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## **THE SINGULAR SUBMODULE OF A FINITELY GENERATED MODULE SPLITS OFF**

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## THE SINGULAR SUBMODULE OF A FINITELY GENERATED MODULE SPLITS OFF

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**A characterization is given of the finitely generated non-singular left  $R$ -modules  $N$  such that  $\text{Ext}_R^1(N, M) = 0$  for every singular left  $R$ -module  $M$ . As a corollary, the rings  $R$ , over which the singular submodule  $Z(A)$  is a direct summand of every finitely generated left  $R$ -module  $A$ , are characterized. This characterization takes on a simplified form whenever  $R$  is commutative. An example is given to show that a ring  $R$ , over which the singular submodule  $Z(A)$  is a direct summand of every left  $R$ -module  $A$ , need not be right semi-hereditary.**

In this paper, all rings  $R$  are assumed to be associative with an identity element, and, unless otherwise stated, all  $R$ -modules will be unitary left  $R$ -modules.

A submodule  $B$  of an  $R$ -module  $A$  is an essential submodule of  $A$  if  $B \cap C \neq 0$  for all nonzero submodules  $C$  of  $A$ . A left ideal  $I$  of  $R$  is essential in  $R$  if it is essential in  $R$  as a submodule of  $R$ . If  $A$  is an  $R$ -module,  $Z(A) = \{a \in A \mid (0 : a) \text{ is essential in } R\}$  is the singular submodule of  $A$ .  $A$  is called a singular module if  $Z(A) = A$ ; and  $A$  is a nonsingular module if  $Z(A) = 0$ . A submodule  $B$  of  $A$  is closed in  $A$  if  $B$  has no proper essential extension in  $A$ . If  $A$  is nonsingular, then a submodule  $B$  of  $A$  is a closed submodule of  $A$  if and only if  $A/B$  is a nonsingular module. A simple  $R$ -module  $S$  is nonsingular if and only if it is projective. For an  $R$ -module  $A$ ,  $\text{Soc}(A)$  denotes the sum of all simple submodules of  $A$  or 0 if  $A$  has no simple submodules.

Motivated by a definition of Kaplansky [6], we say that an  $R$ -module  $N$  is *UF* if  $N$  is a nonsingular module and  $\text{Ext}_R^1(N, M) = 0$  for all singular  $R$ -modules  $M$ . An  $R$ -module  $A$  is said to *split* if  $Z(A)$  is a direct summand of  $A$ . As in [2], a ring  $R$  has the *finitely generated splitting property* (FGSP) if every finitely generated  $R$ -module splits.

We shall use the following result of Cateforis and Sandomierski [2, Proposition 1.11], which is included here for completeness.

LEMMA 1. *For any ring  $R$ , the following statements are equivalent:*

- (a)  $R$  has FGSP.
- (b)  $Z(R) = 0$ , and every finitely generated nonsingular  $R$ -module is *UF*.

An  $R$ -module  $K$  is said to be *almost finitely generated* if  $K = U \oplus V$ , where  $U$  is a finitely generated  $R$ -module and  $V = \text{Soc}(K)$ . Then an  $R$ -module  $N$  is called *almost finitely related* if there exists an exact sequence of  $R$ -modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0 ,$$

where  $F$  is a finitely generated free module and  $K$  is almost finitely generated.

Before stating our main results, we prove several lemmas.

LEMMA 2. *If  $N$  is an almost finitely related  $R$ -module and if*

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0$$

*is any exact sequence of  $R$ -modules with  $F$  a finitely generated free module, then  $K$  is almost finitely generated.*

*Proof.* Since  $N$  is almost finitely related, there exists an exact sequence of  $R$ -modules

$$0 \longrightarrow K_1 \longrightarrow F_1 \longrightarrow N \longrightarrow 0 ,$$

where  $F_1$  is a finitely generated free module and  $K_1$  is almost finitely generated. By a result of Schanuel [9, p. 369],  $K \oplus F_1 \cong K_1 \oplus F$ . Since  $K_1$  and  $F$  are almost finitely generated, then so is  $K \oplus F_1 \cong K_1 \oplus F$ . Therefore  $(K \oplus F_1)/\text{Soc}(K \oplus F_1)$  is finitely generated. Since

$$\frac{K \oplus F_1}{\text{Soc}(K \oplus F_1)} = \frac{K \oplus F_1}{\text{Soc}(K) \oplus \text{Soc}(F_1)} \cong \frac{K}{\text{Soc}(K)} \oplus \frac{F_1}{\text{Soc}(F_1)} ,$$

then  $K/\text{Soc}(K)$  is also finitely generated.

