DIAGONAL SIMILARITY OF IRREDUCIBLE MATRICES TO ROW STOCHASTIC MATRICES

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By using the Perron-Frobenius Theorem it is easily shown that if $A$ is an irreducible matrix then there is a diagonal matrix $D$ with positive main diagonal so that $DAD^{-1} = rS$ where $r$ is a positive scalar and $S$ a stochastic matrix. This paper gives a short proof of this result without direct appeal to the Perron-Frobenius Theorem.

Definitions and Notations. Let $n \geq 2$ be an integer. Let $N = \{1, 2, \cdots, n\}$. An $n \times n$ nonnegative matrix $A$ is said to be reducible if there is a permutation matrix $P$ so that

$$PAP^T = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

where $A_1$ and $A_2$ are square. If $A$ is not reducible we say that $A$ is irreducible. By agreement each $1 \times 1$ matrix is irreducible.

Denote by

$$u(A) = \min \left[ \max_{i,j} a_{ij} \right]$$

where the minimum is over all proper subsets of $N$.

$$r(A) = \max \sum_{i \in N} a_{ik}, \quad p(A) = \min \sum_{i \in N} a_{ik}$$

$$D = \{d = (d_1, d_2, \cdots, d_n) \mid \text{each } d_k > 0 \text{ and } \min d_k = 1\}.$$ 

$$f(d) = \max_{i,j \in N} \left| \sum_{k \in N} d_ia_{ik}d_k^{-1} - \sum_{k \in N} d_ja_{jk}d_k^{-1} \right|$$

where each $d_k > 0$ and $A$ is irreducible. Finally let $S(A)$ denote the positive number so that $S(A)u(A) - r(A) = f(e)$ where $e = (1, 1, \cdots, 1)$.

RESULTS.

LEMMA 1: $f(d) = f(\lambda d)$ for each $\lambda > 0$.

LEMMA 2. If $(d_1, d_2, \cdots, d_n) \in D$, and $\max_{k \in N} d_k > [S(A)]^{n-1}$, then $f(d) > f(e)$.

Proof. Reorder $(d_1, d_2, \cdots, d_n)$ to $(d_{i_1}, d_{i_2}, \cdots, d_{i_n})$ so that $d_{i_1} \geq d_{i_2} \geq \cdots \geq d_{i_n}$. Let $s$ denote the smallest integer so that $(d_{i_k}/d_{i_{k+1}}) > S(A)$. That there is such an $s$ follows since $(d_{i_k}/d_{i_{k+1}}) \leq S(A)$ for each $k \in \{1, 2, \cdots, n - 1\}$ would imply that
Let \( M = \{d_{1i}, d_{2i}, \ldots, d_{ni}\} \). Note that \( M \neq N \). Since \( A \) is irreducible there is an \( a_{pq} = \max_{i \in M, j \in M} a_{ij} > 0 \). Then since \( p \in M \) and \( q \in M \)

\[
\frac{d_p}{d_q} > S(A), \quad \text{for each} \quad k \in N,
\]

\[
\sum_{k \in N} d_p a_{pk} d_k^{-1} > S(A) \cdot u(A), \quad \text{and} \quad \sum_{k \in N} d_i a_{ik} d_k^{-1} \leq r(A).
\]

From this it follows that

\[
f(d) \geq \left| \sum_{k \in N} d_p a_{pk} d_k^{-1} - \sum_{k \in N} d_i a_{ik} d_k^{-1} \right| > S(A) \cdot u(A) - r(A) = f(\varepsilon).\]

**Lemma 3.** \( f \) achieves a minimum in \( D \).

**Proof.** The proof follows from Lemma 2, the fact that \( f \) is continuous on the compact set \( \{d \mid d \in D \text{ and } \max_k d_k \leq [S(A)]^{-1} \} \), and \( e \in D \).

