DIAGONAL SIMILARITY OF IRREDUCIBLE MATRICES TO ROW STOCHASTIC MATRICES

Darald Joe Hartfiel and J. W. Spellmann
DIAGONAL SIMILARITY OF IRREDUCIBLE MATRICES
TO ROW STOCHASTIC MATRICES

D. J. HARTFIELD AND J. W. SPELLMANN

By using the Perron-Frobenius Theorem it is easily shown that if $A$ is an irreducible matrix then there is a diagonal matrix $D$ with positive main diagonal so that $DAD^{-1} = rS$ where $r$ is a positive scalar and $S$ a stochastic matrix. This paper gives a short proof of this result without direct appeal to the Perron-Frobenius Theorem.

Definitions and Notations. Let $n \geq 2$ be an integer. Let $N = \{1, 2, \cdots, n\}$. An $n \times n$ nonnegative matrix $A$ is said to be reducible if there is a permutation matrix $P$ so that $PAP^T = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$ where $A_1$ and $A_2$ are square. If $A$ is not reducible we say that $A$ is irreducible. By agreement each $1 \times 1$ matrix is irreducible.

Denote by $u(A) = \min_{M} \max_{i,j \in M} a_{ij}$ where the minimum is over all proper subsets of $N$.

$r(A) = \max_{i \in N} \sum_{k \in N} a_{ik}$, $p(A) = \min_{i \in N} \sum_{k \in N} a_{ik}$

$D = \{ (d_1, d_2, \cdots, d_n) | \text{each } d_k > 0 \text{ and } \min d_k = 1 \}$. 

$f(d) = \max_{i,j \in N} | \sum_{k \in N} d_ka_{ik}d_{kj}^{-1} - \sum_{k \in N} d_ka_{jk}d_{kj}^{-1} |$ where each $d_k > 0$ and $A$ is irreducible. Finally let $S(A)$ denote the positive number so that $S(A) \cdot u(A) - r(A) = f(e)$ where $e = (1, 1, \cdots, 1)$.

RESULTS.

LEMMA 1: $f(d) = f(\lambda \cdot d)$ for each $\lambda > 0$.

LEMMA 2. If $(d_1, d_2, \cdots, d_n) \in D$, and $\max_{k \in N} d_k > [S(A)]^{n-1}$, then $f(d) > f(e)$.

Proof. Reorder $(d_1, d_2, \cdots, d_n)$ to $(d_{i_1}, d_{i_2}, \cdots, d_{i_n})$ so that $d_{i_1} \geq d_{i_2} \geq \cdots \geq d_{i_n}$. Let $s$ denote the smallest integer so that $(d_{i_s}/d_{i_{s+1}}) > S(A)$. That there is such an $s$ follows since $(d_{i_k}/d_{i_{k+1}}) \leq S(A)$ for each $k \in \{1, 2, \cdots, n-1\}$ would imply that
\[ d_{i_k} = \frac{d_{i_k}}{d_{i_n}} = \prod_{k=1}^{n-1} \left( \frac{d_{i_k}}{d_{i_{k+1}}} \right) \leq (S(A))^{-1}. \]

Let \( M = \{d_{i_1}, d_{i_2}, \ldots, d_{i_n}\} \). Note that \( M \neq N \). Since \( A \) is irreducible there is an \( a_{pq} = \max_{i \in M, j \in M} a_{ij} > 0 \). Then since \( p \in M \) and \( q \in M \)
\[
\frac{d_p}{d_q} > S(A), \quad \frac{d_{i_n}}{d_k} \leq 1 \quad \text{for each} \quad k \in N, \quad \sum_{k \in N} d_p a_{ph} d_k^{-1} > S(A) \cdot u(A), \quad \text{and} \quad \sum_{k \in N} d_{i_n} a_{i_n k} d_k^{-1} \leq r(A).
\]

From this it follows that

\[
f(d) \geq \left| \sum_{k \in N} d_p a_{ph} d_k^{-1} - \sum_{k \in N} d_{i_n} a_{i_n k} d_k^{-1} \right| > S(A) \cdot u(A) - r(A) = f(e).\]

**Lemma 3.** \( f \) achieves a minimum in \( D \).

**Proof.** The proof follows from Lemma 2, the fact that \( f \) is continuous on the compact set \( \{d \mid d \in D \text{ and } \max_k d_k \leq [S(A)]^{-1}\} \), and \( e \in D \).

**Theorem.** The minimum of \( f \) in \( D \) is 0, i.e., \( \min_{d_{i_k} > 0, k \in N} f(d) = 0 \).

**Proof.** We first prove the theorem for positive matrices. Suppose \( A > 0 \) and \( f \) achieves its minimum at \( d^0 = (d_{i_1}^0, d_{i_2}^0, \ldots, d_{i_n}^0) \in D \). Further suppose \( f(d^0) > 0 \). Let \( D_0 = \text{diagonal} (d_{i_1}^0, d_{i_2}^0, \ldots, d_{i_n}^0) \). Let \( D_0 A D_0^{-1} = B \). If \( P \) is a permutation matrix then \( (PD_0 P^T)PAP^T(PD_0^{-1} P^T) = PB P^T \). Hence we may assume that

\[
\sum_{k \in N} b_{1k} \geq \sum_{k \in N} b_{2k} \geq \cdots \geq \sum_{k \in N} b_{nk}.
\]

Let

\[ M_1 = \{i \mid \sum_{k \in N} b_{ik} = \sum_{k \in N} b_{i_k} \} \quad \text{and} \quad M_2 = \{i \mid \sum_{k \in N} b_{ik} = \sum_{k \in N} b_{nk} \}. \]

