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**DIAGONAL SIMILARITY OF IRREDUCIBLE MATRICES TO
ROW STOCHASTIC MATRICES**

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By using the Perron-Frobenius Theorem it is easily shown that if A is an irreducible matrix then there is a diagonal matrix D with positive main diagonal so that $DAD^{-1} = rS$ where r is a positive scalar and S a stochastic matrix. This paper gives a short proof of this result without direct appeal to the Perron-Frobenius Theorem.

Definitions and Notations. Let $n \geq 2$ be an integer. Let $N = \{1, 2, \dots, n\}$. An $n \times n$ nonnegative matrix A is said to be reducible if there is a permutation matrix P so that

$PAP^T = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$ where A_1 and A_2 are square. If A is not reducible we say that A is irreducible. By agreement each 1×1 matrix is irreducible.

Denote by

$$u(A) = \min_M \left[\max_{\substack{i \in M \\ j \notin M}} a_{ij} \right]$$

where the minimum is over all proper subsets of N .

$$r(A) = \max_{i \in N} \sum_{k \in N} a_{ik}, \quad p(A) = \min_{i \in N} \sum_{k \in N} a_{ik}$$

$$D = \{d = (d_1, d_2, \dots, d_n) \mid \text{each } d_k > 0 \text{ and } \min_k d_k = 1\}.$$

$f(d) = \max_{i, j \in N} \left| \sum_{k \in N} d_i a_{ik} d_k^{-1} - \sum_{k \in N} d_j a_{jk} d_k^{-1} \right|$ where each $d_k > 0$ and A is irreducible. Finally let $S(A)$ denote the positive number so that $S(A) \cdot u(A) - r(A) = f(e)$ where $e = (1, 1, \dots, 1)$.

RESULTS.

LEMMA 1: $f(d) = f(\lambda \cdot d)$ for each $\lambda > 0$.

LEMMA 2. If $(d_1, d_2, \dots, d_n) \in D$, and $\max_{k \in N} d_k > [S(A)]^{n-1}$, then $f(d) > f(e)$.

Proof. Reorder (d_1, d_2, \dots, d_n) to $(d_{i_1}, d_{i_2}, \dots, d_{i_n})$ so that $d_{i_1} \geq d_{i_2} \geq \dots \geq d_{i_n}$. Let s denote the smallest integer so that $(d_{i_s}/d_{i_{s+1}}) > S(A)$. That there is such an s follows since $(d_{i_k}/d_{i_{k+1}}) \leq S(A)$ for each $k \in \{1, 2, \dots, n-1\}$ would imply that

$$d_{i_1} = \frac{d_{i_1}}{d_{i_n}} = \prod_{k=1}^{n-1} \left(\frac{d_{i_k}}{d_{i_{k+1}}} \right) \leq [S(A)]^{n-1}.$$

Let $M = \{d_{i_1}, d_{i_2}, \dots, d_{i_s}\}$. Note that $M \neq N$. Since A is irreducible there is an $a_{pq} = \max_{i \in M, j \notin M} a_{ij} > 0$. Then since $p \in M$ and $q \notin M$

$$\frac{d_p}{d_q} > S(A), \quad \frac{d_{i_n}}{d_k} \leq 1 \text{ for each } k \in N,$$

$$\sum_{k \in N} d_p a_{pk} d_k^{-1} > S(A) \cdot u(A), \quad \text{and} \quad \sum_{k \in N} d_{i_n} a_{i_n k} d_k^{-1} \leq r(A).$$

From this it follows that

$$f(d) \geq \left| \sum_{k \in N} d_p a_{pk} d_k^{-1} - \sum_{k \in N} d_{i_n} a_{i_n k} d_k^{-1} \right| > S(A) \cdot u(A) - r(A) = f(e).$$

LEMMA 3. f achieves a minimum in D .

Proof. The proof follows from Lemma 2, the fact that f is continuous on the compact set $\{d \mid d \in D \text{ and } \max_k d_k \leq [S(A)]^{n-1}\}$, and $e \in D$.

