THE SINGULAR SUBMODULE OF A FINITELY GENERATED
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JOHN FUELBERTH AND MARK LAWRENCE TEPLOY
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A characterization is given of the finitely generated non-singular left $R$-modules $N$ such that $\text{Ext}_R(N, M) = 0$ for every singular left $R$-module $M$. As a corollary, the rings $R$, over which the singular submodule $Z(A)$ is a direct summand of every finitely generated left $R$-module $A$, are characterized. This characterization takes a simplified form whenever $R$ is commutative. An example is given to show that a ring $R$, over which the singular submodule $Z(A)$ is a direct summand of every left $R$-module $A$, need not be right semi-hereditary.

In this paper, all rings $R$ are assumed to be associative with an identity element, and, unless otherwise stated, all $R$-modules will be unitary left $R$-modules.

A submodule $B$ of an $R$-module $A$ is an essential submodule of $A$ if $B \cap C \neq 0$ for all nonzero submodules $C$ of $A$. A left ideal $I$ of $R$ is essential in $R$ if it is essential in $R$ as a submodule of $R$. If $A$ is an $R$-module, $Z(A) = \{a \in A \mid (0 : a) \text{ is essential in } R\}$ is the singular submodule of $A$. $A$ is called a singular module if $Z(A) = A$; and $A$ is a nonsingular module if $Z(A) = 0$. A submodule $B$ of $A$ is closed in $A$ if $B$ has no proper essential extension in $A$. If $A$ is nonsingular, then a submodule $B$ of $A$ is a closed submodule of $A$ if and only if $A/B$ is a nonsingular module. A simple $R$-module $S$ is nonsingular if and only if it is projective. For an $R$-module $A$, $\text{Soc}(A)$ denotes the sum of all simple submodules of $A$ or 0 if $A$ has no simple submodules.

Motivated by a definition of Kaplansky [6], we say that an $R$-module $N$ is $UF$ if $N$ is a nonsingular module and $\text{Ext}_R(N, M) = 0$ for all singular $R$-modules $M$. An $R$-module $A$ is said to split if $Z(A)$ is a direct summand of $A$. As in [2], a ring $R$ has the finitely generated splitting property (FGSP) if every finitely generated $R$-module splits.

We shall use the following result of Cateforis and Sandomierski [2, Proposition 1.11], which is included here for completeness.

**Lemma 1.** For any ring $R$, the following statements are equivalent:

(a) $R$ has FGSP.

(b) $Z(R) = 0$, and every finitely generated nonsingular $R$-module is $UF$. 

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An $R$-module $K$ is said to be \textit{almost finitely generated} if $K = U \oplus V$, where $U$ is a finitely generated $R$-module and $V = \text{Soc}(V)$. Then an $R$-module $N$ is called \textit{almost finitely related} if there exists an exact sequence of $R$-modules

$$0 \rightarrow K \rightarrow F \rightarrow N \rightarrow 0,$$

where $F$ is a finitely generated free module and $K$ is almost finitely generated.

Before stating our main results, we prove several lemmas.

**Lemma 2.** \textit{If $N$ is an almost finitely related $R$-module and if}

$$0 \rightarrow K \rightarrow F \rightarrow N \rightarrow 0$$

\textit{is any exact sequence of $R$-modules with $F$ a finitely generated free module, then $K$ is almost finitely generated.}

\textit{Proof.} Since $N$ is almost finitely related, there exists an exact sequence of $R$-modules

$$0 \rightarrow K \rightarrow F \rightarrow N \rightarrow 0,$$

where $F_i$ is a finitely generated free module and $K_i$ is almost finitely generated. By a result of Schanuel [9, p. 369], $K \oplus F_i \cong K_i \oplus F$. Since $K_i$ and $F$ are almost finitely generated, then so is $K \oplus F_i \cong K_i \oplus F$. Therefore $(K \oplus F_i)/\text{Soc}(K \oplus F_i)$ is finitely generated. Since

$$\frac{K \oplus F_i}{\text{Soc}(K \oplus F_i)} \cong \frac{K}{\text{Soc}(K)} \oplus \frac{F_i}{\text{Soc}(F_i)},$$

then $K/\text{Soc}(K)$ is also finitely generated.

