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Myron Goldberg and John W. Moon

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A tournament T_n with n nodes is k-strong if k is the largest integer such that for every partition of the nodes of T_n into two nonempty subsets A and B there are at least k arcs that go from nodes of A to nodes of B and conversely. The main result is that every k-strong tournament has at least k different spanning cycles.

1. Introduction. A tournament T_n consists of a finite set of nodes 1, 2, \cdots , n such that each pair of distinct nodes i and j is joined by exactly one of the arcs \vec{ij} or \vec{ji} . If the arc \vec{ij} is in T_n we say that i beats j or j loses to i and write $i \rightarrow j$. If each node of a subtournament A beats each node of a subtournament B we write $A \rightarrow B$ and let A + B denote the tournament determined by the nodes of A and B. A tournament T_n is k-strong if k is the largest integer such that for every partition of the nodes of T_n into two nonempty subsets A and B there are at least k arcs that go from nodes of A to nodes of B and conversely; a tournament T_n is strong if n = 1 or if it is k-strong for some positive integer k. If a tournament T_n is not strong, or weak, it has a unique expression of the type $T_n = A + A$ $B + \cdots + J$ where the nonempty components A, B, \cdots , J all are strong; we call A and J the top and bottom components of T_n . (The top or bottom component of a strong tournament is the tournament itself.)

An *l*-path is a sequence $\mathscr{P} = \{p_1, p_2, \dots, p_{l+1}\}$ of nodes such that $p_i \to p_{i+1}$ for $1 \leq i \leq l$; we assume the nodes of \mathscr{P} are distinct except that p_{l+1} and p_1 may be the same in which case we call the sequence an *l*-cycle; it is sometimes convenient to regard a single node as a 0-path or a 1-cycle. A spanning path or cycle of T_n is one that involves every node of T_n .

Camion [1] proved that every strong tournament contains a spanning cycle. Our main object is to prove the following result.

THEOREM 1. Any k-strong tournament contains at least k spanning cycles.

More generally, we shall prove the following result.

THEOREM 2. Let p denote any node of any k-strong tournament T_n ; if $3 \leq l \leq n$, then p is contained in at least k l-cycles.

In what follows we assume that the node p and the k-strong tournament T_n are fixed. The case k = 1 is treated, in effect, in [2; p. 6] so we may suppose that $k \ge 2$; since each node of T_n must beat and lose to at least k other nodes, it follows that $2k + 1 \le n$ or $k \le 1/2(n-1)$. Before proving the theorem we make some observations about paths and the structure of the k-strong tournament T_n .

2. Three lemmas. The following result is obvious.

LEMMA 1. Let \mathscr{P} denote an *l*-path from node *u* to node *v*. If node *w* is not contained in \mathscr{P} and $u \to w$ and $w \to v$, then *w* can be inserted in the path to form an (l + 1)-path from *u* to *v*; in particular *w* can be inserted immediately before the first node of \mathscr{P} it beats.

LEMMA 2. If u and v are any nodes of the top and bottom components of a weak tournament W_i and $1 \leq l \leq t - 1$, then there exists an *l*-path in W_i that starts with u and ends with v; furthermore, if $2 \leq l \leq t - 1$ this path may be assumed to contain any given node belonging to any intermediate component of W_i .

This may be proved by applying the following observations to the components of W_i : If a tournament Z_k is strong and $0 \le l \le k-1$, then it contains a spanning cycle and, hence, each node is the first node, and the last node, of at least one *l*-path in Z_k ; and, if $R \to S$, then any *c*-path of R may be followed by any *d*-path of S to form a (c + d + 1)-path of R + S.

LEMMA 3. Let G denote any minimal subtournament of the kstrong tournament T_n whose removal leaves a weak subtournament W of the form W = Q + R + S where Q and S are strong and R may be empty; then each node of G loses to at least one node of S and beats at least one node of Q, and there are at least k arcs from nodes of G to nodes of Q and at least k arcs from nodes of S to nodes of G.

The conclusion in this lemma follows from the fact that G is minimal and T_n is k-strong. The existence of such a subtournament G will be shown before each application of this lemma.

We now proceed to the proof of Theorem 2; we have to use different arguments when l lies in different intervals.

3. Proof when l = 3. Let B and L denote the set of nodes that beat and lose to p, respectively. Since T_n is k-strong B and L are nonempty and there are at least k arcs \vec{uv} that go from a node u of L to a node v of B. The theorem now follows when l = 3 since each such \overrightarrow{uv} determines a different 3-cycle $\{p, u, v, p\}$ containing p.

