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Γ -EXTENSIONS OF IMAGINARY QUADRATIC FIELDS

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Let p be an odd rational prime and $E_0 = \mathcal{O}(\sqrt{-m})$ a quadratic imaginary number field. There is a unique Γ -extension E of E_0 for the prime p which is absolutely abelian. For each positive integer n there is a subfield E_n of E which is cyclic of degree p^n over E_0 and by Iwasawa the exponent of p in the class number of E_n is of the form $\mu p^n + \lambda n + c$ for sufficiently large n . We here examine the analytic formula for the class number of E_n and in the case $p = 3$ give a simple condition implying that $\mu = 0$. It follows easily from this condition that there are infinitely many imaginary quadratic fields which have Γ -extensions for the prime 3 with the invariants $\mu = 0$ while $\lambda \geq 1$.

1. Analytic formula. Let \mathcal{O} be the rationals, p an odd prime, n an integer ≥ 0 , and $\zeta_{p^{n+1}}$ a primitive p^{n+1} root of unity. Let F_n be the subfield of $\mathcal{O}(\zeta_{p^{n+1}})$ of degree p^n over the rationals so that F_n/\mathcal{O} is cyclic, p is the unique ramified prime for the extension, and p is totally ramified. Let $E_0 = \mathcal{O}(\sqrt{-m})$, a quadratic imaginary field where $(m, p) = 1$ and let $E_n = F_n \cdot E_0$, the composite field.

We attempt to study the order, e_n , to which p divides the class number of E_n ,

$$h_{E_n} = p^{e_n} \cdot h' \quad (p, h') = 1$$

by use of the classical analytic formula for an arbitrary number field k :

$$(1) \quad \lim_{s \rightarrow 1} (s-1)\zeta_k(s) = \frac{2^{s+t}\pi^t R_k}{m_k \sqrt{|D_k|}} h_k$$

where, as usual, R_k is the regulator of k ; m_k , the order of the group of roots of unity; D_k , the discriminant of k ; and s and t , the number of real and complex infinite primes of k .

We note the following sequence of lemmas:

LEMMA 1. $m_{E_n} = m_{F_n} = 2$ unless $E_0 = \mathcal{O}(\sqrt{-3})$ or $\mathcal{O}(\sqrt{-1})$.

Proof. By degrees: $[E_n : \mathcal{O}] = 2p^n$.

Note that in the two excluded cases $(p, m_{E_n}) = 1$ if $(p, m) = 1$.

LEMMA 2. $D_{E_n} = D_{F_n}^{\circ} \cdot D_{E_0}^{p^n}$ and $D_{F_n} = p^{t_n}$; $t_n = (n+1)p^n - (p^n - 1)/(p - 1) - 1$.

Proof. First statement is trivial, second is proved as follows.

Note that $\zeta_{p^{n+1}}$ is a distinguished element for the extension $\mathcal{Q}(\zeta_{p^{n+1}})/F_n$ in the relation its different bears to the different of the extension [3]. The computation of the different of $\mathcal{Q}(\zeta_{p^{n+1}})/F_n$ becomes then an exercise in determinants. The result combined with the well known different of $\mathcal{Q}(\zeta_{p^{n+1}})/\mathcal{Q}$ gives the expression above.

LEMMA 3. $R_{E_n} = R_{F_n} \cdot 2^a$ some $a \in \mathbb{Z}$.

Proof. F_n is the maximal real subfield of E_n and the result is then well known [1].

Now let $k = E_n$, respectively F_n , in equation (1) and divide the former by the latter. Taking into account the preceding lemmas this simplifies to:

$$(2) \quad \lim_{s \rightarrow 1} (\zeta_{E_n}(s)/\zeta_{F_n}(s)) = \frac{2^a \pi^{p^n}}{\sqrt{|D_{E_0}|^{p^n}}} \frac{h_{E_n}}{p^{s_n} h_{F_n}}$$

