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A characterization is given of the finitely generated nonsingular left R-modules N such that $\mathrm{Ext}^1_R(N, M) = 0$ for every singular left R-module M. As a corollary, the rings R, over which the singular submodule $Z(A)$ is a direct summand of every finitely generated left R-module A, are characterized. This characterization takes on a simplified form whenever R is commutative. An example is given to show that a ring R, over which the singular submodule $Z(A)$ is a direct summand of every left R-module A, need not be right semi-hereditary.

In this paper, all rings R are assumed to be associative with an identity element, and, unless otherwise stated, all R-modules will be unitary left R-modules.

A submodule B of an R-module A is an essential submodule of A if $B \cap C \neq 0$ for all nonzero submodules C of A. A left ideal I of R is essential in R if it is essential in R as a submodule of R. If A is an R-module, $Z(A) = \{a \in A \mid (0: a)$ is essential in $R\}$ is the singular submodule of A. A is called a singular module if $Z(A) = A$; and A is a nonsingular module if $Z(A) = 0$. A submodule B of A is closed in A if B has no proper essential extension in A. If A is nonsingular, then a submodule B of A is a closed submodule of A if and only if A/B is a nonsingular module. A simple R-module S is nonsingular if and only if it is projective. For an R-module A, $\mathrm{Soc}(A)$ denotes the sum of all simple submodules of A or 0 if A has no simple submodules.

Motivated by a definition of Kaplansky [6], we say that an R-module N is *UF* if N is a nonsingular module and $\mathrm{Ext}^1_R(N, M) = 0$ for all singular R-modules M. An R-module A is said to *split* if $Z(A)$ is a direct summand of A. As in [2], a ring R has the *finitely generated splitting property* (FGSP) if every finitely generated R-module splits.

We shall use the following result of Cateforis and Sandomierski [2, Proposition 1.11], which is included here for completeness.

LEMMA 1. *For any ring R, the following statements are equivalent:*

(a) *R has* FGSP.

(b) *$Z(R) = 0$, and every finitely generated nonsingular R-module is UF.*

An R-module K is said to be *almost finitely generated* if $K = U \oplus V$, where U is a finitely generated R-module and $V = \mathrm{Soc}(V)$. Then an R-module N is called *almost finitely related* if there exists an exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0\ ,$$

where F is a finitely generated free module and K is almost finitely generated.

Before stating our main results, we prove several lemmas.

LEMMA 2. *If N is an almost finitely related R-module and if*

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0$$

is any exact sequence of R-modules with F a finitely generated free module, then K is almost finitely generated.

Proof. Since N is almost finitely related, there exists an exact sequence of R-modules

$$0 \longrightarrow K_1 \longrightarrow F_1 \longrightarrow N \longrightarrow 0\ ,$$

where F_1 is a finitely generated free module and K_1 is almost finitely generated. By a result of Schanuel [9, p. 369], $K \oplus F_1 \cong K_1 \oplus F$. Since K_1 and F are almost finitely generated, then so is $K \oplus F_1 \cong K_1 \oplus F$. Therefore $(K \oplus F_1)/\mathrm{Soc}(K \oplus F_1)$ is finitely generated. Since

$$\frac{K \oplus F_1}{\mathrm{Soc}(K \oplus F_1)} = \frac{K \oplus F_1}{\mathrm{Soc}(K) \oplus \mathrm{Soc}(F_1)} \cong \frac{K}{\mathrm{Soc}(K)} \oplus \frac{F_1}{\mathrm{Soc}(F_1)}\ ,$$

then $K/\mathrm{Soc}(K)$ is also finitely generated.

Now we write $K = Rx_1 + Rx_2 + \cdots + Rx_m + \mathrm{Soc}(K)$, where $x_1, x_2, \cdots, x_m \in K$. Let $W = (\mathrm{Soc}(K)) \cap (Rx_1 + Rx_2 + \cdots + Rx_m)$. Then there exists an R-module V such that $\mathrm{Soc}(K) = W \oplus V$. It follows that $K = (Rx_1 + Rx_2 + \cdots + Rx_m) \oplus V$, and hence K is almost finitely generated.

A finitely generated nonsingular R-module N is called *finitely generated torsion inducing* (FGTI) if N has the following property: If M is any finitely generated R-module with $M/Z(M) \cong N$, then $Z(M)$ is finitely generated.