Now we write $K = Rx_1 + Rx_2 + \cdots + Rx_m + \text{Soc}(K)$, where $x_1, x_2, \cdots, x_m \in K$. Let $W = (\text{Soc}(K)) \cap (Rx_1 + Rx_2 + \cdots + Rx_m)$. Then there exists an $R$-module $V$ such that $\text{Soc}(K) = W \oplus V$. It follows that $K = (Rx_1 + Rx_2 + \cdots + Rx_m) \oplus V$, and hence $K$ is almost finitely generated.

A finitely generated nonsingular $R$-module $N$ is called \textit{finitely generated torsion inducing} (FGTI) if $N$ has the following property: If $M$ is any finitely generated $R$-module with $M/Z(M) \cong N$, then $Z(M)$ is finitely generated.

**Lemma 3.** Let $Z(R) = 0$, and let $0 \rightarrow K \rightarrow F \rightarrow N \rightarrow 0$ be an exact sequence of $R$-modules, where $F$ is a finitely generated free module. If $N$ is nonsingular, then the following statements hold:

(a) \textit{If $N$ is FGTI and if $K/\text{Soc}(K)$ is a direct sum of countably generated modules, then $N$ is almost finitely related.}
(b) If $N$ is almost finitely related, then $N$ is an FGTI module.

Proof. To show (a), we need to show that $K$ is almost finitely generated. By hypothesis, $Y = K/\text{Soc}(K) = \bigoplus_{\alpha \in \mathcal{A}} M_{\alpha}$, where each $M_{\alpha}$ is a countably generated $R$-module. First we show that $Y$ is, in fact, countably generated also. Let $\mathcal{B} = \{ \alpha \in \mathcal{A} \mid M_{\alpha}$ contains a proper essential submodule$\}$. Thus if $\alpha \in \mathcal{A} - \mathcal{B}$, then $M_{\alpha}$ is a direct sum of singular simple $R$-modules or zero. For each $\alpha \in \mathcal{B}$, let $L_{\alpha}$ be a proper essential submodule of $M_{\alpha}$. Define $L = \bigoplus_{\alpha \in \mathcal{B}} L_{\alpha}$, and let $J$ be a submodule of $K$ containing $\text{Soc}(K)$ such that $J/\text{Soc}(K) = L$. Since

$$Z(F/J) \cong Z((F/\text{Soc}(K))/(J/\text{Soc}(K))) \cong Y/L \cong K/J,$$

then $K/J$ is a singular module; but since $Z(F/K) = 0$, it follows that $Z(F/J) = K/J$. By hypothesis, $N$ is a FGTI module; hence

$$K/J \cong \left( \bigoplus_{\alpha \in \mathcal{A}} (M_{\alpha}/L_{\alpha}) \right) \bigoplus \left( \bigoplus_{\alpha \in \mathcal{B}} M_{\alpha}\right)$$

is a finitely generated $R$-module. Therefore all but finitely many of the $M_{\alpha}(\alpha \in \mathcal{A})$ must be 0, and hence $K/\text{Soc}(K)$ is countably generated.

Thus there exist $x_{i} \in K$ ($i = 1, 2, \cdots$) such that $K = \sum_{i=1}^{m} Rx_{i} + \text{Soc}(K)$. We will show that there exists a positive integer $m$ such that $K = \sum_{i=1}^{m} Rx_{i} + \text{Soc}(K)$. If this were not the case, then for each positive integer $n$, there exists a least positive integer $k(n)$ such that $x_{k(n)} \in Rx_{1} + Rx_{2} + \cdots + Rx_{n} + \text{Soc}(K)$. By Zorn’s lemma, choose $K_{n}$ maximal with respect to $x_{k(n)} \in K_{n}$ and

$$Rx_{1} + Rx_{2} + \cdots + Rx_{n} + \text{Soc}(K) \subseteq K_{n} \subseteq K.$$

It follows that $(Rx_{k(n)} + K_{n})/K_{n}$ is an essential, simple, nonprojective submodule of $K/K_{n}$. Since $K/K_{n}$ is an essential extension of a singular simple module, then $K/K_{n}$ is also a singular module.

