) *Furthermore, all differential operators coming into* $E^i_{p,q}$ $(i > 0)$ *are zero and all differential operators issuing* *forth from* $E^i_{p,q}$ *are zero except for* $i = q - 1$.

We show this by arguing on the total degree j $(2c(X) + 2 \leqq j \leqq 3cX)$.

(a) $p + q = 2c(X) + 2 \Rightarrow p = c(X) + 1$. All differential operators with range $E^i_{cX+1\ cX+1}$ are zero for $i > 0$ since $E^1_{cX+1-i\ cX+1+i+1} = 0$ for all $i > 0$. Similarly all d^i: $E^i_{cX+1\ cX+1} \rightarrow E^i_{cX+1+i, cX+1-i-1}$ are zero for $i \leqq c(X) - 1$ since the latter group is zero in that range.

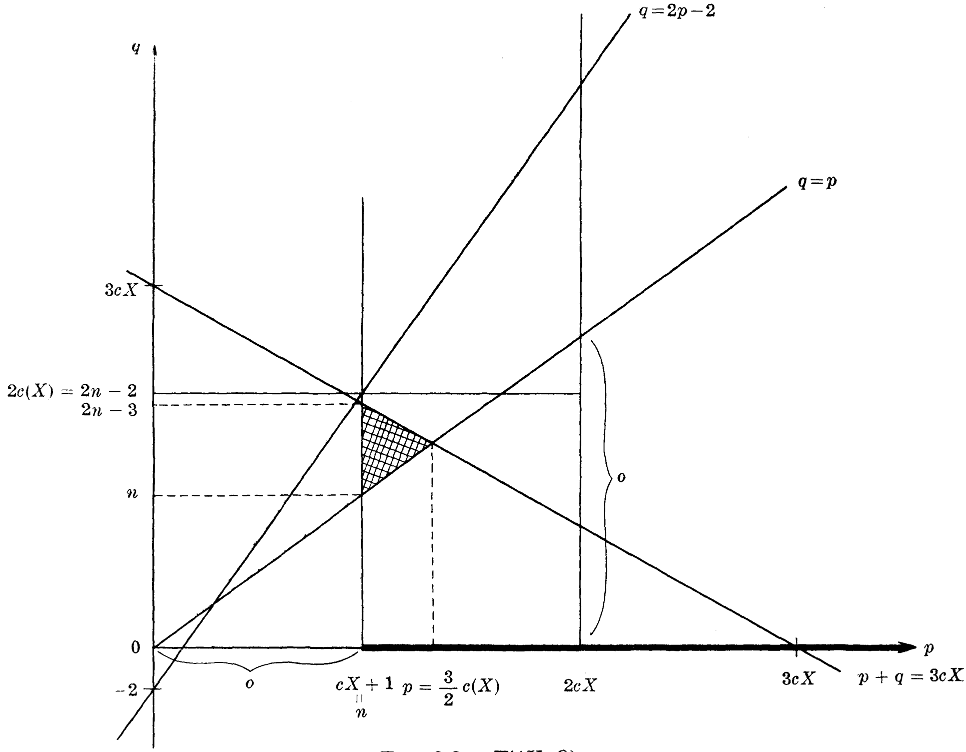


FIG. 3.2. $E^1(X; Q)$.

(b) Suppose $j > 2c(X) + 2$. Consider $p + q = j \leqq 3c(X)$, where $c(X) + 1 \leqq p \leqq [j/2]$, and the following commutative diagram

$$\begin{array}{ccc} E^1_{p-1,q+2} & \xrightarrow{E^1(i\circ f)_{p-1}} & \hat{E}^1_{p-1,q+2} \\ \downarrow d_p & & \downarrow \hat{d}_p \\ E^1_{p,q} & \xrightarrow{E^1(i\circ f)_p} & \hat{E}^1_{p,q} \\ \downarrow d_{p+1} & & \downarrow \hat{d}_{p+1} \\ E^1_{p+1,q-2} & \xrightarrow{E^1(i\circ f)_{p+1}} & \hat{E}^1_{p+1,q-2} \end{array}$$

where $E^1 \equiv E^1(X; Q)$, $\hat{E}^1 \equiv E^1(\widehat{X \otimes Q}; Q)$. $E^1(i\circ f)_k$ $(k = p-1, p, p+1)$ is an isomorphism by 3.9 since the total degree in each case is $\leqq 3c(X)+1$.

Since $\hat{d}_i = 0$, we have $d_i = 0$ for $i = p, p+1$. Thus $E^1_{p,q} = E^2_{p,q}$ for (p, q) satisfying the above. Similar arguments imply $E^i_{p,q} = E^1_{p,q}$ for $i = 3, 4, \cdots, q-2$.

