

# Pacific Journal of Mathematics

**A MAP-THEORETIC APPROACH TO DAVENPORT-SCHINZEL  
SEQUENCES**

RONALD C. MULLIN AND RALPH GORDON STANTON

## A MAP-THEORETIC APPROACH TO DAVENPORT-SCHINZEL SEQUENCES

R. C. MULLIN AND R. G. STANTON

An  $(n, d)$  Davenport-Schinzel Sequence (more briefly, a *DS* sequence) is a sequence of symbols selected from  $1, 2, \dots, n$ , with the properties that (1) no two adjacent symbols are identical, (2) no subsequence of the form  $abab\dots$  has length greater than  $d$ , (3) no symbol can be added to the end of the sequence, without violating (1) or (2). It is shown that the set of  $(n, 3)$  *DS* sequences is in one-to-one correspondence with the set of rooted planar maps on  $n$  vertices in which every edge of the map is incident with the root face. The number of such sequences and the number of such sequences of longest possible length  $2n - 1$  is explicitly determined.

As an illustration of *DS* sequences, take  $n = 4$ ,  $d = 3$ . Then it is simple to enumerate all *DS* sequences in normal form (the symbols occur in increasing order). The results follow, and we see that there are 11  $(4, 3)$  *DS* sequences.

1213141  
121341  
1213431  
123141  
1232141  
123241  
1232421  
123421  
123431  
1234321  
123431.

It is obvious from the definition that a normal  $(n, 3)$  *DS* sequence must begin and end with 1.

The concept of a *DS* sequence was introduced in [1], and various results were obtained. For subsequent developments, one may consult [2], [3], and [4].

Among all *DS* sequences for fixed  $n$  and  $d$ , some will have greatest lengths. We define the number  $N_d(n)$  to be this greatest length. For example, in the preceding example,  $N_3(4) = 7$ , and there are 5 sequences of this greatest length.

The results of [1], [2], [3], [4], basically concern the value of  $N_a(n)$ . In this paper, we consider  $(n, 3)$  sequences (it was shown in [1] that  $N_3(n) = 2n - 1$ ). We set up a correspondence with rooted planar maps, obtain an enumeration of  $(n, 3)$  *DS* sequences, and show that the number of *DS* sequences of greatest length is given by the familiar Catalan sequence 1, 1, 2, 5, 14,  $\dots$ .

2. *Face maps.* By a *rooted planar map* we mean a planar map which is either the vertex map or a map in which an edge is distinguished as the root edge, and positive and negative ends and left and right sides are specified for this root edge. The vertex at the negative end is called the root vertex, and the face on the left is called the root face. For convenience, we shall henceforth assume that the map has no loops or multiple joins. A *face map* is rooted planar map in which every edge is incident with the root face. (Thus, apart from the rooting, the map can be considered to be a typical face of any planar map.)

Given any face map, we can obtain a sequence of integers, the *face sequence*, from it as follows. If the map is the vertex map, the sequence is 1. Otherwise the integer 1 is assigned to the root vertex. Then, in the root face, starting with the root vertex and proceeding along the root edge, the next vertex encountered is labelled 2. Every time an unlabelled vertex is encountered, it is assigned the least positive integer which has not been used as a label; this is done until one returns to the root vertex. The sequence is then constructed by listing the labels as they are encountered as one traces out the root face, beginning 1, 2,  $\dots$ , and ending with the symbol 1, when the root vertex is encountered for the last time before retracing the root edge. With the exception of vertex map, such a sequence will have length  $n + 2$ , where  $n$  is the number of edges in the map, isthmuses being counted twice.

We now point out a 1 - 1 correspondence between face sequences and normal Davenport-Schinzel sequences of type  $(n, 3)$ . Indeed, consider any face map. As it is traversed in the above fashion, any unordered pair of distinct vertices is linked by a unique sequence of edges, which we shall call the arc  $ab$ , which is traversed precisely once. The arc  $ab$  is, of course, identical with the arc  $ba$ . Now if the face sequence of a face map were to contain a subsequence  $abab$ , for  $a \neq b$ , then the arc  $ab$  would be traced twice, and this would be a contradiction. Also, since there are no loops, no 2 consecutive symbols are identical. Hence properties (1) and (2) of *DS* properties are satisfied. Also, every vertex occurs in the sequence; thus the addition of any symbol  $a$  other than 1 introduces a subsequence  $1a$

1  $a$ , and this violates (2); on the other hand, the addition of 1 violates (1). Hence  $DS$  property (3) is also satisfied, and we sum up our results in

**THEOREM 1.** *Every face sequence is a  $DS$  sequence.*

We now demonstrate the converse result, namely.