**Theorem.** The minimum of \( f \) in \( D \) is 0, i.e., \( \min_{d_k > 0, k \in N} f(d) = 0 \).

**Proof.** We first prove the theorem for positive matrices. Suppose \( A > 0 \) and \( f \) achieves its minimum at \( d^0 = (d_{1i}^0, d_{2i}^0, \ldots, d_{ni}^0) \in D \). Further suppose \( f(d^0) > 0 \). Let \( D_0 = \text{diagonal} (d_{1i}^0, d_{2i}^0, \ldots, d_{ni}^0) \). Let \( D_0 A D_0^{-1} = B \). If \( P \) is a permutation matrix then \( (PD_0 P^T)PAP^T(PD_0^{-1} P^T) = PBP^T \). Hence we may assume that

\[
\sum_{k \in N} b_{ik} \geq \sum_{k \in N} b_{ik} \geq \cdots \geq \sum_{k \in N} b_{nk}.
\]

Let

\[
M_1 = \left\{ i \mid \sum_{k \in N} b_{ik} = \sum_{k \in N} b_{1k} \right\} \quad \text{and} \quad M_2 = \left\{ i \mid \sum_{k \in N} b_{ik} = \sum_{k \in N} b_{nk} \right\}.
\]

Let

\[
d_k = \begin{cases} (1 - \varepsilon) & k \in M_1 \\ (1 - \varepsilon)^{-1} & k \in M_2 \\ 1 & \text{otherwise}. \end{cases}
\]

Consider \( DBD^{-1} \) and let \( g(\varepsilon) = \sum_{k \in N} d_k b_{ik} d_k^{-1} - \sum_{k \in N} d_j b_{jk} d_k^{-1} \quad i \in M_1, \ j \in M_2. \)

Then

\[
g'(0) = - \sum_{k \in M_1} b_{ik} - 2 \sum_{k \in M_2} b_{ik} - 2 \sum_{k \in M_1} b_{jk} - \sum_{k \in M_2} b_{jk} < 0.
\]
Hence for sufficiently small \( \varepsilon \),

\[
f_\varepsilon[d_1d_1^\varepsilon, d_2d_2^\varepsilon, \ldots, d_nd_n^\varepsilon] < f(d)\,.
\]

However, this contradicts \( f \) having its minimum at \( d^0 \). Therefore, if \( A > 0 \), \( \min_{d_k > 0 \ k \in \mathbb{N}} f(d) = 0 \).

Now suppose \( A \) is irreducible. For each positive integer \( k \), let \( A_k = A + (1/k)J \) where \( J \) is the \( n \times n \) matrix of ones so that \( \lim_{m \to \infty} A_m = A \). For each \( A_m \) there is a diagonal matrix \( D_m = \text{diag.}(d_1^m, d_2^m, \ldots, d_n^m) \), \((d_1^m, d_2^m, \ldots, d_n^m) \in D, \) so that \( D_mA_mD_m^{-1} \) has equal row sums. Further

\[1 \leq d_k^m \leq [S(A_m)]^{n-1} \text{ for each } k \in \mathbb{N}.
\]

The \( S(A_m) \)'s are easily seen to be bounded, and hence the \( D_m \)'s are bounded having a limit point \( D \). Let \( \{D_m\} \) denote a subsequence of \( \{D_m\} \) so that \( \lim_{m \to \infty} D_m = D \). Then \( \lim_{m \to \infty} D_mA_mD_m^{-1} = DAD^{-1} \) which has all its row sums equal. Hence \( \min_{d_k > 0 \ k \in \mathbb{N}} f(d) = 0 \).

**COROLLARY.** If \( A \) is an irreducible matrix then there is a diagonal matrix \( D \) with positive main diagonal so that \( DAD^{-1} = rS \) where \( S \) is a row stochastic matrix and \( r \) a positive number.

We also include the following corollary to Lemma 2.

**COROLLARY.** If \( A \) is irreducible with Perron eigenvector \( x = (x_1, x_2, \ldots, x_n) \) then \( \max_{i,j} x_i/x_j \leq [S(A)]^{n-1} = (2r(A) - p(A)/u(A))^{n-1} \).

We include this bound as the bound involves the quantity \( u(A) \) which to our knowledge is new.

**Reference**


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