LEMMA 3. *Let $Z(R) = 0$, and let $0 \to K \to F \to N \to 0$ be an exact sequence of R-modules, where F is a finitely generated free module. If N is nonsingular, then the following statements hold:*

(a) *If N is FGTI and if $K/\mathrm{Soc}(K)$ is a direct sum of countably generated modules, then N is almost finitely related.*

(b) *If N is almost finitely related, then N is an* FGTI *module.*

Proof. To show (a), we need to show that K is almost finitely generated. By hypothesis, $Y = K/\mathrm{Soc}(K) = \bigoplus\sum_{\alpha\in\mathscr{A}} M_\alpha$, where each M_α is a countably generated R-module. First we show that Y is, in fact, countably generated also. Let $\mathscr{B} = \{\alpha \in \mathscr{A} \mid M_\alpha$ contains a proper essential submodule$\}$. Thus if $\alpha \in \mathscr{A} - \mathscr{B}$, then M_α is a direct sum of singular simple R-modules or zero. For each $\alpha \in \mathscr{B}$, let L_α be a proper essential submodule of M_α. Define $L = \bigoplus\sum_{\alpha\in\mathscr{B}} L_\alpha$, and let J be a submodule of K containing $\mathrm{Soc}(K)$ such that $J/\mathrm{Soc}(K) = L$. Since

$$Z(F/J) \cong Z((F/\mathrm{Soc}(K))/(J/\mathrm{Soc}(K))) \supseteq Y/L \cong K/J\,,$$

then K/J is a singular module; but since $Z(F/K) = 0$, it follows that $Z(F/J) = K/J$. By hypothesis, N is a FGTI module; hence

$$K/J \cong (\bigoplus\textstyle\sum_{\alpha\in\mathscr{B}}(M_\alpha/L_\alpha)) \oplus (\bigoplus\sum_{\alpha\in\mathscr{A}-\mathscr{B}} M_\alpha)$$

is a finitely generated R-module. Therefore all but finitely many of the $M_\alpha(\alpha \in \mathscr{A})$ must be 0, and hence $K/\mathrm{Soc}(K)$ is countably generated.

Thus there exist $x_i \in K$ $(i = 1, 2, \cdots)$ such that $K = \sum_{i=1}^{\infty} Rx_i + \mathrm{Soc}(K)$. We will show that there exists a positive integer m such that $K = \sum_{i=1}^{m} Rx_i + \mathrm{Soc}(K)$. If this were not the case, then for each positive integer n, there exists a least positive integer $k(n)$ such that $x_{k(n)} \notin Rx_1 + Rx_2 + \cdots + Rx_n + \mathrm{Soc}(K)$. By Zorn's lemma, choose K_n maximal with respect to $x_{k(n)} \notin K_n$ and

$$Rx_1 + Rx_2 + \cdots + Rx_n + \mathrm{Soc}(K) \subseteq K_n \subseteq K\,.$$

It follows that $(Rx_{k(n)} + K_n)/K_n$ is an essential, simple, nonprojective submodule of K/K_n. Since K/K_n is an essential extension of a singular simple module, then K/K_n is also a singular module.

Define $\varphi\colon K \to \bigoplus\sum_{n=1}^{\infty} K/K_n\colon x \to \sum_{n=1}^{\infty}\varphi_n(x)$, where $\varphi_n\colon K \to K/K_n$ is the natural map. If $x \in K$, then $x = \sum_{i=1}^{t} r_i x_i \in \sum_{i=1}^{t} Rx_i \subseteq K_n$ for all $n \geqq t$. Thus $\varphi_n(x) = 0$ for all $n \geqq t$, and hence φ is well-defined. If $H = \ker\varphi$, then $K/H \cong \mathrm{im}\,\varphi$ is not finitely generated (as $\varphi_n(x_{k(n)}) \neq 0$ for each integer n). Moreover, since $\mathrm{im}\,\varphi$ is a submodule of the singular module $\bigoplus\sum_{n=1}^{\infty} K/K_n$, then $K/H \cong \mathrm{im}\,\varphi$ is also a singular module. Since K is a closed submodule of F, then $Z(F/H) = K/H$. But then F/H does not have a finitely generated singular submodule, and $(F/H)/Z(F/H) \cong F/K \cong N$. This contradicts the hypothesis that N is a FGTI module. Thus $K = \sum_{i=1}^{m} Rx_i + \mathrm{Soc}(K)$ for some positive integer m.