Let

\[ d_k = \begin{cases} 1 - \varepsilon & \text{if } k \in M_1 \\ (1 - \varepsilon)^{-1} & \text{if } k \in M_2 \\ 1 & \text{otherwise}. \end{cases} \]

Consider \( DBD^{-1} \) and let \( g(\varepsilon) \)
\[ = \sum_{k \in N} d_{i_k} b_{i_k} d_k^{-1} - \sum_{k \in N} d_{i_j} b_{i_j} d_k^{-1} \quad \text{for } i \in M_1, j \in M_2. \]

Then

\[ g'(0) = -\sum_{k \in M_1} b_{i_k} - 2 \sum_{k \in M_2} b_{i_k} - 2 \sum_{k \in M_1} b_{i_k} - \sum_{k \in M_2} b_{i_k} < 0. \]
Hence for sufficiently small $\varepsilon$,
\[ f^{*}[d_1d_0^{n}, d_2d_0^{n}, \ldots, d_{n}d_0^{n}] < f(d^0) . \]

However, this contradicts $f$ having its minimum at $d^0$. Therefore, if $A > 0$, $\min_{d_k > 0 \in \mathbb{K}} f(d) = 0$.

Now suppose $A$ is irreducible. For each positive integer $k$, let $A_k = A + (1/k)J$ where $J$ is the $n \times n$ matrix of ones so that $\lim_{m \to \infty} A_m = A$. For each $A_m$ there is a diagonal matrix $D_m = \text{diag.} (d_1^m, d_2^m, \ldots, d_n^m)$, $(d_1^m, d_2^m, \ldots, d_n^m) \in D$, so that $D_m A_m D_m^{-1}$ has equal row sums. Further
\[ 1 \leq d_k^m \leq [S(A_m)]^{n-1} \text{ for each } k \in N. \]

The $S(A_m)$'s are easily seen to be bounded, and hence the $D_m$'s are bounded having a limit point $D$. Let $\{D_m\}$ denote a subsequence of $\{D_m\}$ so that $\lim_{m \to \infty} D_m = D$. Then $\lim_{m \to \infty} D_m A_m D_m^{-1} = DAD^{-1}$ which has all its row sums equal. Hence $\min_{d_k > 0 \in \mathbb{K}} f(d) = 0$.

**COROLLARY.** If $A$ is an irreducible matrix then there is a diagonal matrix $D$ with positive main diagonal so that $DAD^{-1} = rS$ where $S$ is a row stochastic matrix and $r$ a positive number.

We also include the following corollary to Lemma 2.

**COROLLARY.** If $A$ is irreducible with Perron eigenvector $x = (x_1, x_2, \ldots, x_n)$ then $\max_{i,j} x_i/x_j \leq [S(A)]^{n-1} = ((2r(A) - p(A)/u(A))^{n-1}.

We include this bound as the bound involves the quantity $u(A)$ which to our knowledge is new.

**REFERENCE**


Received December 4, 1970 and in revised form May 6, 1971.

**TEXAS A & M UNIVERSITY**
Alex Bacopoulos and Athanassios G. Kartsatos, On polynomials approximating the solutions of nonlinear differential equations .......... 1
Monte Boisen and Max Dean Larsen, Prüfer and valuation rings with zero divisors ........................................................................ 7
James J. Bowe, Neat homomorphisms ........................................ 13
David W. Boyd and Hershy Kisilevsky, The Diophantine equation $u(u + 1)(u + 2)(u + 3) = v(v + 1)(v + 2)$ .................................. 23
George Ulrich Brauer, Summability and Fourier analysis ............... 33
Robin B. S. Brooks, On removing coincidences of two maps when only one, rather than both, of them may be deformed by a homotopy ....... 45
Frank Castagna and Geert Caleb Ernst Prins, Every generalized Petersen graph has a Tait coloring ........................................... 53
Micheal Neal Dyer, Rational homology and Whitehead products ........ 59
John Fuelberth and Mark Lawrence Teply, The singular submodule of a finitely generated module splits off ........................................ 73
Robert Gold, Γ-extensions of imaginary quadratic fields .................. 83
Myron Goldberg and John W. Moon, Cycles in k-strong tournaments .... 89
Darald Joe Hartfiel and J. W. Spellmann, Diagonal similarity of irreducible matrices to row stochastic matrices .............................. 97
Wayland M. Hubbart, Some results on blocks over local fields .......... 101
Alan Loeb Kostinsky, Projective lattices and bounded homomorphisms 111
Kenneth O. Leland, Maximum modulus theorems for algebras of operator valued functions ...................................................... 121
Jerome Irving Malitz and William Nelson Reinhardt, Maximal models in the language with quantifier “there exist uncountably many” .... 139
John Douglas Moore, Isometric immersions of space forms in space forms ..................................................................................... 157
Ronald C. Mullin and Ralph Gordon Stanton, A map-theoretic approach to Davenport-Schinzel sequences ........................................ 167
Chull Park, On Fredholm transformations in Yeh-Wiener space .......... 173
Stanley Poreda, Complex Chebyshev alterations ............................. 197
Ray C. Shiflett, Extreme Markov operators and the orbits of Ryff 201
Robert L. Snider, Lattices of radicals ............................................. 207
Ralph Richard Summerhill, Unknotting cones in the topological category ......................................................................................... 221
Charles Irvin Vinsonhaler, A note on two generalizations of QF − 3 .... 229
William Patterson Wardlaw, Defining relations for certain integrally parameterized Chevalley groups ............................................. 235
William Jennings Wickless, Abelian groups which admit only nilpotent multiplications ............................................................ 251