$$s_n = \frac{1}{2} t_n = \frac{1}{2} ((n+1)p^n - (p^n - 1)/(p - 1) - 1).$$

On the other hand $\zeta_{E_n}(s) = \prod L(s, \chi)$ where the product is taken over all Dirichlet characters belonging to the extension E_n/\mathcal{Q} . Since $g(E_n/\mathcal{Q}) \cong \mathcal{X}/2 + \mathcal{X}/p^n$ we can write $\zeta_{E_n}(s) = \prod L(s, \chi_0^i \chi_1^j)$, $i = 0, 1$; $j = 0, \dots, p^n - 1$ where χ_0, χ_0^2 are the characters belonging to E_0/\mathcal{Q} while $\chi_1^0, \dots, \chi_1^{p^n-1}$ are the characters belonging to F_n/\mathcal{Q} . Hence $\zeta_{F_n}(s) = \prod L(s, \chi_1^j)$, $j = 0, \dots, p^n - 1$ and therefore $\zeta_{E_n}(s)/\zeta_{F_n}(s) = \prod L(s, \chi_0 \chi_1^j)$, $j = 0, \dots, p^n - 1$. Furthermore the χ_1^k , $k = 0, \dots, p^{n-1} - 1$ are the characters belonging to F_{n-1}/\mathcal{Q} and therefore

$$(3) \quad \frac{\zeta_{E_n}(s)/\zeta_{F_n}(s)}{\zeta_{E_{n-1}}(s)/\zeta_{F_{n-1}}(s)} = \prod_{\substack{0 \leq j < p^n \\ (j, p) = 1}} L(s, \chi_0 \chi_1^j).$$

Note in passing that χ_1 is an even character and takes on the p^n th roots of unity as values. Comparing (2) and (3) we may write

$$(4) \quad \prod_{\substack{0 \leq j < p^n \\ (j, p) = 1}} L(1, \chi_0 \chi_1^j) = \frac{h_{E_n} \cdot h_{F_{n-1}} \pi^{c(p^n)}}{h_{F_n} h_{E_{n-1}} p^{(s_n - s_{n-1})} \sqrt{|D_{E_0}|^{c(p^n)}}}.$$

Note that χ_0 is primitive modulo $d = D_{E_0} =$ the conductor of E_0/\mathcal{Q} , while χ_1^j , $(j, p) = 1$ is primitive modulo $p^{n+1} =$ the conductor of F_n/\mathcal{Q} . It follows that $\chi_0 \chi_1^j$, $(j, p) = 1$ is primitive with modulus $w = dp^{n+1}$ and is an odd character. It is well known then that

$$(5) \quad L(1, \chi_0 \chi_1^j) = \frac{\pi i \tau(\chi_0 \chi_1^j)}{w^2} \sum_{\substack{0 < k < w \\ (k, w) = 1}} \chi_0 \bar{\chi}_1^j(k) k$$

where $\tau(\chi_0 \chi_1^j)$ is the classical Gauss sum and $|\tau(\chi_0 \chi_1^j)| = \sqrt{w}$. Comparing now (4) and (5) and taking absolute values we see

$$(6) \quad \frac{\left| \prod_{\substack{(j, p) = 1 \\ 0 < j < p^n}} \sum_{\substack{(k, w) = 1 \\ 0 < k < w}} \chi_0 \bar{\chi}_1^j(k) k \right|}{d^{\varphi(p^n)} p^{(n+1)\varphi(p^n)}} = \frac{h_{E_n} h_{F_{n-1}}}{h_{F_n} h_{E_{n-1}}}.$$

Next we examine the sum appearing in (6).

$$\begin{aligned} S_j &= \sum_{0 < k < w} \chi_0 \bar{\chi}_1^j(k) k = \sum_{\alpha=0}^{d-1} p^{n+1-\alpha} \sum_{i=0}^{p^{n+1}-1} \chi_0 \bar{\chi}_1^j(i + \alpha p^{n+1})(i + \alpha p^{n+1}) \\ &= \sum_{\alpha=0}^{d-1} p^{n+1-\alpha} \sum_{i=0}^{p^{n+1}-1} \chi_0(i + \alpha p^{n+1}) \bar{\chi}_1^j(i) i + \alpha p^{n+1} \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i) \chi_0(i + \alpha p^{n+1}). \end{aligned}$$

But since

$$\sum_{\alpha=0}^{d-1} \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i) \chi_0(i + \alpha p^{n+1}) i = \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i) i \sum_{\alpha=0}^{d-1} \chi_0(i + \alpha p^{n+1}) = 0$$

we have

$$S_j = p^{n+1} \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i) \sum_{\alpha=0}^{d-1} \alpha \chi_0(i + \alpha p^{n+1}).$$

