Pacific Journal of Mathematics

DIAGONAL SIMILARITY OF IRREDUCIBLE MATRICES TO ROW STOCHASTIC MATRICES

DARALD JOE HARTFIEL AND J. W. SPELLMANN

Vol. 40, No. 1 September 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 1, 1972

DIAGONAL SIMILARITY OF IRREDUCIBLE MATRICES TO ROW STOCHASTIC MATRICES

D. J. HARTFIEL AND J. W. SPELLMANN

By using the Perron-Frobenius Theorem it is easily shown that if A is an irreducible matrix then there is a diagonal matrix D with positive main diagonal so that $DAD^{-1} = rS$ where r is a positive scalar and S a stochastic matrix. This paper gives a short proof of this result without direct appeal to the Perron-Frobenius Theorem.

Definitions and Notations. Let $n \geqq 2$ be an integer. Let $N = \{1, 2, \cdots, n\}$. An $n \times n$ nonnegative matrix A is said to be reducible if there is a permutation matrix P so that

$PAP^T = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$ where A_1 and A_2 are square. If A is not reducible we say that A is irreducible. By agreement each 1×1 matrix is irreducible.

Denote by

$$u(A) = \min_M \left[\max_{\substack{i \in M \\ j \notin M}} a_{ij} \right]$$

where the minimum is over all proper subsets of N.

$$r(A) = \max_{i \in N} \sum_{k \in N} a_{ik}\,, \qquad p(A) = \min_{i \in N} \sum_{k \in N} a_{ik}$$
$$D = \{d = (d_1, d_2, \cdots, d_n) \mid \text{each } d_k > 0 \text{ and } \min_k d_k = 1\}\,.$$

$f(d) = \max_{i\ j \in N} \mid \sum_{k \in N} d_i a_{ik} d_k^{-1} - \sum_{k \in N} d_j a_{jk} d_k^{-1} \mid$ where each $d_k > 0$ and A is irreducible. Finally let $S(A)$ denote the positive number so that $S(A) \cdot u(A) - r(A) = f(e)$ where $e = (1, 1, \cdots, 1)$.

RESULTS.

LEMMA 1: *$f(d) = f(\lambda \cdot d)$ for each $\lambda > 0$.*

LEMMA 2. *If $(d_1, d_2, \cdots, d_n) \in D$, and $\max_{k \in N} d_k > [S(A)]^{n-1}$, then $f(d) > f(e)$.*

Proof. Reorder $(d_1, d_2, \cdots, d_n)$ to $(d_{i_1}, d_{i_2}, \cdots, d_{i_n})$ so that $d_{i_1} \geqq d_{i_2} \geqq \cdots \geqq d_{i_n}$. Let s denote the smallest integer so that $(d_{i_s}/d_{i_{s+1}}) > S(A)$. That there is such an s follows since $(d_{i_k}/d_{i_{k+1}}) \leqq S(A)$ for each $k \in \{1, 2, \cdots, n-1\}$ would imply that

$$d_{i_1} = \frac{d_{i_1}}{d_{i_n}} = \prod_{k=1}^{n-1}\left(\frac{d_{i_k}}{d_{i_{k+1}}}\right) \leqq [S(A)]^{n-1} .$$

Let $M = \{d_{i_1}, d_{i_2}, \cdots, d_{i_s}\}$. Note that $M \neq N$. Since A is irreducible there is an $a_{pq} = \max_{i \in M, j \notin M} a_{ij} > 0$. Then since $p \in M$ and $q \notin M$

$$\frac{d_p}{d_q} > S(A),\ \frac{d_{i_n}}{d_k} \leqq 1 \text{ for each } k \in N ,$$

$$\sum_{k \in N} d_p a_{pk} d_k^{-1} > S(A) \cdot u(A) , \quad \text{and} \quad \sum_{k \in N} d_{i_n} a_{i_n k} d_k^{-1} \leqq r(A) .$$

From this it follows that

$$f(d) \geqq \left| \sum_{k \in N} d_p a_{pk} d_k^{-1} - \sum_{k \in N} d_{i_n} a_{i_n k} d_k^{-1} \right| > S(A) \cdot u(A) - r(A) = f(e) .$$

LEMMA 3. *f achieves a minimum in D.*

Proof. The proof follows from Lemma 2, the fact that f is continuous on the compact set $\{d \mid d \in D \text{ and } \max_k d_k \leqq [S(A)]^{n-1}\}$, and $e \in D$.