Define $\varphi : K \to \bigoplus_{n=1}^{\infty} K/K_{n}$: $x \to \sum_{n=1}^{\infty} \varphi_{n}(x)$, where $\varphi_{n} : K \to K/K_{n}$ is the natural map. If $x \in K$, then $x = \sum_{i=1}^{t} r_{i}x_{i} \in \sum_{i=1}^{t} Rx_{i} \subseteq K_{n}$ for all $n \geq t$. Thus $\varphi_{n}(x) = 0$ for all $n \geq t$, and hence $\varphi$ is well-defined. If $H = \ker \varphi$, then $K/H \cong \text{im} \varphi$ is not finitely generated (as $\varphi_{n}(x_{k(n)}) = 0$ for each integer $n$). Moreover, since $\text{im} \varphi$ is a submodule of the singular module $\bigoplus_{n=1}^{\infty} K/K_{n}$, then $K/H \cong \text{im} \varphi$ is also a singular module. Since $K$ is a closed submodule of $F$, then $Z(F/H) = K/H$. But then $F/H$ does not have a finitely generated singular submodule, and $(F/H)/Z(F/H) \cong F/K \cong N$. This contradicts the hypothesis that $N$ is a FGTI module. Thus $K = \sum_{i=1}^{m} Rx_{i} + \text{Soc}(K)$ for some positive integer $m$.

Now the argument used in the last paragraph of the proof of
Lemma 2 shows that $K$ is almost finitely generated. Therefore (a) holds.

Now we prove (b). Let $M$ be a finitely generated $R$-module such that $M/Z(M) \cong N$. Let $y_1, y_2, \ldots, y_n$ be a set of generators of $M$, and let $F$ be a free $R$-module with basis $u_1, u_2, \ldots, u_n$. Then there exists a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \rightarrow & K & \rightarrow & F & \rightarrow & N & \rightarrow & 0 \\
\downarrow{\lambda} & & \downarrow{\mu} & & \downarrow{\nu} & & & & \\
0 & \rightarrow & Z(M) & \rightarrow & M & \rightarrow & M/Z(M) & \rightarrow & 0,
\end{array}
$$

where $\mu: F \rightarrow M$ via $\mu(u_i) = y_i$ is an epimorphism and $\nu$ is an isomorphism. Then $\lambda$ must be an epimorphism. By the hypothesis and Lemma 2, $K = U \oplus V$, where $U$ is a finitely generated $R$-module and $V = \text{Soc} (V)$. Since $\lambda(V)$ is isomorphic to a submodule of the nonsingular, semi-simple module $V$ and since $Z(M)$ is singular, then $\lambda(V) = 0$. Thus $Z(M)$ is an epimorphic image of the finitely generated module $U$. Consequently, $Z(M)$ is a finitely generated module.

**Remarks.** (1) If $R$ is a left hereditary ring, then any closed submodule $K$ of a finitely generated free module $F$ is projective. So it follows from [7, Theorem 1] that $K/\text{Soc} (K)$ is a direct sum of countably generated modules. Thus for a left hereditary ring $R$, a finitely generated nonsingular $R$-module $N$ is FGTI if and only if $N$ is almost finitely related.

(2) Suppose that $N$, $F$, and $K$ are as in the hypothesis of Lemma 3. If $N$ is FGTI and $\text{Soc} (K)$ is essential in $K$, then $K/\text{Soc} (K)$ is finitely generated. So we can conclude the following result from Lemma 3: If $R$ is a nonsingular ring with essential socle, then a finitely generated nonsingular FGTI module is almost finitely related.