We now make the following assumption for the sake of simplifying notation and proofs: (A) $p^{n+1} \equiv 1(d)$. It then follows that

$$S_j = p^{n+1} \sum_i \bar{\chi}_1^j(i) \sum_{\alpha} \chi_0(i\alpha + \alpha).$$

Letting $w_k = \sum_{\alpha=0}^{d-1} \alpha \chi_0(\alpha + k)$ one can easily deduce that $w_0 = w_1$, $w_{k+d} = w_k$, and $w_k = w_0 + d \sum_{\alpha=0}^{k-1} \chi_0(\alpha)$. Then

$$\begin{aligned} S_j &= p^{n+1} \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i) w_0 + d \sum_{\alpha=0}^{i-1} \chi_0(\alpha) \\ &= p^{n+1} w_0 \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i) + d \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i) \sum_{\alpha=0}^{i-1} \chi_0(\alpha) \\ &= d p^{n+1} \sum_{i=0}^{p^{n+1}-1} \bar{\chi}_1^j(i) \cdot \alpha_i; \quad \text{where } \alpha_i = \sum_{\alpha=0}^{i-1} \chi_0(\alpha). \end{aligned}$$

Comparing this last result with (6) we see that

$$(7) \quad \prod_{\substack{(j, p) = 1 \\ 0 < j < p^{n+1}}} \sum_{i=0}^{p^{n+1}-1} \alpha_i \bar{\chi}_1^j(i) = \frac{h_{E_n} h_{F_{n-1}}}{h_{F_n} h_{E_{n-1}}},$$

and again $\alpha_i = \sum_{\alpha=0}^{i-1} \chi_0(\alpha)$.

We reduce our concern now to the power of p occurring in each

member of (7). By results of Iwasawa $(p, h_{F_n}) = (p, h_{F_{n-1}}) = 1$ while for sufficiently large n : $\text{ord}_p(h_{E_n}) = \mu p^n + \lambda n + c$, $\text{ord}_p(h_{E_{n-1}}) = \mu p^{n-1} + \lambda(n-1) + c$ ([2]). Therefore

$$(8) \quad \text{ord}_p \prod_{0 < j < p^{n+1}} \sum_{i=0}^{p^{n+1}-1} \alpha_i \bar{\chi}_1^j(i) = \mu \varphi(p^n) + \lambda.$$

It is clear that $\alpha_i \in \mathcal{R}$ and hence $\sum_{i=0}^{p^{n+1}-1} \alpha_i \bar{\chi}_1^j(i)$ is an integer in $\mathcal{O}(\zeta_{p^n})$. In fact, $\prod \sum \alpha_i \bar{\chi}_1^j(i)$ is simply the absolute norm of this integer. Hence

$$(9) \quad \begin{aligned} \mu \varphi(p^n) + \lambda &= \text{ord}_p \mathcal{N}_{\mathcal{O}} \left(\sum_{i=0}^{p^{n+1}-1} \alpha_i \chi_1(i) \right) \\ &= \text{ord}_p \sum_{i=0}^{p^{n+1}-1} \alpha_i \chi_1(i). \end{aligned}$$

Here p is the unique prime of $\mathcal{O}(\zeta_{p^n})$ dividing p .

We now rewrite $\sum \alpha_i \chi_1(i)$ in terms of an integral basis of $\mathcal{O}(\zeta_{p^n})$. Let g be a primitive root modulo p^{n+1} , i.e. \bar{g} generates $(\mathcal{R}/p^{n+1})^*$. Then $\sum_{i=0}^{p^{n+1}-1} \alpha_i \chi_1(i) = \sum_{s=0}^{\varphi(p^{n+1})-1} \alpha_{g^s} \chi_1(g^s)$ where $0 < g_s < p^{n+1}$ and $g_s \equiv g^s (p^{n+1})$. Then $\eta = \chi_1(g)$ is a primitive p^n th root of unity and

$$\sum_{s=0}^{\varphi(p^{n+1})-1} \chi_1(g^s) \alpha_{g^s} = \sum_{s=0}^{\varphi(p^{n+1})-1} \eta^s \alpha_{g^s}.$$