4. Proof when l = 4. If w is any node that beats p, let B, L, M, and N denote the set of nodes that beat both w and p, lose to both w and p, beat w and lose to p, and lose to w and beat p, respectively. If $L = \phi$, then M must contain at least k nodes and N must contain at least k - 1 nodes, since p and w must each beat at least k nodes. In this case there are at least $k(k-1) \ge k$ different 4-cycles of the type $\{p, u, w, v, p\}$ containing p, where $u \in M$ and $v \in N$. We may suppose, therefore, that $L \neq \phi$.

There are at least k arcs of the type \overrightarrow{uv} where $u \in L$ and $v \in B \cup M \cup N$. If $v \in B \cup M$, then the 4-cycle $\{p, u, v, w, p\}$ contains p. If $v \in N$ and v beats some other node y of N, then the 4-cycle $\{p, u, v, y, p\}$ contains p; if there is no such node y but u loses to some other node z of L, then the 4-cycle $\{p, z, u, v, p\}$ contains p. Thus, there are at least k different 4-cycles containing p except, possibly, when there exists an arc \overrightarrow{uv} from L to N such that u beats the remaining nodes of L and v loses to the remaining nodes of N; there is at most one such arc \overrightarrow{uv} so in this case the preceding construction provides at least k - 1 4-cycles containing p.

If $z \in M$, then $\{p, z, w, v, p\}$ is a new 4-cycle containing p. Thus we may suppose that $M = \phi$; this implies L has at least k nodes since p beats at least k nodes. If there exists an arc \overline{zy} where $z \neq$ $u, z \in L$, and $y \in B$ then $\{p, u, z, y, p\}$ is a new 4-cycle containing p. Thus we may suppose that u is the only node of L that beats any nodes of B. This implies, since T_n is k-strong, that there must be at least k arcs of the type \overline{zy} where $z \neq u, z \in L$, and $y \in N$. In this case, however, there are at least k 4-cycles of the type $\{p, u, z, y, p\}$ containing p. This completes the proof of the theorem when l = 4.

5. Proof when $5 \leq l \leq n - k + l$. Let \mathscr{C} denote any (l-2)-cycle containing p; such a cycle exists, either by virtue of an induction hypothesis or as a consequence of the result cited in the introduction. Let B and L denote the set of nodes that beat and lose to every node of \mathscr{C} , respectively, and let M denote the set of the remaining nodes of T_n that aren't in \mathscr{C} .

If $L \neq \phi$, there exist at least k arcs of the type \vec{uv} where $u \in L$ and $v \in B \cup M$. For each such node v there exists at least one node q of \mathscr{C} such that $v \to q$. If we insert the nodes u and v immediately before q in \mathscr{C} we obtain an *l*-cycle containing p; different arcs \vec{uv} clearly yield different *l*-cycles. A similar argument may be applied to B if $B \neq \phi$ so we may now assume that $L = B = \phi$ and $M \neq \phi$.

If $u \in M$, then there exists a pair of consecutive nodes r and s of \mathscr{C} such that $r \to u$ and $u \to s$. Thus u can be inserted between r and s in \mathscr{C} to form an (l-1)-cycle \mathscr{C}_1 containing p. Any other node v of M can now be inserted between some pair of consecutive nodes of \mathscr{C}_1 to form an l-cycle \mathscr{C}_2 containing p. Different cycles \mathscr{C}_2 are formed when different pairs of nodes of M are inserted in \mathscr{C} . Thus, there are at least $\binom{n-(l-2)}{2} \ge \binom{k+1}{2} \ge k$ different l-cycles containing p when $5 \le l \le n-k+1$. (This argument can be applied for somewhat larger values of l as well.)

6. Proof when $n - k + 2 \leq l \leq n - 1$. Let T_l denote any subtournament of T_n with l nodes that contains the node p. If T_l is strong, then it contains an *l*-cycle containing p, by Camion's theorem. Thus, if each such subtournament T_l is strong, then p is contained in at least $\binom{n-1}{l-1} \geq n-1 > k$ *l*-cycles in T_n .

We may suppose, therefore, that there exists a minimal subtournament G of T_n , with $g \leq n - l$ nodes, whose removal leaves a weak subtournament W containing node p. Then W can be expressed in the form W = Q + R + S where Q and S are strong and R may be empty.

There are at least $k \operatorname{arcs} \overrightarrow{xq}$ in T_n that go from a node x of G to a node q of Q, and for each such node x there exists at least one node s of S such that $s \to x$; this follows from Lemma 3. We shall show that for each such pair of nodes q and s, there exists an (l - 2)-path \mathscr{P} in W that starts with q, contains the node p, and ends with s.