(3) There seems to be some independent interest in determining when the singular submodule of a finitely generated module is finitely generated. Indeed, Pierce [8, p. 109] asks questions along this line. Lemma 3 and the first of this remark shed some light in this direction.

We shall use $\text{hd}(N)$ to denote the projective homological dimension of an $R$-module $N$.

We now need an obvious generalization of a result of Kaplansky, [6, Theorem 1]:

**Lemma 4.** If $N$ is a UF $R$-module, then $\text{hd}(N) \leq 1$.

**Proof.** Let $N$ be a UF $R$-module, and let $M$ be any $R$-module. The exact sequence
induces the exact sequence

\[
\text{Ext}_R^1(N, E(M)/M) \longrightarrow \text{Ext}_R^2(N, M) \longrightarrow \text{Ext}_R^3(N, E(M)) = 0,
\]

where \(E(M)\) denotes the injective hull of \(M\). Since \(N\) is \(UF\) we have \(\text{Ext}_R^1(N, E(M)/M) = 0\); and hence \(\text{Ext}_R^2(N, M) = 0\) by exactness.

We now give a characterization of \(UF\) modules.

**Theorem 1.** Let \(Z(R) = 0\), and let \(N\) be a finitely generated \(R\)-module. Then \(N\) is \(UF\) if and only if the following conditions are satisfied:

(i) \(N\) is an almost finitely related, nonsingular module.

(ii) \(hd(N) \leq 1\).

(iii) \(\text{Tor}_i^R(\text{Hom}_Z(A, D), N) = 0\), where \(A\) is any singular \(R\)-module, \(D\) is any divisible Abelian group, and \(Z\) denotes the ring of integers.

**Proof.** We develop a diagram (see (*)), which we use in both directions of the proof. For any finitely generated \(R\)-module \(N\), there is an exact sequence

\[
0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0,
\]

where \(F\) is a finitely generated free \(R\)-module. If \(D\) is any divisible Abelian group and if \(A\) is any singular \(R\)-module, then \(\text{Hom}_Z(A, D)\) is a right \(R\)-module. Hence there is an exact sequence

\[
0 \longrightarrow \text{Tor}_i^R(\text{Hom}_Z(A, D), N) \longrightarrow \text{Hom}_Z(A, D) \otimes_R K
\]

\[
\longrightarrow \text{Hom}_Z(A, D) \otimes_R F.
\]

The exact sequence

\[
\text{Hom}_R(F, A) \longrightarrow \text{Hom}_R(K, A) \longrightarrow \text{Ext}_R^1(N, A) \longrightarrow 0
\]

induces an exact sequence

\[
0 \longrightarrow \text{Hom}_Z(\text{Ext}_R^1(N, A), D) \longrightarrow \text{Hom}_Z(\text{Hom}_R(K, A), D)
\]

\[
\longrightarrow \text{Hom}_Z(\text{Hom}_R(F, A), D).
\]

It is well-known [1, p. 120] that there exists a homomorphism \(\psi\) and an isomorphism \(\beta\) making the following diagram commutative:

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Tor}_i^R(\text{Hom}_Z(A, D), N) \\
& & \downarrow \psi \\
& & \text{Hom}_Z(A, D) \otimes_R K \\
& & \downarrow \beta \\
0 & \longrightarrow & \text{Hom}_Z(\text{Ext}_R^1(N, A), D) \\
& & \text{Hom}_Z(\text{Hom}_R(K, A), D) \\
& & \text{Hom}_Z(\text{Hom}_R(F, A), D) \
\end{array}
\]
"only if": Let \( N \) be a finitely generated \( UF \) \( R \)-module. Then there exists an exact sequence

\[
0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0
\]