Now the argument used in the last paragraph of the proof of

Lemma 2 shows that K is almost finitely generated. Therefore (a) holds.

Now we prove (b). Let M be a finitely generated R-module such that $M/Z(M) \cong N$. Let $y_1, y_2, \cdots, y_n$ be a set of generators of M, and let F be a free R-module with basis $u_1, u_2, \cdots, u_n$. Then there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & N & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\lambda} & & \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\nu} & & \\ 0 & \longrightarrow & Z(M) & \longrightarrow & M & \longrightarrow & M/Z(M) & \longrightarrow & 0, \end{array}$$

where $\mu: F \rightarrow M$ via $\mu(u_i) = y_i$ is an epimorphism and ν is an isomorphism. Then λ must be an epimorphism. By the hypothesis and Lemma 2, $K = U \oplus V$, where U is a finitely generated R-module and $V = \mathrm{Soc}(V)$. Since $\lambda(V)$ is isomorphic to a submodule of the nonsingular, semi-simple module V and since $Z(M)$ is singular, then $\lambda(V) = 0$. Thus $Z(M)$ is an epimorphic image of the finitely generated module U. Consequently, $Z(M)$ is a finitely generated module.

REMARKS. (1) If R is a left hereditary ring, then any closed submodule K of a finitely generated free module F is projective. So it follows from [7, Theorem 1] that $K/\mathrm{Soc}(K)$ is a direct sum of countably generated modules. Thus for a left hereditary ring R, a finitely generated nonsingular R-module N is FGTI if and only if N is almost finitely related.

(2) Suppose that N, F, and K are as in the hypothesis of Lemma 3. If N is FGTI and $\mathrm{Soc}(K)$ is essential in K, then $K/\mathrm{Soc}(K)$ is finitely generated. So we can conclude the following result from Lemma 3: If R is a nonsingular ring with essential socle, then a finitely generated nonsingular FGTI module is almost finitely related.

(3) There seems to be some independent interest in determining when the singular submodule of a finitely generated module is finitely generated. Indeed, Pierce [8, p. 109] asks questions along this line. Lemma 3 and the first of this remark shed some light in this direction.

We shall use $hd(N)$ to denote the projective homological dimension of an R-module N.

We now need an obvious generalization of a result of Kaplansky, [6, Theorem 1]:

LEMMA 4. *If N is a UF R-module, then $hd(N) \leqq 1$.*

Proof. Let N be a *UF* R-module, and let M be any R-module. The exact sequence

$$0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0$$

induces the exact sequence

$$\mathrm{Ext}^1_R(N, E(M)/M) \longrightarrow \mathrm{Ext}^2_R(N, M) \longrightarrow \mathrm{Ext}^2_R(N, E(M)) = 0 ,$$

where $E(M)$ denotes the injective hull of M. Since N is UF we have $\mathrm{Ext}^1_R(N, E(M)/M) = 0$; and hence $\mathrm{Ext}^2_R(N, M) = 0$ by exactness.

We now give a characterization of UF modules.

THEOREM 1. *Let* $Z(R) = 0$, *and let* N *be a finitely generated* R*-module. Then* N *is* UF *if and only if the following conditions are satisfied:*

(i) N *is an almost finitely related, nonsingular module.*

(ii) $hd(N) \leqq 1$.

(iii) $\mathrm{Tor}^R_1(\mathrm{Hom}_Z(A, D), N) = 0$, *where* A *is any singular* R*-module,* D *is any divisible Abelian group, and* Z *denotes the ring of integers.*

Proof. We develop a diagram (see (*)), which we use in both directions of the proof. For any finitely generated R-module N, there is an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0 ,$$

where F is a finitely generated free R-module. If D is any divisible Abelian group and if A is any singular R-module, then $\mathrm{Hom}_Z(A, D)$ is a right R-module. Hence there is an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Tor}^R_1(\mathrm{Hom}_Z(A, D), N) &\longrightarrow \mathrm{Hom}_Z(A, D) \otimes_R K \\ &\longrightarrow \mathrm{Hom}_Z(A, D) \otimes_R F . \end{aligned}$$