Now we write  $K = Rx_1 + Rx_2 + \cdots + Rx_m + \text{Soc}(K)$ , where  $x_1, x_2, \dots, x_m \in K$ . Let  $W = (\text{Soc}(K)) \cap (Rx_1 + Rx_2 + \cdots + Rx_m)$ . Then there exists an  $R$ -module  $V$  such that  $\text{Soc}(K) = W \oplus V$ . It follows that  $K = (Rx_1 + Rx_2 + \cdots + Rx_m) \oplus V$ , and hence  $K$  is almost finitely generated.

A finitely generated nonsingular  $R$ -module  $N$  is called *finitely generated torsion inducing* (FGTI) if  $N$  has the following property: If  $M$  is any finitely generated  $R$ -module with  $M/Z(M) \cong N$ , then  $Z(M)$  is finitely generated.

LEMMA 3. *Let  $Z(R) = 0$ , and let  $0 \rightarrow K \rightarrow F \rightarrow N \rightarrow 0$  be an exact sequence of  $R$ -modules, where  $F$  is a finitely generated free module. If  $N$  is nonsingular, then the following statements hold:*

(a) *If  $N$  is FGTI and if  $K/\text{Soc}(K)$  is a direct sum of countably generated modules, then  $N$  is almost finitely related.*

(b) *If  $N$  is almost finitely related, then  $N$  is an FGTI module.*

*Proof.* To show (a), we need to show that  $K$  is almost finitely generated. By hypothesis,  $Y = K/\text{Soc}(K) = \bigoplus \sum_{\alpha \in \mathcal{A}} M_\alpha$ , where each  $M_\alpha$  is a countably generated  $R$ -module. First we show that  $Y$  is, in fact, countably generated also. Let  $\mathcal{B} = \{\alpha \in \mathcal{A} \mid M_\alpha \text{ contains a proper essential submodule}\}$ . Thus if  $\alpha \in \mathcal{A} - \mathcal{B}$ , then  $M_\alpha$  is a direct sum of singular simple  $R$ -modules or zero. For each  $\alpha \in \mathcal{B}$ , let  $L_\alpha$  be a proper essential submodule of  $M_\alpha$ . Define  $L = \bigoplus \sum_{\alpha \in \mathcal{A}} L_\alpha$ , and let  $J$  be a submodule of  $K$  containing  $\text{Soc}(K)$  such that  $J/\text{Soc}(K) = L$ . Since

$$Z(F/J) \cong Z((F/\text{Soc}(K))/(J/\text{Soc}(K))) \cong Y/L \cong K/J,$$

then  $K/J$  is a singular module; but since  $Z(F/K) = 0$ , it follows that  $Z(F/J) = K/J$ . By hypothesis,  $N$  is a FGTI module; hence

$$K/J \cong (\bigoplus \sum_{\alpha \in \mathcal{B}} (M_\alpha/L_\alpha)) \oplus (\bigoplus \sum_{\alpha \in \mathcal{A} - \mathcal{B}} M_\alpha)$$

is a finitely generated  $R$ -module. Therefore all but finitely many of the  $M_\alpha (\alpha \in \mathcal{A})$  must be 0, and hence  $K/\text{Soc}(K)$  is countably generated.