of left \( R \)-modules, where \( F \) is a finitely generated free module. By Lemma 4, \( K \) is a projective \( R \)-module; thus \( K = \bigoplus \sum_{\alpha \in \mathcal{A}} K_{\alpha} \), where each \( K_{\alpha} \) is countably generated by [7, Theorem 1]. Since

\[
\frac{K}{\text{Soc}(K)} = \left( \bigoplus \sum_{\alpha \in \mathcal{A}} \frac{K_{\alpha}}{\text{Soc}(K_{\alpha})} \right) \cong \left( \bigoplus \sum_{\alpha \in \mathcal{A}} \frac{K_{\alpha}}{\text{Soc}(K_{\alpha})} \right),
\]

then \( K/\text{Soc}(K) \) is a direct sum of countably generated \( R \)-modules.

Since a \( UF \) module is FGTI, then Lemma 3 (a) implies that \( N \) is almost finitely related, i.e., (i) holds.

Lemma 4 implies that \( \text{hd}(N) \leq 1 \); so (ii) holds.

Now we show that (iii) holds. Let \( A, D, F, \) and \( K \) be chosen as in (*). Then by (i), \( K = U \oplus V \), where \( U \) is finitely generated and \( V = \text{Soc}(V) \). But for any nonsingular simple \( R \)-module \( S \), \( \text{Hom}_{R}(S, A) = 0 \) (as \( A \) is singular). Thus by [1, VI. Prop. 5.2], \( \text{Hom}_{R}(A, D) \otimes_{R} S \cong \text{Hom}_{R}(\text{Hom}_{R}(S, A), D) = 0 \). Therefore \( \text{Hom}_{R}(A, D) \otimes_{R} V = 0 \), and \( \text{Hom}_{R}(V, A) = 0 \). Hence there exist obvious isomorphisms \( \sigma \) and \( \tau \) making the diagram

\[
\begin{align*}
\text{Hom}_{R}(A, D) \otimes_{R} K & \longrightarrow \text{Hom}_{R}(A, D) \otimes_{R} U \\
\psi' | & \text{Hom}_{R}(K, A, D) \longrightarrow \text{Hom}_{R}(\text{Hom}_{R}(U, A), D)
\end{align*}
\]

commute, where \( \psi' \) is the restriction of \( \psi \) in (*) to \( \text{Hom}_{R}(A, D) \otimes_{R} U \). By [1, VI. Prop. 5.2] \( \psi' \) is an isomorphism; whence \( \psi \) is forced to be an isomorphism also. By the commutativity of (*) and the fact that \( \text{Ext}^{1}_{R}(N, A) = 0 \), it is now easy to obtain \( \text{Tor}^{1}_{R}(\text{Hom}_{R}(A, D), N) = 0 \).

"if": Let \( A, D, F, K \) be as in (*). Since \( \text{hd}(N) \leq 1 \) and \( N \) is almost finitely related, then \( K \) is an almost finitely generated projective \( R \)-module. By the argument used in the preceding paragraph, \( \psi \) is an isomorphism in (*). From the commutativity of (*) and the fact that \( \text{Tor}^{1}_{R}(\text{Hom}_{R}(A, D), N) = 0 \), we now obtain \( \text{Hom}_{R}(\text{Ext}^{1}_{R}(N, A), D) = 0 \). Since \( D \) is any divisible Abelian group, then \( \text{Ext}^{1}_{R}(N, A) = 0 \) for every singular module \( A \). Thus \( N \) is a \( UF \) module.

As a corollary, we have the following result for left hereditary rings:

**Corollary 1.** Let \( R \) be a left hereditary ring whose maximal
quotient ring $\varphi Q$ (see [3], [11]) is $R$-flat. Then the following statements are equivalent for any finitely generated nonsingular $R$-module $N$:

(a) $N$ is a UF module.
(b) $N$ is almost finitely related.
(c) $N$ is a FGTI module.