The exact sequence

$$\mathrm{Hom}_R(F, A) \longrightarrow \mathrm{Hom}_R(K, A) \longrightarrow \mathrm{Ext}^1_R(N, A) \longrightarrow 0$$

induces an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_Z(\mathrm{Ext}^1_R(N, A), D) &\longrightarrow \mathrm{Hom}_Z(\mathrm{Hom}_R(K, A), D) \\ &\longrightarrow \mathrm{Hom}_Z(\mathrm{Hom}_R(F, A), D) . \end{aligned}$$

It is well-known [1, p. 120] that there exists a homomorphism ψ and an isomorphism β making the following diagram commutative:

$$(*) \qquad \begin{array}{ccccccc} 0 \longrightarrow \mathrm{Tor}^R_1(\mathrm{Hom}_Z(A, D), N) & \longrightarrow & \mathrm{Hom}_Z(A, D) \otimes_R K & \longrightarrow & \mathrm{Hom}_Z(A, D) \otimes_R F \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\beta} \\ 0 \longrightarrow \mathrm{Hom}_Z(\mathrm{Ext}^1_R(N, A), D) & \longrightarrow & \mathrm{Hom}_Z(\mathrm{Hom}_R(K, A), D) & \longrightarrow & \mathrm{Hom}_Z(\mathrm{Hom}_R(F, A), D) . \end{array}$$

"only if": Let N be a finitely generated UF R-module. Then there exists an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0$$

of left R-modules, where F is a finitely generated free module. By Lemma 4, K is a projective R-module; thus $K = \bigoplus \sum_{\alpha \in \mathscr{A}} K_\alpha$, where each K_α is countably generated by [7, Theorem 1]. Since

$$\frac{K}{\mathrm{Soc}\,(K)} = \frac{\bigoplus \sum_{\alpha \in \mathscr{A}} K_\alpha}{\bigoplus \sum_{\alpha \in \mathscr{A}} \mathrm{Soc}\,(K_\alpha)} \cong \bigoplus \sum_{\alpha \in \mathscr{A}} \frac{K_\alpha}{\mathrm{Soc}\,(K_\alpha)}\,,$$

then $K/\mathrm{Soc}\,(K)$ is a direct sum of countably generated R-modules. Since a UF module is FGTI, then Lemma 3 (a) implies that N is almost finitely related, i.e., (i) holds.

Lemma 4 implies that $hd(N) \leqq 1$; so (ii) holds.

Now we show that (iii) holds. Let A, D, F, and K be chosen as in (*). Then by (i), $K = U \oplus V$, where U is finitely generated and $V = \mathrm{Soc}\,(V)$. But for any nonsingular simple R-module S, $\mathrm{Hom}_R\,(S, A) = 0$ (as A is singular). Thus by [1, VI. Prop. 5.2], $\mathrm{Hom}_Z\,(A, D) \otimes_R S \cong \mathrm{Hom}_Z\,(\mathrm{Hom}_R\,(S, A), D) = 0$. Therefore $\mathrm{Hom}_Z\,(A, D) \otimes_R V = 0$, and $\mathrm{Hom}_R\,(V, A) = 0$. Hence there exist obvious isomorphisms σ and τ making the diagram

$$\begin{array}{ccc} \mathrm{Hom}_Z\,(A, D) \otimes_R K & \xrightarrow{\sigma} & \mathrm{Hom}_Z\,(A, D) \otimes_R U \\ \psi \big\downarrow & & \big\downarrow \psi' \\ \mathrm{Hom}_Z\,(\mathrm{Hom}_R\,(K, A), D) & \xrightarrow{\tau} & \mathrm{Hom}_Z\,(\mathrm{Hom}_R\,(U, A), D) \end{array}$$

commute, where ψ' is the restriction of ψ in (*) to $\mathrm{Hom}_Z\,(A, D) \otimes_R U$. By [1, VI. Prop. 5.2] ψ' is an isomorphism; whence ψ is forced to be an isomorphism also. By the commutativity of (*) and the fact that $\mathrm{Ext}^1_R\,(N, A) = 0$, it is now easy to obtain $\mathrm{Tor}^R_1\,(\mathrm{Hom}_Z\,(A, D), N) = 0$.