Since $1, \eta, \dots, \eta^{\varphi(p^n)-1}$ form a \mathcal{R} -basis for the integers of $\mathcal{O}(\zeta_{p^n})$ we may rewrite this last sum, using identities of the form $1 + \eta^{p^{n-1}} + \dots + \eta^{(p-1)p^{n-1}} = 0$, as

$$\sum_{s=0}^{\varphi(p^{n+1})-1} \eta^s \alpha_{g^s} = \sum_{s=0}^{\varphi(p^n)-1} \eta^s \sum_{i=0}^{p-2} (\alpha_{g_{s+i p^n}} - \alpha_{g_{\varphi(p^n)+t+i p^n}})$$

where $0 < t < p^{n-1}$ and $t \equiv s(p^{n-1})$. It follows from (9) then that

$$(10) \quad \mu \varphi(p^n) + \lambda = \text{ord}_p \sum_{s=0}^{\varphi(p^n)-1} \eta^s \sum_{i=0}^{p-2} (\alpha_{g_{s+i p^n}} - \alpha_{g_{\varphi(p^n)+t+i p^n}}).$$

For sufficiently large n the left member of (10) is $\geq \varphi(p^n)$ if and only if $\mu > 0$. However the right member is greater than $\varphi(p^n)$ if and only if $p^{\varphi(p^n)} = (p)$ divides the algebraic integer in brackets. Since this integer is written in terms of an integral basis it is divisible by (p) if and only if the coefficients of η^s is divisible by p for every s . Hence $\mu > 0$ if and only if p divides

$$(11) \quad \sum_{i=0}^{p-2} (\alpha_{g_{s+i p^n}} - \alpha_{g_{\varphi(p^n)+t+i p^n}}) \quad s = 0, 1, \dots, \varphi(p^n) - 1.$$

2. Special case of $p = 3$. If we specialize to $p = 3$, $s = 0$ we

may proceed in the following manner. For $p = 3, s = 0$ equation (11) reads

$$(12) \quad \alpha_{g_0} + \alpha_{g_{3^n}} - \alpha_{g_{(3^n)}} - \alpha_{g_{3^{2n} + \varphi(3^n)}} \cdot$$

Clearly $g_0 = 1, g_{3^n} = 3^{n+1} - 1$; while for appropriate choice of g we have $g_{\varphi(3^n)} = 3^n + 1$ (resp. $2 \cdot 3^n + 1$) and $g_{\varphi(3^{2n} + 3^n)} = 2 \cdot 3^n - 1$ (resp. $3^n - 1$). Hence (12) reads, letting $M(m) = \sum_{\alpha=0}^m \chi_0(\alpha)$,

$$(13) \quad \begin{aligned} &M(0) + M(3^{n+1}) - M(3^n) - M(2 \cdot 3^n - 2) \\ &(\text{resp. } M(0) + M(3^{n+1} - 2) - M(2 \cdot 3^n) - M(3^n - 2)) \cdot \end{aligned}$$

Clearly $M(0) = 0$ and recalling that (A) $3^{n+1} \equiv 1 \pmod{d}$ we see that $M(3^{n+1} - 2) = M(d - 1) = 0$ as well. Since $\chi_0(-1) = -1$ we have the trivial but useful identity $M(m) = M(kd - m - 1), kd - m - 1 > 0$. By this it follows that $M(2 \cdot 3^n - 2) = M(kd + 1 - 3^n - 2) = M(kd - 3^n - 1) = M(3^n)$ (resp. $M(3^n - 2) = M(2 \cdot 3^n)$). Hence (13) reduces to $-2M(3^n)$ (resp. $-2M(2 \cdot 3^n)$) and so $\mu > 0$ if and only if $M(3^n) \equiv 0 \pmod{3}$ (resp. $M(2 \cdot 3^n) \equiv 0 \pmod{3}$).