Thus there exist  $x_i \in K$  ( $i = 1, 2, \dots$ ) such that  $K = \sum_{i=1}^{\infty} Rx_i + \text{Soc}(K)$ . We will show that there exists a positive integer  $m$  such that  $K = \sum_{i=1}^m Rx_i + \text{Soc}(K)$ . If this were not the case, then for each positive integer  $n$ , there exists a least positive integer  $k(n)$  such that  $x_{k(n)} \notin Rx_1 + Rx_2 + \dots + Rx_n + \text{Soc}(K)$ . By Zorn's lemma, choose  $K_n$  maximal with respect to  $x_{k(n)} \notin K_n$  and

$$Rx_1 + Rx_2 + \dots + Rx_n + \text{Soc}(K) \subseteq K_n \subseteq K.$$

It follows that  $(Rx_{k(n)} + K_n)/K_n$  is an essential, simple, nonprojective submodule of  $K/K_n$ . Since  $K/K_n$  is an essential extension of a singular simple module, then  $K/K_n$  is also a singular module.

Define  $\varphi: K \rightarrow \bigoplus \sum_{n=1}^{\infty} K/K_n: x \rightarrow \sum_{n=1}^{\infty} \varphi_n(x)$ , where  $\varphi_n: K \rightarrow K/K_n$  is the natural map. If  $x \in K$ , then  $x = \sum_{i=1}^t r_i x_i \in \sum_{i=1}^t Rx_i \subseteq K_n$  for all  $n \geq t$ . Thus  $\varphi_n(x) = 0$  for all  $n \geq t$ , and hence  $\varphi$  is well-defined. If  $H = \ker \varphi$ , then  $K/H \cong \text{im } \varphi$  is not finitely generated (as  $\varphi_n(x_{k(n)}) \neq 0$  for each integer  $n$ ). Moreover, since  $\text{im } \varphi$  is a submodule of the singular module  $\bigoplus \sum_{n=1}^{\infty} K/K_n$ , then  $K/H \cong \text{im } \varphi$  is also a singular module. Since  $K$  is a closed submodule of  $F$ , then  $Z(F/H) = K/H$ . But then  $F/H$  does not have a finitely generated singular submodule, and  $(F/H)/Z(F/H) \cong F/K \cong N$ . This contradicts the hypothesis that  $N$  is a FGTI module. Thus  $K = \sum_{i=1}^m Rx_i + \text{Soc}(K)$  for some positive integer  $m$ .

Now the argument used in the last paragraph of the proof of

Lemma 2 shows that  $K$  is almost finitely generated. Therefore (a) holds.

Now we prove (b). Let  $M$  be a finitely generated  $R$ -module such that  $M/Z(M) \cong N$ . Let  $y_1, y_2, \dots, y_n$  be a set of generators of  $M$ , and let  $F$  be a free  $R$ -module with basis  $u_1, u_2, \dots, u_n$ . Then there exists a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & N \longrightarrow 0 \\ & & \lambda \downarrow & & \mu \downarrow & & \nu \downarrow \\ 0 & \longrightarrow & Z(M) & \longrightarrow & M & \longrightarrow & M/Z(M) \longrightarrow 0, \end{array}$$

where  $\mu: F \rightarrow M$  via  $\mu(u_i) = y_i$  is an epimorphism and  $\nu$  is an isomorphism. Then  $\lambda$  must be an epimorphism. By the hypothesis and Lemma 2,  $K = U \oplus V$ , where  $U$  is a finitely generated  $R$ -module and  $V = \text{Soc}(V)$ . Since  $\lambda(V)$  is isomorphic to a submodule of the nonsingular, semi-simple module  $V$  and since  $Z(M)$  is singular, then  $\lambda(V) = 0$ . Thus  $Z(M)$  is an epimorphic image of the finitely generated module  $U$ . Consequently,  $Z(M)$  is a finitely generated module.

REMARKS. (1) If  $R$  is a left hereditary ring, then any closed submodule  $K$  of a finitely generated free module  $F$  is projective. So it follows from [7, Theorem 1] that  $K/\text{Soc}(K)$  is a direct sum of countably generated modules. Thus for a left hereditary ring  $R$ , a finitely generated nonsingular  $R$ -module  $N$  is FGTI if and only if  $N$  is almost finitely related.

(2) Suppose that  $N$ ,  $F$ , and  $K$  are as in the hypothesis of Lemma 3. If  $N$  is FGTI and  $\text{Soc}(K)$  is essential in  $K$ , then  $K/\text{Soc}(K)$  is finitely generated. So we can conclude the following result from Lemma 3: If  $R$  is a nonsingular ring with essential socle, then a finitely generated nonsingular FGTI module is almost finitely related.