"if": Let A, D, F, K be as in (*). Since $hd(N) \leqq 1$ and N is almost finitely related, then K is an almost finitely generated projective R-module. By the argument used in the preceding paragraph, ψ is an isomorphism in (*). From the commutativity of (*) and the fact that $\mathrm{Tor}^R_1\,(\mathrm{Hom}_Z\,(A, D), N) = 0$, we now obtain $\mathrm{Hom}_Z\,(\mathrm{Ext}^1_R\,(N, A), D) = 0$. Since D is any divisible Abelian group, then $\mathrm{Ext}^1_R\,(N, A) = 0$ for every singular module A. Thus N is a UF module.

As a corollary, we have the following result for left hereditary rings:

COROLLARY 1. *Let R be a left hereditary ring whose maximal* *quotient ring* ${}_RQ$ (see [3], [11]) *is R-flat. Then the following statements are equivalent for any finitely generated nonsingular R-module N*:

(a) *N is a UF module.*

(b) *N is almost finitely related.*

(c) *N is a FGTI module.*

Proof. The equivalence of (b) and (c) is clear from Remark (1) following Lemma 3. The equivalence of (a) and (b) will follow immediately from Theorem 1 if we show that the ring hypothesis implies every nonsingular R-module is R-flat. But this follows from [11, Cor. 2.5] and [11, Theorem 2.1].

An immediate consequence of Lemma 1 and Theorem 1 is the following characterization of FGSP:

COROLLARY 2. *A ring R has FGSP if and only if the following statements hold*:

(a) $Z(R) = 0$.

(b) *Every finitely generated nonsingular R-module is almost finitely related.*

(c) $hd(N) \leqq 1$ *for every finitely generated nonsingular R-module N.*

(d) $\mathrm{Tor}_1^R(\mathrm{Hom}_Z(A, D), N) = 0$, *where N is any finitely generated nonsingular R-module, D is any divisible Abelian group, and Z denotes the ring of integers.*

Combining Corollaries 1 and 2, the reader can easily see that a left hereditary ring R, whose maximal left quotient ring ${}_RQ$ is flat, has FGSP if and only if every finitely generated nonsingular R-module is almost finitely related. We shall see in Corollary 6 that Corollary 2 also takes on a particularly nice form whenever R is a commutative ring.

A submodule K of an R-module M is said to be an *almost summand* of M if $K = U \oplus V$, where U is a direct summand of M and $V = \mathrm{Soc}(V)$. The next theorem gives a relationship between UF R-modules and almost summands of free R-modules.

THEOREM 2. *Let* $Z(R) = 0$, *and let* $N \cong F/K$ *be a finitely generated nonsingular R-module, where F is a finitely generated free R-module. If K is an almost summand of F, then N is UF. Moreover, if N is R-flat, then the converse holds.*

Proof. To prove the first statement, it suffices to show that any homomorphism $f\colon K \to A$ can be lifted to a homomorphism $g\colon F \to A$, where A is any singular module. Now $K = U \oplus V$, where

$F = U \oplus W$ for some submodule W of F and $V = \mathrm{Soc}(V)$. Since $Z(A) = A$ and $Z(K) = 0$, then $f(\mathrm{Soc}(K)) = 0$. If $x \in K \cap W$, it follows from the direct sum decompositions that $x \in \mathrm{Soc}(K)$, and hence $f(x) = 0$. So the desired lifting of f is given by $g(u + w) = f(u)$ for all $u \in U$ and all $w \in W$.

Now assume N is an R-flat UF module. By Theorem 1, $K = U \oplus V$, where U is finitely generated and $V = \mathrm{Soc}(V)$ is projective. Then there is an exact sequence

$$0 \longrightarrow K/U \longrightarrow F/U \longrightarrow F/K \longrightarrow 0$$

with K/U and F/K R-flat. Thus F/U is also R-flat. But F/U is finitely related (*see* [5, p. 459]) and therefore projective by [5, p. 459]. Consequently U is a direct summand of F, and $K = U \oplus V$ is an almost summand of F.

The following corollary is an immediate consequence of Lemma 1 and Theorem 2.