Again by (A): $M(2 \cdot 3^n) = M(kd + 1 - 3^n) = M(3^n - 2) = M(3^n) - \chi_0(3^n) - \chi_0(3^n - 1)$. Since both congruences above must be satisfied it follows that $\mu > 0$ if and only if $\chi_0(3^n) + \chi_0(3^n - 1) \equiv 0 \pmod{3}$. Multiplying by $\chi_0(3) \neq 0$ we have $[\chi_0(3^n) + \chi_0(3^n - 1)] = \chi_0(3) = \chi_0(1) - \chi_0(2)$. Hence we may finally state in the language of Iwasawa

THEOREM. *Let $E_\infty = \bigcup E_n$ be the absolutely abelian Γ -extension for the prime 3 of $\mathcal{O}(\sqrt{-m})$; $(m, 3) = 1$. If 2 does not split in $\mathcal{O}(\sqrt{-m})/\mathcal{O}$ then the invariant μ equals 0.*

EXAMPLE 1. $E_0 = \mathcal{O}(\sqrt{-5})$. Since $\chi_0(3) = +1, 3$ splits in $\mathcal{O}(\sqrt{-5})/\mathcal{O}$ and it is easy to see from the structure of the genus field for E_n/E_0 that $\lambda \geq 1$. On the other hand, $\chi_0(2) = 0$ and therefore $\mu = 0$. Obviously all $\mathcal{O}(\sqrt{-m})$ for $m \equiv 7, 10 \pmod{12}$ behave in this manner.

EXAMPLE 2. $E_0 = \mathcal{O}(\sqrt{-23})$. This field has class number 3 and is therefore of some interest. Unfortunately $\chi_0(2) = 1$, but we may use the remark above that $\mu > 0$ if and only if $M(3^n) \equiv 0 \pmod{3}$. By (A): $M(3^n) = M(3^{-1}) = M(8)$ in this case. But $M(8) = 4 \not\equiv 0 \pmod{3}$ and so again $\mu = 0$.

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Alex Bacopoulos and Athanassios G. Kartsatos, <i>On polynomials approximating the solutions of nonlinear differential equations</i>	1
Monte Boisen and Max Dean Larsen, <i>Prüfer and valuation rings with zero divisors</i>	7
James J. Bowe, <i>Neat homomorphisms</i>	13
David W. Boyd and Hershy Kisilevsky, <i>The Diophantine equation $u(u + 1)(u + 2)(u + 3) = v(v + 1)(v + 2)$</i>	23
George Ulrich Brauer, <i>Summability and Fourier analysis</i>	33
Robin B. S. Brooks, <i>On removing coincidences of two maps when only one, rather than both, of them may be deformed by a homotopy</i>	45
Frank Castagna and Geert Caleb Ernst Prins, <i>Every generalized Petersen graph has a Tait coloring</i>	53
Micheal Neal Dyer, <i>Rational homology and Whitehead products</i>	59
John Fuelberth and Mark Lawrence Teply, <i>The singular submodule of a finitely generated module splits off</i>	73
Robert Gold, <i>Γ-extensions of imaginary quadratic fields</i>	83
Myron Goldberg and John W. Moon, <i>Cycles in k-strong tournaments</i>	89
Darald Joe Hartfiel and J. W. Spellmann, <i>Diagonal similarity of irreducible matrices to row stochastic matrices</i>	97
Wayland M. Hubbart, <i>Some results on blocks over local fields</i>	101
Alan Loeb Kostinsky, <i>Projective lattices and bounded homomorphisms</i>	111
Kenneth O. Leland, <i>Maximum modulus theorems for algebras of operator valued functions</i>	121
Jerome Irving Malitz and William Nelson Reinhardt, <i>Maximal models in the language with quantifier "there exist uncountably many"</i>	139
John Douglas Moore, <i>Isometric immersions of space forms in space forms</i>	157
Ronald C. Mullin and Ralph Gordon Stanton, <i>A map-theoretic approach to Davenport-Schinzel sequences</i>	167
Chull Park, <i>On Fredholm transformations in Yeh-Wiener space</i>	173
Stanley Poreda, <i>Complex Chebyshev alterations</i>	197
Ray C. Shiflett, <i>Extreme Markov operators and the orbits of Ryff</i>	201
Robert L. Snider, <i>Lattices of radicals</i>	207
Ralph Richard Summerhill, <i>Unknotting cones in the topological category</i>	221
Charles Irvin Vinsonhaler, <i>A note on two generalizations of QF-3</i>	229
William Patterson Wardlaw, <i>Defining relations for certain integrally parameterized Chevalley groups</i>	235
William Jennings Wickless, <i>Abelian groups which admit only nilpotent multiplications</i>	251