(3) There seems to be some independent interest in determining when the singular submodule of a finitely generated module is finitely generated. Indeed, Pierce [8, p. 109] asks questions along this line. Lemma 3 and the first of this remark shed some light in this direction.

We shall use  $hd(N)$  to denote the projective homological dimension of an  $R$ -module  $N$ .

We now need an obvious generalization of a result of Kaplansky, [6, Theorem 1]:

LEMMA 4. *If  $N$  is a  $UF$   $R$ -module, then  $hd(N) \leq 1$ .*

*Proof.* Let  $N$  be a  $UF$   $R$ -module, and let  $M$  be any  $R$ -module. The exact sequence

$$0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0$$

induces the exact sequence

$$\text{Ext}_R^1(N, E(M)/M) \longrightarrow \text{Ext}_R^2(N, M) \longrightarrow \text{Ext}_R^2(N, E(M)) = 0 ,$$

where  $E(M)$  denotes the injective hull of  $M$ . Since  $N$  is  $UF$  we have  $\text{Ext}_R^1(N, E(M)/M) = 0$ ; and hence  $\text{Ext}_R^2(N, M) = 0$  by exactness.

We now give a characterization of  $UF$  modules.

**THEOREM 1.** *Let  $Z(R) = 0$ , and let  $N$  be a finitely generated  $R$ -module. Then  $N$  is  $UF$  if and only if the following conditions are satisfied:*

- (i)  *$N$  is an almost finitely related, nonsingular module.*
- (ii)  *$hd(N) \leq 1$ .*
- (iii)  *$\text{Tor}_1^R(\text{Hom}_Z(A, D), N) = 0$ , where  $A$  is any singular  $R$ -module,  $D$  is any divisible Abelian group, and  $Z$  denotes the ring of integers.*

*Proof.* We develop a diagram (see (\*)), which we use in both directions of the proof. For any finitely generated  $R$ -module  $N$ , there is an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0 ,$$

where  $F$  is a finitely generated free  $R$ -module. If  $D$  is any divisible Abelian group and if  $A$  is any singular  $R$ -module, then  $\text{Hom}_Z(A, D)$  is a right  $R$ -module. Hence there is an exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Tor}_1^R(\text{Hom}_Z(A, D), N) &\longrightarrow \text{Hom}_Z(A, D) \otimes_R K \\ &\longrightarrow \text{Hom}_Z(A, D) \otimes_R F . \end{aligned}$$

The exact sequence

$$\text{Hom}_R(F, A) \longrightarrow \text{Hom}_R(K, A) \longrightarrow \text{Ext}_R^1(N, A) \longrightarrow 0$$

induces an exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Hom}_Z(\text{Ext}_R^1(N, A), D) &\longrightarrow \text{Hom}_Z(\text{Hom}_R(K, A), D) \\ &\longrightarrow \text{Hom}_Z(\text{Hom}_R(F, A), D) . \end{aligned}$$

It is well-known [1, p. 120] that there exists a homomorphism  $\psi$  and an isomorphism  $\beta$  making the following diagram commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Tor}_1^R(\text{Hom}_Z(A, D), N) & \longrightarrow & \text{Hom}_Z(A, D) \otimes_R K & \longrightarrow & \text{Hom}_Z(A, D) \otimes_R F \\ (*) & & & & \psi \downarrow & & \beta \downarrow \\ 0 & \longrightarrow & \text{Hom}_Z(\text{Ext}_R^1(N, A), D) & \longrightarrow & \text{Hom}_Z(\text{Hom}_R(K, A), D) & \longrightarrow & \text{Hom}_Z(\text{Hom}_R(F, A), D) . \end{array}$$