Proof. The equivalence of (b) and (c) is clear from Remark (1) following Lemma 3. The equivalence of (a) and (b) will follow immediately from Theorem 1 if we show that the ring hypothesis implies every nonsingular $R$-module is $R$-flat. But this follows from [11, Cor. 2.5] and [11, Theorem 2.1].

An immediate consequence of Lemma 1 and Theorem 1 is the following characterization of FGSP:

**Corollary 2.** A ring $R$ has FGSP if and only if the following statements hold:

(a) $Z(R) = 0$.
(b) Every finitely generated nonsingular $R$-module is almost finitely related.
(c) $\text{hd}(N) \leq 1$ for every finitely generated nonsingular $R$-module $N$.
(d) $\text{Tor}_i^R(\text{Hom}_R(A, D), N) = 0$, where $N$ is any finitely generated nonsingular $R$-module, $D$ is any divisible Abelian group, and $Z$ denotes the ring of integers.

Combining Corollaries 1 and 2, the reader can easily see that a left hereditary ring $R$, whose maximal left quotient ring $\varphi Q$ is flat, has FGSP if and only if every finitely generated nonsingular $R$-module is almost finitely related. We shall see in Corollary 6 that Corollary 2 also takes on a particularly nice form whenever $R$ is a commutative ring.

A submodule $K$ of an $R$-module $M$ is said to be an almost summand of $M$ if $K = U \oplus V$, where $U$ is a direct summand of $M$ and $V = \text{Soc}(V)$. The next theorem gives a relationship between UF $R$-modules and almost summands of free $R$-modules.

**Theorem 2.** Let $Z(R) = 0$, and let $N \cong F/K$ be a finitely generated nonsingular $R$-module, where $F$ is a finitely generated free $R$-module. If $K$ is an almost summand of $F$, then $N$ is UF. Moreover, if $N$ is $R$-flat, then the converse holds.

Proof. To prove the first statement, it suffices to show that any homomorphism $f: K \rightarrow A$ can be lifted to a homomorphism $g: F \rightarrow A$, where $A$ is any singular module. Now $K = U \oplus V$, where
$F = U \oplus W$ for some submodule $W$ of $F$ and $V = \text{Soc}(V)$. Since $Z(A) = A$ and $Z(K) = 0$, then $f(\text{Soc}(K)) = 0$. If $x \in K \cap W$, it follows from the direct sum decompositions that $x \in \text{Soc}(K)$, and hence $f(x) = 0$. So the desired lifting of $f$ is given by $g(u + w) = f(u)$ for all $u \in U$ and all $w \in W$.

Now assume $N$ is an $R$-flat $UF$ module. By Theorem 1, $K = U \oplus V$, where $U$ is finitely generated and $V = \text{Soc}(V)$ is projective. Then there is an exact sequence

$$0 \rightarrow K/U \rightarrow F/U \rightarrow F/K \rightarrow 0$$

with $K/U$ and $F/K$ $R$-flat. Thus $F/U$ is also $R$-flat. But $F/U$ is finitely related (see [5, p. 459]) and therefore projective by [5, p. 459]. Consequently $U$ is a direct summand of $F$, and $K = U \oplus V$ is an almost summand of $F$.

The following corollary is an immediate consequence of Lemma 1 and Theorem 2.

**Corollary 3.** If $Z(R) = 0$ and if every closed submodule of a finitely generated free $R$-module $F$ is an almost summand of $F$, then $R$ has FGSP. Moreover, if every (finitely generated) nonsingular $R$-module is flat, then the converse holds.

The next corollary is a partial generalization of [11, Corollary 2.7].

**Corollary 4.** If $R$ is a right semi-hereditary ring having a maximal left quotient ring $Q$ (see [3], [11]), which is a two-sided quotient ring of $R$, then the following statements are equivalent:

(a) $R$ has FGSP.

(b) $Z(R) = 0$, and every closed submodule of a finitely generated free $R$-module $F$ is an almost summand of $F$.