If $p \in R$, then the existence of \mathscr{S} follows immediately from Lemma 2 since W has n-g nodes and $2 \leq l-2 \leq n-g-1$. If $p \in Q$, let \mathscr{P}_1 denote any spanning path of Q that starts with q. We observe that if Q has m nodes then $m \leq l-3$ since otherwise node s would lose to at least $l-2 \geq (n-k+2)-2 = n-k$ nodes and this is impossible since T_n is k-strong. Let \mathscr{P}_2 denote any (l-m-2)-path of R+S that ends with s; the existence of \mathscr{P}_2 follows from Lemma 2 since R+S has n-g-m nodes and $1 \leq l-m-2 \leq n-g-m-1$. If $\mathscr{P} = \mathscr{P}_1 + \mathscr{P}_2$ then \mathscr{P} is an (l-2)-path in W with the required properties and we can also find such a path when $p \in S$ by a similar argument.

This suffices to complete the proof when $n - k + 2 \leq l \leq n - 1$ since $\{x\} + \mathscr{P} + \{x\}$ is an *l*-cycle containing p and it is clear that different arcs \overrightarrow{qx} yield different *l*-cycles.

7. Proof when l = n; a special case. Since T_n is k-strong, there exists a partition of the nodes of T_n into two subsets A and B such that precisely k arcs go from nodes of A to nodes of B. At least one of these subsets has more than k nodes; if the nodes in this subset that are incident with the k arcs that go from A to B are removed, then the subtournament determined by the remaining nodes is weak. It follows, therefore, that there exists a smallest subtournament G, with at most k nodes, whose removal leaves a weak subtournament W of the form W = Q + R + S where Q and S are strong and R may be empty. We may now apply Lemma 3 to T_n . There are at least k arcs that go from a node of G to a node of Q and we shall prove the case l = n of the theorem, in general, by constructing a different *n*-cycle of T_n for each such arc; the node p plays no special role in this case since it automatically belongs to every *n*-cycle. First, however, we dispose of a special case.

Suppose R is empty and $Q = \{q\}$ so that $W = \{q\} + S$. Then G must have precisely k nodes all of which beat q for otherwise there wouldn't be k arcs going from G to Q. Consequently, S has $n-1-k \ge k$ nodes. There must be at least k nodes S that don't lose to all nodes of G for otherwise these nodes would determine a subtournament smaller than G whose removal from T_n would leave a weak subtournament.

Let s denote any node of S that beats some node x of G. It follows from Lemma 2, that there exists a spanning path \mathscr{P}_1 of W that starts with q and ends with s and a path \mathscr{P}_2 in G that starts with x and contains all nodes of G except those belonging to components of G that are above the component X that contains x. Hence, the cycle $\mathscr{C} = \mathscr{P}_2 + \mathscr{P}_1 + \{x\}$ contains all nodes of T_n except those nodes, if any, belonging to components of G above X. These nodes, however, can all be inserted in \mathscr{C} by Lemma 1, since they all beat x and lose to at least one node of S. The node s in the resulting n-cycle is the last node of S that occurs before the node q. Thus, in this way we can construct a different n-cycle for each of the at least k nodes of S that beat some nodes of G. Similarly, the theorem holds when W = Q + S and S consists of a single node.

8. Proof when l = n; the general case. Let \overrightarrow{xq} denote any arc that goes from a node x of G to a node q of Q in the tournament T_n . Next, let \overrightarrow{sy} denote any arc that goes from a node s of S to a node y of the top component of G; if the component X of G containing x is the top component of G let y be the immediate successor of x in some fixed spanning cycle of X unless $X = \{x\}$ in which case let y = x. Finally, let $\mathscr{P}(q, s)$ denote some spanning path of W that starts with q and ends with s and let $\mathscr{P}(y, x)$ denote a path from

y to x in G that contains all the nodes in components of G that are not below x; it is not difficult to see that these paths exist and that we may suppose q loses to the last node of Q other than itself that occurs in $\mathscr{P}(q, s)$.

Insert as many as possible of the nodes in the components of G below X between q and s in the path $\mathscr{P}(q, s)$ to form an augmented path $\mathscr{P}'(q, s)$ and let $\mathscr{P}(f, g)$ denote any spanning path, starting and ending with some nodes f and g, of the subtournament F determined by those nodes that can't be so inserted; it may be that $\mathscr{P}(f, g)$ is empty or consists of a single node. If t is any node of f, then (i) $t \to q$, (ii) $s \to t$, and (iii) $t \to u$, where u is the immediate successor of q in $\mathscr{P}'(q, s)$. The node t beats at least one node of Q and loses to at least one node of S; hence, by Lemma 1, it could be inserted in $\mathscr{P}'(q, s)$ unless (i) and (ii) hold. Since t doesn't beat itself or node x, and since there are at most k-2 other nodes of G, it must be that t beats at least one other node of W besides q if it is to beat at least k nodes altogether; this implies (iii) in view of Lemma 1.