“only if”: Let  $N$  be a finitely generated  $UF$   $R$ -module. Then there exists an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0$$

of left  $R$ -modules, where  $F$  is a finitely generated free module. By Lemma 4,  $K$  is a projective  $R$ -module; thus  $K = \bigoplus \sum_{\alpha \in \mathcal{A}} K_\alpha$ , where each  $K_\alpha$  is countably generated by [7, Theorem 1]. Since

$$\frac{K}{\text{Soc}(K)} = \frac{\bigoplus \sum_{\alpha \in \mathcal{A}} K_\alpha}{\bigoplus \sum_{\alpha \in \mathcal{A}} \text{Soc}(K_\alpha)} \cong \bigoplus \sum_{\alpha \in \mathcal{A}} \frac{K_\alpha}{\text{Soc}(K_\alpha)},$$

then  $K/\text{Soc}(K)$  is a direct sum of countably generated  $R$ -modules. Since a  $UF$  module is FGTI, then Lemma 3 (a) implies that  $N$  is almost finitely related, i.e., (i) holds.

Lemma 4 implies that  $hd(N) \leq 1$ ; so (ii) holds.

Now we show that (iii) holds. Let  $A$ ,  $D$ ,  $F$ , and  $K$  be chosen as in (\*). Then by (i),  $K = U \oplus V$ , where  $U$  is finitely generated and  $V = \text{Soc}(V)$ . But for any nonsingular simple  $R$ -module  $S$ ,  $\text{Hom}_R(S, A) = 0$  (as  $A$  is singular). Thus by [1, VI. Prop. 5.2],  $\text{Hom}_Z(A, D) \otimes_R S \cong \text{Hom}_Z(\text{Hom}_R(S, A), D) = 0$ . Therefore  $\text{Hom}_Z(A, D) \otimes_R V = 0$ , and  $\text{Hom}_R(V, A) = 0$ . Hence there exist obvious isomorphisms  $\sigma$  and  $\tau$  making the diagram

$$\begin{array}{ccc} \text{Hom}_Z(A, D) \otimes_R K & \xrightarrow{\sigma} & \text{Hom}_Z(A, D) \otimes_R U \\ \psi \downarrow & & \psi' \downarrow \\ \text{Hom}_Z(\text{Hom}_R(K, A), D) & \xrightarrow{\tau} & \text{Hom}_Z(\text{Hom}_R(U, A), D) \end{array}$$

commute, where  $\psi'$  is the restriction of  $\psi$  in (\*) to  $\text{Hom}_Z(A, D) \otimes_R U$ . By [1, VI. Prop. 5.2]  $\psi'$  is an isomorphism; whence  $\psi$  is forced to be an isomorphism also. By the commutativity of (\*) and the fact that  $\text{Ext}_R^1(N, A) = 0$ , it is now easy to obtain  $\text{Tor}_1^R(\text{Hom}_Z(A, D), N) = 0$ .

“if”: Let  $A$ ,  $D$ ,  $F$ ,  $K$  be as in (\*). Since  $hd(N) \leq 1$  and  $N$  is almost finitely related, then  $K$  is an almost finitely generated projective  $R$ -module. By the argument used in the preceding paragraph,  $\psi$  is an isomorphism in (\*). From the commutativity of (\*) and the fact that  $\text{Tor}_1^R(\text{Hom}_Z(A, D), N) = 0$ , we now obtain  $\text{Hom}_Z(\text{Ext}_R^1(N, A), D) = 0$ . Since  $D$  is any divisible Abelian group, then  $\text{Ext}_R^1(N, A) = 0$  for every singular module  $A$ . Thus  $N$  is a  $UF$  module.

As a corollary, we have the following result for left hereditary rings:

**COROLLARY 1.** *Let  $R$  be a left hereditary ring whose maximal*

quotient ring  ${}_R Q$  (see [3], [11]) is  $R$ -flat. Then the following statements are equivalent for any finitely generated nonsingular  $R$ -module  $N$ :

- (a)  $N$  is a  $UF$  module.
- (b)  $N$  is almost finitely related.
- (c)  $N$  is a  $FGTI$  module.

*Proof.* The equivalence of (b) and (c) is clear from Remark (1) following Lemma 3. The equivalence of (a) and (b) will follow immediately from Theorem 1 if we show that the ring hypothesis implies every nonsingular  $R$ -module is  $R$ -flat. But this follows from [11, Cor. 2.5] and [11, Theorem 2.1].