**THEOREM 2.** *Every normal  $DS$  sequence is the face sequence of a face map.*

*Proof.* We use induction on the number of symbols in a normal  $DS$  sequence. The result is trivially true for the set  $A_1$  of normal  $DS$  sequences on the symbol 1. Indeed,  $A_1$  has only one member, the sequence 1, which corresponds to the vertex map. Similarly for  $A_2$ , the set of normal  $DS$  sequences on 12, which contains only 121, which corresponds to the rooted edge.

Nor let us assume that the result is valid for  $A_i$ , the set of normal  $DS$  sequences on  $1, 2, \dots$ , for  $i \leq n - 1$ . We establish the result for  $A_n$ . Clearly the sequence  $1, 2, 3, \dots, n, 1$ , is a member of  $A_n$ , and corresponds to the rooted  $n$ -gon. Let  $X$  be any other member of  $A_n$ . It is evident that  $X$  contains either at least three 1's or some repeated member of  $2, 3, \dots, n$ . If  $X$  contains three or more 1's, then write  $X = PQ$ , where  $P = 1, 2, \dots, 1$ ,  $Q = w, \dots, 1$ , and  $P$  contains only two 1's. Let  $R = IQ$ , and let  $S$  be the normalized version of  $R$ . For example, if  $X$  is 12314151, then  $P$  is 1231,  $Q = 4151$ ,  $R = 14151$ , and  $S$  is 12131. Evidently  $R$  and  $S$  are  $DS$  sequences on fewer than  $n$  symbols, and as such correspond to face maps  $M$  and  $N$  respectively. Let  $x$  and  $y$  be the root vertices of  $M$  and  $N$  respectively, and  $(xa)$  and  $(yb)$  be the root edges of  $M$  and  $N$  respectively. Choose vertices  $c$  and  $d$  such that  $cxa$  and  $dyb$  are the angles of  $M$  and  $N$  respectively which lie to the left of the root edges. (Occasionally this may also be the right.) We then embed  $N$  homeomorphically in the root face of  $M$ , identifying  $x$  and  $y$ , and carry out the embedding so that  $N$  lies in the angle  $cxa$ , and so that  $c(xy)b$  is an angle of the resultant map,  $P$ , where  $(xy)$  is the vertex obtained by identifying  $x$  and  $y$ . The root edge, together with its positive and negative ends and left and right sides, is taken as the rooting for  $P$ . It is clear that  $P$  is a face map, and that its face sequence is  $X$ . If, on the other hand,  $X$  contains only two 1's, and some other symbol occurs twice, let  $a$  be the smallest symbol greater than 1 which is repeated. Write  $X$  as  $PQR$ , where  $Q$  is of the form  $ab \dots a$ ,  $b \neq a$ , and  $Q$  contains only two  $a$ 's, whereas  $P$  contains none. Let  $T = PSR$ , and let  $U$  and  $V$  be the normalized version of  $Q$  and

$T$  respectively. Let  $M$  and  $N$  be the face maps of  $T$  and  $Q$  respectively. Then by embedding  $N$  in the root face of  $M$  at the angle of  $M$  corresponding to the first occurrence of  $a$  in  $T$ , in a fashion analogous to that described previously, one obtains a map  $P$  whose face sequence is  $X$ . It may be verified that distinct  $DS$  sequences give rise to distinct maps in the above construction. Thus, if  $Y_i$  is the set of face sequences corresponding to maps with  $i$  vertices,  $|A_i| \leq |Y_i|$ ; however, since every face sequence is  $DS$ , then  $|Y_i| \leq |A_i|$ , and the two sets are in 1 - 1 correspondence. Indeed, we have shown that a sequence is a face sequence if and only if it is  $DS$ .