THEOREM. *The minimum of f in D is* 0, *i.e.*, $\mathrm{Min}_{d_k > 0\ k \in N} f(d) = 0$.

Proof. We first prove the theorem for positive matrices. Suppose $A > 0$ and f achieves its minimum at $d^0 = (d_1^0, d_2^0, \cdots, d_n^0) \in D$. Further suppose $f(d^0) > 0$. Let $D_0 = \text{diagonal } (d_1^0, d_2^0, \cdots, d_n^0)$. Let $D_0 A D_0^{-1} = B$. If P is a permutation matrix then $(PD_0P^T)PAP^T(PD_0^{-1}P^T) = PBP^T$. Hence we may assume that

$$\sum_{k \in N} b_{1k} \geqq \sum_{k \in N} b_{2k} \geqq \cdots \geqq \sum_{k \in N} b_{nk} .$$

Let

$$M_1 = \left\{ i \,\middle|\, \sum_{k \in N} b_{ik} = \sum_{k \in N} b_{1k} \right\} \qquad M_2 = \left\{ i \,\middle|\, \sum_{k \in N} b_{ik} = \sum_{k \in N} b_{nk} \right\} .$$

Let

$$d_k = \begin{cases} 1 - \varepsilon & k \in M_1 \\ (1 - \varepsilon)^{-1} & k \in M_2 \\ 1 & \text{otherwise} . \end{cases}$$

Consider DBD^{-1} and let $g(\varepsilon)$
$= \sum_{k \in N} d_i b_{ik} d_k^{-1} - \sum_{k \in N} d_j b_{jk} d_k^{-1} \qquad i \in M_1,\ j \in M_2.$

Then

$$g'(0) = -\sum_{\substack{k \notin M_1 \\ k \notin M_2}} b_{ik} - 2\sum_{k \in M_2} b_{ik} - 2\sum_{k \in M_1} b_{jk} - \sum_{\substack{k \notin M_1 \\ k \notin M_2}} b_{jk} < 0 .$$

Hence for sufficiently small ε,

$$f_A[d_1d_1^0, d_2d_2^0, \cdots, d_nd_n^0] < f(d^0) .$$

However, this contradicts f having its minimum at d^0. Therefore, if $A > 0$, $\min_{d_k>0\ k\in N} f(d) = 0$.

Now suppose A is irreducible. For each positive integer k, let $A_k = A + (1/k)J$ where J is the $n \times n$ matrix of ones so that $\lim_{m\to\infty} A_m = A$. For each A_m there is a diagonal matrix $D_m = \text{diag.}\ (d_1^m, d_2^m, \cdots, d_n^m)$, $(d_1^m, d_2^m, \cdots, d_n^m) \in D$, so that $D_m A_m D_m^{-1}$ has equal row sums. Further

$$1 \leqq d_k^m \leqq [S(A_m)]^{n-1} \text{ for each } k \in N .$$

The $S(A_m)$'s are easily seen to be bounded, and hence the D_m's are bounded having a limit point D. Let $\{D_{m'}\}$ denote a subsequence of $\{D_m\}$ so that $\lim_{m\to\infty} D_{m'} = D$. Then $\lim_{m\to\infty} D_{m'}A_{m'}D_{m'}^{-1} = DAD^{-1}$ which has all its row sums equal. Hence $\min_{d_k>0\ k\in N} f(d) = 0$.

COROLLARY. *If A is an irreducible matrix then there is a diagonal matrix D with positive main diagonal so that $DAD^{-1} = rS$ where S is a row stochastic matrix and r a positive number.*

We also include the following corollary to Lemma 2.

COROLLARY. *If A is irreducible with Perron eigenvector $x = (x_1, x_2, \cdots, x_n)$ then* $\max_{i\ j} x_i/x_j \leqq [S(A)]^{n-1} = ((2r(A) - p(A)/u(A))^{n-1}$.

We include this bound as the bound involves the quantity $u(A)$ which to our knowledge is new.

REFERENCE

1. F. R. Gantmacher, *The Theory of Matrices*. Chelsea Publishing Co. New York, N. Y. (1969).

Received December 4, 1970 and in revised form May 6, 1971.

TEXAS A & M UNIVERSITY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 40, No. 1 September, 1972