An immediate consequence of Lemma 1 and Theorem 1 is the following characterization of FGSP:

**COROLLARY 2.** *A ring  $R$  has FGSP if and only if the following statements hold:*

- (a)  $Z(R) = 0$ .
- (b) *Every finitely generated nonsingular  $R$ -module is almost finitely related.*
- (c)  $hd(N) \leq 1$  for every finitely generated nonsingular  $R$ -module  $N$ .
- (d)  $Tor_1^R(\text{Hom}_Z(A, D), N) = 0$ , where  $N$  is any finitely generated nonsingular  $R$ -module,  $D$  is any divisible Abelian group, and  $Z$  denotes the ring of integers.

Combining Corollaries 1 and 2, the reader can easily see that a left hereditary ring  $R$ , whose maximal left quotient ring  ${}_R Q$  is flat, has FGSP if and only if every finitely generated nonsingular  $R$ -module is almost finitely related. We shall see in Corollary 6 that Corollary 2 also takes on a particularly nice form whenever  $R$  is a commutative ring.

A submodule  $K$  of an  $R$ -module  $M$  is said to be an *almost summand* of  $M$  if  $K = U \oplus V$ , where  $U$  is a direct summand of  $M$  and  $V = \text{Soc}(V)$ . The next theorem gives a relationship between  $UF$   $R$ -modules and almost summands of free  $R$ -modules.

**THEOREM 2.** *Let  $Z(R) = 0$ , and let  $N \cong F/K$  be a finitely generated nonsingular  $R$ -module, where  $F$  is a finitely generated free  $R$ -module. If  $K$  is an almost summand of  $F$ , then  $N$  is  $UF$ . Moreover, if  $N$  is  $R$ -flat, then the converse holds.*

*Proof.* To prove the first statement, it suffices to show that any homomorphism  $f: K \rightarrow A$  can be lifted to a homomorphism  $g: F \rightarrow A$ , where  $A$  is any singular module. Now  $K = U \oplus V$ , where



$F = U \oplus W$  for some submodule  $W$  of  $F$  and  $V = \text{Soc}(V)$ . Since  $Z(A) = A$  and  $Z(K) = 0$ , then  $f(\text{Soc}(K)) = 0$ . If  $x \in K \cap W$ , it follows from the direct sum decompositions that  $x \in \text{Soc}(K)$ , and hence  $f(x) = 0$ . So the desired lifting of  $f$  is given by  $g(u + w) = f(u)$  for all  $u \in U$  and all  $w \in W$ .

Now assume  $N$  is an  $R$ -flat  $UF$  module. By Theorem 1,  $K = U \oplus V$ , where  $U$  is finitely generated and  $V = \text{Soc}(V)$  is projective. Then there is an exact sequence

$$0 \longrightarrow K/U \longrightarrow F/U \longrightarrow F/K \longrightarrow 0$$

with  $K/U$  and  $F/K$   $R$ -flat. Thus  $F/U$  is also  $R$ -flat. But  $F/U$  is finitely related (see [5, p. 459]) and therefore projective by [5, p. 459]. Consequently  $U$  is a direct summand of  $F$ , and  $K = U \oplus V$  is an almost summand of  $F$ .

The following corollary is an immediate consequence of Lemma 1 and Theorem 2.

**COROLLARY 3.** *If  $Z(R) = 0$  and if every closed submodule of a finitely generated free  $R$ -module  $F$  is an almost summand of  $F$ , then  $R$  has FGSP. Moreover, if every (finitely generated) nonsingular  $R$ -module is flat, then the converse holds.*

The next corollary is a partial generalization of [11, Corollary 2.7].

**COROLLARY 4.** *If  $R$  is a right semi-hereditary ring having a maximal left quotient ring  $Q$  (see [3], [11]), which is a two-sided quotient ring of  $R$ , then the following statements are equivalent:*

- (a)  $R$  has FGSP.
- (b)  $Z(R) = 0$ , and every closed submodule of a finitely generated free  $R$ -module  $F$  is an almost summand of  $F$ .