**COROLLARY.** *The greatest length for a  $DS$  sequence on  $n$  symbols is  $2n - 1$ ; such sequences correspond to rooted planar trees, in which every edge is an isthmus.*

3. The number of  $DS$  sequences. It is well known that the number of topologically distinct rooted plane trees on  $n$  vertices is given by the Catalan number  $(2n - 2)/(n - 1)! n!$ ; thus this is the number of  $DS$  sequences of greatest length on  $n$  symbols. We also determine the number of normal  $DS$  sequences on  $n$  symbols by enumerating face maps with  $n$  vertices.

Let  $f_n$  represent the number of such face maps, and define a generating function  $F(x) = \sum_{n=1}^{\infty} f_n x_n$ . Let  $F_k$  be the set of face maps whose root edge is contained in a  $k$ -gon. (We let  $k = 2$  correspond to the case in which the root edge is an isthmus.) Every member of  $F_k$  is completely determined by the ordered set of maps which occur in the angles of the  $k$ -gon as one proceeds around it in the direction induced by the root. Thus the generating function for  $F_k$  is  $(F(x))^k$ . Hence  $F(x)$  satisfies the equation

$$(1) \quad F(x) = x + \sum_{k=2}^{\infty} (F(x))^k,$$

where the term  $x$  corresponds to the vertex map.

We then see that

$$(2) \quad F(x) = x + (F(x))^2/(1 - F(x)).$$

Hence,

$$(3) \quad 2(F(x))^2 - (1 + x)F(x) + x = 0,$$

$$(4) \quad F(x) = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4},$$

where the radical denotes the series with constant term 1. The coefficients  $f_n$  may be determined by direct expansion of (4), or by

applying Lagrange's theorem to (2). In each case, the resulting expression is a finite summation with alternating signs, which is undesirable for calculation. Thus we use another approach, based on the fact that  $y = F(x) - 1 - x$  satisfies the differential equation

$$(5) \quad (1 - 6x + x^2)y' - y(x - 3) = 0 .$$

If we use the standard method for obtaining a series solution for (5), where  $a_n$  is the coefficient of  $x^n$ , we find that  $a_n$  satisfies the difference equation

$$(6) \quad (n + 1)a_{n+1} - (6n - 3)a_n + (n - 2)a_{n-1} = 0 .$$

By using initial conditions  $a_0 = -1/4, a_1 = 3/4$ , then, for  $n \geq 2, a_n = f_n$ , and thus  $f_n$  may be easily calculated for small values of  $n$ . We find that  $f_1 = 1, f_2 = 1, f_3 = 3, f_4 = 11, f_5 = 45$ , and  $f_6 = 197$ . Moreover, using the Laplace method for solving linear difference equations in terms of integrals, one can show that for  $n \geq 2$ ,

$$(7) \quad f_n = \frac{1}{4\pi} \int_{-2\sqrt{2}}^{2\sqrt{2}} (t + 3)^{n-2} \sqrt{8 - t^2} dt .$$

This gives an explicit solution for  $f_n, n \geq 2$ . However, using the fact that  $\sqrt{8 - t^2}$  is an even function, we find that

$$(8) \quad f_n = \frac{1}{2\pi} \sum_{k=0}^{n-2} \binom{n-2}{2k} 3^{n-2k-2} \int_0^{2\sqrt{2}} t^{2k} \sqrt{8 - t^2} dt .$$

But

$$(9) \quad \int_0^{2\sqrt{2}} t^{2k} \sqrt{8 - t^2} dt = 8^{k+1} \int_0^{\pi/2} (\sin^{2k} \theta - \sin^{2k+2} \theta) d\theta ,$$

by which, in virtue of the fact that

$$\int_0^{\pi/2} \sin^n x dx = \sqrt{\pi} \Gamma[(n + 1)/2] / 2\Gamma(n/2 + 1) ,$$

where  $\Gamma(x)$  is the Gamma function, we obtain a formula for  $f_n, n \geq 2$ , as a sum of positive terms, namely,

$$(10) \quad f_n = \sum_{k=0}^{\infty} 3^{n-3-2k} 2^k \binom{n-2}{2k} \frac{2k!}{k!(k+1)!} ,$$

which is convenient for nonrecursive calculation for small values of  $n$ .

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