THEOREM. The minimum of f in D is 0, i.e., $\text{Min}_{d_k > 0, k \in N} f(d) = 0$.

Proof. We first prove the theorem for positive matrices. Suppose $A > 0$ and f achieves its minimum at $d^0 = (d_1^0, d_2^0, \dots, d_n^0) \in D$. Further suppose $f(d^0) > 0$. Let $D_0 = \text{diagonal}(d_1^0, d_2^0, \dots, d_n^0)$. Let $D_0 A D_0^{-1} = B$. If P is a permutation matrix then $(P D_0 P^T) P A P^T (P D_0^{-1} P^T) = P B P^T$. Hence we may assume that

$$\sum_{k \in N} b_{1k} \geq \sum_{k \in N} b_{2k} \geq \dots \geq \sum_{k \in N} b_{nk}.$$

Let

$$M_1 = \left\{ i \mid \sum_{k \in N} b_{ik} = \sum_{k \in N} b_{1k} \right\} \quad M_2 = \left\{ i \mid \sum_{k \in N} b_{ik} = \sum_{k \in N} b_{nk} \right\}.$$

Let

$$d_k = \begin{cases} 1 - \varepsilon & k \in M_1 \\ (1 - \varepsilon)^{-1} & k \in M_2 \\ 1 & \text{otherwise} \end{cases}.$$

Consider DBD^{-1} and let $g(\varepsilon)$

$$= \sum_{k \in N} d_i b_{ik} d_k^{-1} - \sum_{k \in N} d_j b_{jk} d_k^{-1} \quad i \in M_1, j \in M_2.$$

Then

$$g'(0) = - \sum_{\substack{k \in M_1 \\ k \notin M_2}} b_{ik} - 2 \sum_{k \in M_2} b_{ik} - 2 \sum_{k \in M_1} b_{jk} - \sum_{\substack{k \in M_1 \\ k \notin M_2}} b_{jk} < 0.$$

Hence for sufficiently small ε ,

$$f_A[d_1d_1^0, d_2d_2^0, \dots, d_nd_n^0] < f(d^0).$$

However, this contradicts f having its minimum at d^0 . Therefore, if $A > 0$, $\min_{d_k > 0, k \in N} f(d) = 0$.

Now suppose A is irreducible. For each positive integer k , let $A_k = A + (1/k)J$ where J is the $n \times n$ matrix of ones so that $\lim_{m \rightarrow \infty} A_m = A$. For each A_m there is a diagonal matrix $D_m = \text{diag.}(d_1^m, d_2^m, \dots, d_n^m)$, $(d_1^m, d_2^m, \dots, d_n^m) \in D$, so that $D_m A_m D_m^{-1}$ has equal row sums. Further

$$1 \leq d_k^m \leq [S(A_m)]^{n-1} \text{ for each } k \in N.$$

The $S(A_m)$'s are easily seen to be bounded, and hence the D_m 's are bounded having a limit point D . Let $\{D_{m'}\}$ denote a subsequence of $\{D_m\}$ so that $\lim_{m' \rightarrow \infty} D_{m'} = D$. Then $\lim_{m' \rightarrow \infty} D_{m'} A_{m'} D_{m'}^{-1} = DAD^{-1}$ which has all its row sums equal. Hence $\min_{d_k > 0, k \in N} f(d) = 0$.

COROLLARY. *If A is an irreducible matrix then there is a diagonal matrix D with positive main diagonal so that $DAD^{-1} = rS$ where S is a row stochastic matrix and r a positive number.*

We also include the following corollary to Lemma 2.

COROLLARY. *If A is irreducible with Perron eigenvector $x = (x_1, x_2, \dots, x_n)$ then $\max_i x_i/x_j \leq [S(A)]^{n-1} = ((2r(A) - p(A)/u(A))^{n-1}$.*

We include this bound as the bound involves the quantity $u(A)$ which to our knowledge is new.

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