*Proof.* By Corollary 3, we need to show that if  $R$  has FGSP, then every nonsingular  $R$ -module is flat. Since  $Z(R) = 0$  by Lemma 1 and since  $Q$  is two-sided, then every finitely generated nonsingular  $R$ -module is torsionless by [3, Theorem 1.1]. However  $R$  is right semi-hereditary; hence every torsionless  $R$ -module is flat by [5, Theorem 4.1].

**COROLLARY 5.** *Let  $R$  be a commutative ring with  $Z(R) = 0$ . Let  $N \cong F/K$ , where  $F$  is a finitely generated free  $R$ -module. Then  $N$  is  $UF$  if and only if  $N$  is a nonsingular module and  $K$  is an almost summand of  $F$ .*

*Proof.* By Theorem 2, it suffices to show that any  $UF$   $R$ -module is  $R$ -flat. But this follows from the proof of the corollary to [2, Proposition 1.11].

Pierce [8, p. 109] asks when a finitely generated module over a commutative regular ring splits. Corollary 5 sheds some light in this direction. Moreover, since the hypothesis, “ $R$  is a commutative ring with  $Z(R) = 0$ ,” is used only to establish that nonsingular modules are flat, the conclusion of Corollary 5 holds true for any regular ring  $R$ . Corollary 5 also generalizes [10, Theorem 3.3], which deals with the structure of rings for which cyclic modules split.

In [2] Cateforis and Sandomierski have suggested the question of determining all commutative rings with FGSP. The final corollary extends [10, Theorem 3.3] to give an answer to this question.

**COROLLARY 6.** *If  $R$  is a commutative ring, then the following statements are equivalent:*

- (a)  $R$  has FGSP.
- (b)  $Z(R) = 0$ , and every closed submodule of a finitely generated free  $R$ -module  $F$  is an almost summand of  $F$ .
- (c)  $R$  is semi-hereditary, and every finitely generated non-singular module is almost finitely related.

*Proof.* The equivalence of (a) and (b) follows from Lemma 1 and Corollary 5. In view of the corollary to [2, Proposition 1.11], (c) is an immediate consequence of (a) and (b). Assuming (c), the last two sequences in the proof of Corollary 4 show that all non-singular modules are flat. Hence (b) follows by a slight modification of the argument used in the last part of the proof of Theorem 2.

The authors conjecture that a ring  $R$  has FGSP if and only if  $Z(R) = 0$  and every closed submodule of a finitely generated free module  $F$  is an almost summand of  $F$ .

In view of the preceding corollaries and the corollary to [2, Proposition 1.11], the reader might conjecture that the messy “Tor condition” in Corollary 2 (d) can be replaced by the nicer condition, “ $R$  is right semi-hereditary,” or by the stronger condition, “all non-singular  $R$ -modules are flat.” However, the following example shows that a ring  $R$  with FGSP need not be right semi-hereditary.

**EXAMPLE.** Let  $F$  be a field, and let  $T$  be the  $F$ -subalgebra of  $\prod_{n=1}^{\infty} F^{(n)}$  generated by  $\bigoplus \sum_{n=1}^{\infty} F^{(n)}$  and the identity of  $\prod_{n=1}^{\infty} F^{(n)}$ , where  $F^{(n)} \cong F$  for all  $n$ . Let  $I = \bigoplus \sum_{n=1}^{\infty} F^{(n)}$ , and let  $S = T/I$ . If  $R$  is the ring of all  $2 \times 2$  matrices of the form

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b \in S; c \in T \right\},$$

then Chase [4, Proposition 3.1] has shown that  $R$  is a left semi-hereditary ring, which is not a right semi-hereditary ring. Hence  $Z(R) = 0$ , and it is straight forward to check that the only proper essential left ideal of  $R$  is the maximal left ideal

$$J = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b \in S; c \in I \right\}.$$

Thus if  $A$  is any singular  $R$ -module, then  $A$  is a direct sum of copies of the simple module  $R/J$ . It follows that each singular module is injective, and hence every  $R$ -module splits. Thus  $R$  has FGSP, but  $R$  is not right semi-hereditary.

*Added in proof.* K. R. Goodearl has constructed an example (unpublished) which shows that the conjecture following Corollary 6 is not true.

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