(c) $d^i\colon E^i_{p,q} \longrightarrow E^i_{p+1,q-i-1}$ is zero for $i > q-1$ since $q - i - 1 < 0 \Rightarrow E_{p+i,q-i-1} = 0$. $d^i\colon E^i_{p-i,q+i-1} \longrightarrow E^i_{p,q}$ is zero for $i \geqq q-1$ since $i \geqq q-1$, $q \geqq p \Rightarrow p - i \leqq p - q + 1 \Rightarrow E^1_{p-i,q+i-1} = 0$.

Thus the only (possibly) nonzero differential operator for each (p, q) satisfying $c(X) + 1 \leqq p \leqq 2\cdot c(X)$, $p \leqq q \leqq 3c(X) - p$ is

$$d^{q-1}\colon E^{q-1}_{p,q} \longrightarrow E^{q-1}_{p+q-1,0} .$$

But this has been identified by Kahn in [7], Theorem 9.1, as the (rational) Whitehead product: If $q > p$

$$\begin{array}{ccc} \pi_p \otimes \pi_q \otimes Q & \xrightarrow{[\,,\,]\otimes id} & \pi_{p+q-1} \otimes Q \\ \uparrow\approx & & \uparrow\approx \\ E^{q-1}_{p,q} & \xrightarrow{d^{p-1}} & E^{q-1}_{p+q-1,0} \end{array} \qquad (q > p)$$

or, if $q = p$

$$\begin{array}{ccc} S_p(\pi_p \otimes Q) & \xrightarrow{[\,,\,]\otimes id} & \pi_{2p-1} \otimes Q \\ \uparrow\approx & & \uparrow\approx \\ E^{q-1}_{q,q} & \xrightarrow{d^{q-1}} & E^{q-1}_{2q-1,0} \end{array}$$

where [,] is the Whitehead product.

We have thus proved the following.

THEOREM 3.10. *Let* $c(X) \geqq 2$. *If* $p + q \leqq 3\cdot(X)$ *and* $q \geqq p$, *then*
(a) $d^i\colon E^i_{p-i,q+i+1} \longrightarrow E^i_{p,q}$ *is zero for all* $i > 0$.
(b) $d^i\colon E^i_{p,q} \longrightarrow E^i_{p+i,q-i-1}$ *is zero for* $i = 1, 2, \cdots, q-2, q, q+1, \cdots$
(c) $d^{q-1}\colon E^{q-1}_{p,q} \longrightarrow E^{q-1}_{p+q-1,0}$ *is the rational Whitehead product.*

4. **Applications.** We are now in a position to compute $H_i(X; Q)$ $(i \leqq 3\cdot c(X))$ completely in terms of the graded homotopy group $\Pi = \{\pi_i \otimes Q \mid 1 \leqq i \leqq 3\cdot c(X)\}$ and the rational Whitehead product on this group. For $i \leqq 2\cdot c(X)$ this is given by the rational Hurewicz theorem (3.4). Let

$$\mathrm{Ker}_{ij} = \begin{cases} \mathrm{Ker}\,\{\pi_j \otimes \pi_{i-j} \otimes Q \xrightarrow{[\,,\,]\otimes id} \pi_{i-1} \otimes Q\}\,, & c(X) < j \leqq \left[\dfrac{i-1}{2}\right] \\ \mathrm{Ker}\,\{S(\pi_{i/2} \otimes Q) \xrightarrow{[\,,\,]\otimes id} \pi_{i-1} \otimes Q\}\,, & \text{if } i \text{ even, } j = \left[\dfrac{i}{2}\right] \\ 0\,, & \text{if } i \text{ odd, } j = \left[\dfrac{i}{2}\right]. \end{cases}$$

and

$$\mathrm{Ker}_i = \bigoplus_{c(X)<j\leqq[i/2]} \mathrm{Ker}_{ij} \qquad (\oplus \text{ denotes direct sum}),$$

where [,] is the Whitehead product.

Furthermore, let

$$\mathrm{Im}_{ij} = \begin{cases} \mathrm{im}\,\{\pi_j \otimes \pi_{i+1-j} \otimes Q \xrightarrow{[\,,\,]\otimes id} \pi_i \otimes Q\}\,, & \text{if } c(X) < j \leqq \left[\dfrac{i}{2}\right] \\ \mathrm{im}\,\{S(\pi_{(i+1)/2} \otimes Q) \xrightarrow{[\,,\,]\otimes id} \pi_i \otimes Q\}\,, & \text{if } i+1 \text{ even, } j = \left[\dfrac{i+1}{2}\right] \\ 0\,, & \text{if } i+1 \text{ odd, } j = \left[\dfrac{i+1}{2}\right] \end{cases}$$

and (since $\mathrm{Im}_{ij} \subset \pi_i \otimes Q$ for each j)

$$\mathrm{Im}_i = \sum_{c(X)<j\leqq[(i+1)/2]} \mathrm{Im}_{ij} \subset \pi_i \otimes Q. \ (+ \text{ denotes sum, not necessarily direct})$$