COROLLARY 3. *If* $Z(R) = 0$ *and if every closed submodule of a finitely generated free R-module F is an almost summand of F, then R has FGSP. Moreover, if every (finitely generated) nonsingular R-module is flat, then the converse holds.*

The next corollary is a partial generalization of [11, Corollary 2.7].

COROLLARY 4. *If R is a right semi-hereditary ring having a maximal left quotient ring Q* (see [3], [11]), *which is a two-sided quotient ring of R, then the following statements are equivalent:*

(a) *R has* FGSP.

(b) $Z(R) = 0$, *and every closed submodule of a finitely generated free R-module F is an almost summand of F.*

Proof. By Corollary 3, we need to show that if R has FGSP, then every nonsingular R-module is flat. Since $Z(R) = 0$ by Lemma 1 and since Q is two-sided, then every finitely generated nonsingular R-module is torsionless by [3, Theorem 1.1]. However R is right semi-hereditary; hence every torsionless R-module is flat by [5, Theorem 4.1].

COROLLARY 5. *Let R be a commutative ring with* $Z(R) = 0$. *Let* $N \cong F/K$, *where F is a finitely generated free R-module. Then N is UF if and only if N is a nonsingular module and K is an almost summand of F.*

Proof. By Theorem 2, it suffices to show that any UF R-module is R-flat. But this follows from the proof of the corollary to [2, Proposition 1.11].

Pierce [8, p. 109] asks when a finitely generated module over a commutative regular ring splits. Corollary 5 sheds some light in this direction. Moreover, since the hypothesis, "R is a commutative ring with $Z(R) = 0$," is used only to establish that nonsingular modules are flat, the conclusion of Corollary 5 holds true for any regular ring R. Corollary 5 also generalizes [10, Theorem 3.3], which deals with the structure of rings for which cyclic modules split.

In [2] Cateforis and Sandomierski have suggested the question of determining all commutative rings with FGSP. The final corollary extends [10, Theorem 3.3] to give an answer to this question.

COROLLARY 6. *If R is a commutative ring, then the following statements are equivalent:*

(a) *R has* FGSP.

(b) $Z(R) = 0$, *and every closed submodule of a finitely generated free R-module F is an almost summand of F.*

(c) *R is semi-hereditary, and every finitely generated nonsingular module is almost finitely related.*

Proof. The equivalence of (a) and (b) follows from Lemma 1 and Corollary 5. In view of the corollary to [2, Proposition 1.11], (c) is an immediate consequence of (a) and (b). Assuming (c), the last two sequences in the proof of Corollary 4 show that all nonsingular modules are flat. Hence (b) follows by a slight modification of the argument used in the last part of the proof of Theorem 2.

The authors conjecture that a ring R has FGSP if and only if $Z(R) = 0$ and every closed submodule of a finitely generated free module F is an almost summand of F.

In view of the preceding corollaries and the corollary to [2, Proposition 1.11], the reader might conjecture that the messy "Tor condition" in Corollary 2 (d) can be replaced by the nicer condition, "R is right semi-hereditary," or by the stronger condition, "all nonsingular R-modules are flat." However, the following example shows that a ring R with FGSP need not be right semi-hereditary.

EXAMPLE. Let F be a field, and let T be the F-subalgebra of $\prod_{n=1}^{\infty} F^{(n)}$ generated by $\bigoplus \sum_{n=1}^{\infty} F^{(n)}$ and the identity of $\prod_{n=1}^{\infty} F^{(n)}$, where $F^{(n)} \cong F$ for all n. Let $I = \bigoplus \sum_{n=1}^{\infty} F^{(n)}$, and let $S = T/I$. If R is the ring of all 2×2 matrices of the form

$$\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b \in S; c \in T\right\},$$

then Chase [4, Proposition 3.1] has shown that R is a left semi-hereditary ring, which is not a right semi-hereditary ring. Hence $Z(R) = 0$, and it is straight forward to check that the only proper essential left ideal of R is the maximal left ideal

$$J = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b \in S; c \in I\right\}.$$

Thus if A is any singular R-module, then A is a direct sum of copies of the simple module R/J. It follows that each singular module is injective, and hence every R-module splits. Thus R has FGSP, but R is not right semi-hereditary.

Added in proof. K. R. Goodearl has constructed an example (unpublished) which shows that the conjecture following Corollary 6 is not true.

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