**Proof.** By Corollary 3, we need to show that if $R$ has FGSP, then every nonsingular $R$-module is flat. Since $Z(R) = 0$ by Lemma 1 and since $Q$ is two-sided, then every finitely generated nonsingular $R$-module is torsionless by [3, Theorem 1.1]. However $R$ is right semi-hereditary; hence every torsionless $R$-module is flat by [5, Theorem 4.1].

**Corollary 5.** Let $R$ be a commutative ring with $Z(R) = 0$. Let $N \cong F/K$, where $F$ is a finitely generated free $R$-module. Then $N$ is UF if and only if $N$ is a nonsingular module and $K$ is an almost summand of $F$. 
Proof. By Theorem 2, it suffices to show that any $UF R$-module is $R$-flat. But this follows from the proof of the corollary to [2, Proposition 1.11].

Pierce [8, p. 109] asks when a finitely generated module over a commutative regular ring splits. Corollary 5 sheds some light in this direction. Moreover, since the hypothesis, “$R$ is a commutative ring with $Z(R) = 0$,” is used only to establish that nonsingular modules are flat, the conclusion of Corollary 5 holds true for any regular ring $R$. Corollary 5 also generalizes [10, Theorem 3.3], which deals with the structure of rings for which cyclic modules split.

In [2] Cateforis and Sandomierski have suggested the question of determining all commutative rings with FGSP. The final corollary extends [10, Theorem 3.3] to give an answer to this question.

Corollary 6. If $R$ is a commutative ring, then the following statements are equivalent:

(a) $R$ has FGSP.
(b) $Z(R) = 0$, and every closed submodule of a finitely generated free $R$-module $F$ is an almost summand of $F$.
(c) $R$ is semi-hereditary, and every finitely generated nonsingular module is almost finitely related.

Proof. The equivalence of (a) and (b) follows from Lemma 1 and Corollary 5. In view of the corollary to [2, Proposition 1.11], (c) is an immediate consequence of (a) and (b). Assuming (c), the last two sequences in the proof of Corollary 4 show that all nonsingular modules are flat. Hence (b) follows by a slight modification of the argument used in the last part of the proof of Theorem 2.

The authors conjecture that a ring $R$ has FGSP if and only if $Z(R) = 0$ and every closed submodule of a finitely generated free module $F$ is an almost summand of $F$.

In view of the preceding corollaries and the corollary to [2, Proposition 1.11], the reader might conjecture that the messy “Tor condition” in Corollary 2 (d) can be replaced by the nicer condition, “$R$ is right semi-hereditary,” or by the stronger condition, “all nonsingular $R$-modules are flat.” However, the following example shows that a ring $R$ with FGSP need not be right semi-hereditary.

Example. Let $F$ be a field, and let $T$ be the $F$-subalgebra of $\prod_{n=1}^{\infty} F^{(n)}$ generated by $\bigoplus \sum_{n=1}^{\infty} F^{(n)}$ and the identity of $\prod_{n=1}^{\infty} F^{(n)}$, where $F^{(n)} \cong F$ for all $n$. Let $I = \bigoplus \sum_{n=1}^{\infty} F^{(n)}$, and let $S = T/I$. If $R$ is the ring of all $2 \times 2$ matrices of the form
then Chase [4, Proposition 3.1] has shown that $R$ is a left semi-hereditary ring, which is not a right semi-hereditary ring. Hence $Z(R) = 0$, and it is straightforward to check that the only proper essential left ideal of $R$ is the maximal left ideal

$$J = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b \in S; c \in T \right\}.$$

Thus if $A$ is any singular $R$-module, then $A$ is a direct sum of copies of the simple module $R/J$. It follows that each singular module is injective, and hence every $R$-module splits. Thus $R$ has FGSP, but $R$ is not right semi-hereditary.

*Added in proof.* K. R. Goodearl has constructed an example (unpublished) which shows that the conjecture following Corollary 6 is not true.

**REFERENCES**


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