THEOREM 4.1. *If* $2c(X) < i \leqq 3\cdot c(X)$, *then*

$$H_i(X; Q) \approx \mathrm{Ker}_i \oplus (\pi_i \otimes Q/\mathrm{Im}_i)$$

Proof. 3.4, $3.10 \Rightarrow E^\infty_{i,0} \approx (\pi_i \otimes Q/\mathrm{Im}_i)$ and $E^\infty_{p,q}(c(X) < p \leqq [i/2],$ $p + q = i) \approx \mathrm{Ker}_{ip}$. These are the only nonzero terms of total degree i. Since all extensions split we have

$$\begin{aligned} H_i(X; Q) &\approx E^\infty_{i,0} \oplus \bigoplus_{c(X)<p\leqq[i/2]} E^\infty_{p,i-p} \\ &\approx (\pi_i \otimes Q/\mathrm{Im}_i) \oplus \mathrm{Ker}_i\,. \end{aligned}$$

Since Kahn [7] has identified the edge homomorphism with the Hurewicz homomorphism we see

THEOREM 4.2. *If* $i \leqq 3\cdot c(X)$ *and* $h_i \otimes 1\colon \pi_i(X) \otimes Q \to H_i(X; Q)$ *is the Hurewicz homomorphism, then*

(a) $\mathrm{Ker}\, h_i \otimes 1 = \mathrm{Im}_i$

(b) $\mathrm{coker}\, h_i \otimes 1 = \mathrm{Ker}_i$

Proof. This follows because $h_i \otimes 1$ is the natural map

$$\pi_i \otimes Q \to \mathrm{Ker}_i \oplus (\pi_i \otimes Q/\mathrm{Im}_i)\,.$$

COROLLARY 4.3. *If* $i \leqq 3\cdot c(X)$, *then*

(a) $h_i \otimes 1$ *is a monomorphism* $\Leftrightarrow \mathrm{Im}_i = 0$

(b) $h_i \otimes 1$ *is an epimorphism* $\Leftrightarrow \mathrm{Ker}_i = 0$.

Note. By Proposition 2.1 (respectively, 4.1) of [1], $h_i \otimes 1$ is epic (respectively, monic) $\Leftrightarrow$ the i^{th} k'-invariant (k-invariant) of any homology (Postnikov) decomposition is of finite order. 4.3 gives another such characterization. This gives, for instance, the following theorem.

THEOREM 4.4 *If $\pi_i(X; Q) = 0$ for $i > 3\cdot c(X)$, then all k-invariants are of finite order $\Leftrightarrow$ all rational Whitehead products vanish.*

Finally, since it is usually easier to compute $H_i(X; Q)$ than it is the Whitehead product, we will use these relations (4.1 and 4.2) to give information about the Whitehead products themselves.

THEOREM 4.5. *Let $i \leqq 3\cdot c(X)$ and consider the following statements:*

(a) *$\pi_i \otimes Q$ is generated by Whitehead products.*

(b) *For all r such that $c(X) < r \leqq [(i-1)/2]$, $\pi_r \otimes \pi_{i-r} \otimes Q \rightarrow \pi_{i-1} \otimes Q$ is injective.*

(c) *If i even, $S(\pi_{i/2} \otimes Q) \rightarrow \pi_{i-1} \otimes Q$ is injective. The following are true.*

(d) $h_i \otimes 1 = 0 \Leftrightarrow$ (a)

(e) $\operatorname{coker} h_i \otimes 1 = 0 \Leftrightarrow$ (b) *and* (c)

(f) $H_i(X; Q) = 0 \Leftrightarrow$ (a), (b) *and* (*c*).

REFERENCES

1. M. Arkowitz, and C. R. Curjel, *The Hurewicz homomorphism and finite homotopy invariants*, Trans. Amer. Math. Soc., **110** (1964), 538-551.
2. H. Cartan, and J.-P. Serre, *Espaces fibres et groupes d'homotopie, II, Applications*, C. R. Acad. Sci. Paris, **234** (1952), 393-395.
3. M. Dyer, *Replacing Postnikov systems by simpler ones*, (unpublished).
4. S. Eilenberg, and S. MacLane, *Relations between homology and homotopy groups oj spaces*, Ann. of Math., (2) **46** (1945), 480-509.
5. S.-T. Hu, *Homotopy Theory*, Academic Press, New York, 1959.
6. D. W. Kahn, *Induced maps for Postnikov systems*, Trans. Amer. Math. Soc., **107** (1963), 432-450.
7. ———, *The spectral sequence of a Postnikov system*, Comm. Math. Helv., **40** (1966), 196-198.
8. J.-P. Meyer, *Whitehead products and Postnikov systems*, Amer. J. Math., **82** (1960).
9. D. Sullivan, *Geometric Topology, Part I*, Mass. Inst. of Tech. Notes, 1970.

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