If at least one node of the component of G immediately below X is in $\mathscr{P}'(q, s)$ or if X is the bottom component of G let

$$\mathscr{C} = \mathscr{C}(x,q) = \{x\} + \mathscr{P}(f,g) + \mathscr{P}'(q,s) + \mathscr{P}(y,x).$$

This is an *n*-cycle in view of the preceding remarks; we shall call it a type I cycle. The nodes *s* and *q* can be identified as the last node of *S* and the first node of *Q* encountered in traversing the cycle from any node of *S* to any node of *Q*. The node *x* can be identified as the last node between *s* and *q* in \mathcal{C} that belongs to a component *X* of *G* with the property that no node of *X* or any component of *G* above *X* is between *q* and *s* in \mathcal{C} . Thus different arcs \vec{xq} determine different type I cycles, if they determine any at all.

Let us now suppose that X is not the bottom component of G and that no node of the component immediately below X belongs to $\mathscr{P}'(q, s)$. In this case we are unable to identify the node x used in defining the cycle $\mathscr{C}(x, q)$ so we must use a different construction.

Let $\mathscr{P}(u, v)$ denote the nonempty path such that $\mathscr{P}'(q, s) = \{q\} + \mathscr{P}(u, v) + \{s\}$. Node x does not lose to itself or to the node f (which definitely exists in the present case), so it must lose to at least two nodes of $\mathscr{P}'(q, s)$ if it is to lose to at least k nodes altogether; but $x \to q$, so x must lose to at least one node of $\mathscr{P}(u, v)$. If t is any other node of $\mathscr{P}(u, x)$ then t does not lose to itself, its immediate successor in $\mathscr{P}(y, x)$, or to f; hence, t must lose to at least three nodes of $\mathscr{P}'(q, s)$ if it is to lose to k nodes altogether. It follows that every node of $\mathscr{P}(y, x)$ loses to at least one node of $\mathscr{P}(u, v)$.

If every node of $\mathscr{P}(y, x)$ beats v then these nodes can all be

inserted in the path $\mathscr{P}(u, v)$ to form an augmented path $\mathscr{P}'(u, v)$ by Lemma 1; this can be done in such a way that the nodes of $\mathscr{P}(y, x)$ occur in the same order in $\mathscr{P}'(u, v)$ as they do in $\mathscr{P}(y, x)$. In this case let

$$\mathscr{C} = \mathscr{C}(x, q) = \mathscr{P}(f, g) + \{q\} + \mathscr{P}'(u, v) + \{s, f\}.$$

That this is an *n*-cycle follows from properties (i) and (ii) of the nodes F, among other things; we shall call this a type II cycle. The nodes s and q can be identified in the same way as before. The node x can be identified as the last node between q and s that comes from G and beats f, the immediate successor of s in \mathscr{C} (we use the assumption about the nodes in the component of G containing f here). Thus, different arcs \vec{xq} determine different type II cycles, if they determine any at all. We can distinguish between cycles of types I and II because the node following s belongs to the top component of G in a type I cycle but not in a type II cycle.

If not all nodes of $\mathscr{P}(y, x)$ beat v, let w denote the first node of this path that loses to v. The nodes, if any, of $\mathscr{P}(y, x)$ that precede W can be inserted, as before, in $\mathscr{P}(u, v)$ to form an augmented path $\mathscr{P}'(u, v)$. If $\mathscr{P}(w, x)$ denotes the subpath determined by the remaining nodes of $\mathscr{P}(y, x)$, let

$$\mathscr{C} = \mathscr{C}(x, q) = \{x, q, s, f\} + \mathscr{P}(f, g) + \mathscr{P}'(u, v) + \mathscr{P}(w, x) .$$

That this is an *n*-cycle follows from properties (i) and (iii) of the nodes of F; we shall call this a type III cycle. There are at most two nodes of Q that are immediately followed by a node of S in C. If there is only one such node then this node must be q, and if there are two then q is the node that loses to the other one. Thus we can identify the node q in C and x is the immediate predecessor of q. Hence, different arcs xq determine different type III cycles, if they determine any at all.

It remains to show that we can distinguish a type III cycle from a type I or II cycle. Some node of Q is followed immediately by a node of S in a type III cycle but not in a type I or II cycle when R, the subtournament determined by the intermediate components of W, is nonempty. Thus we may suppose that W = Q + S where the strong components Q and S have at least three nodes each, in view of the case treated in §7. In this case, however, the first node of Q that occurs after a node of S is the same for all nodes of S in a type I or II cycle but not in a type III cycle.

Thus, in the general case, we can construct a different *n*-cycle $\mathscr{C}(x, q)$ corresponding to each arc \overrightarrow{xq} from a node of G to a node of Q. As there are at least k such arcs, this completes the proof of the theorem.

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