COMPLEX CHEBYSHEV ALTERATIONS

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P. Chebyshev's famous Alternation Theorem for best uniform approximation to continuous real valued functions on an interval is generalized to include best approximation to a class of continuous complex valued functions on an ellipse.

1. Preliminary remarks and definitions. For a continuous complex valued function \( f \) defined on a compact set \( E \) in the plane and, for \( n \in \mathbb{Z}^+ \), let \( p_n(f, E) \) denote the polynomial of degree \( n \), of best uniform approximation to \( f \) on \( E \) and let:

\[
\rho_n(f, E) = \max_{z \in E} |f(z) - p_n(f, E)(z)|.
\]

Chebyshev's Alternation Theorem [1, p. 29] states that if \( f \) is a continuous real valued function on an interval \([a, b]\), then \( p_n = p_n(f, [a, b]) \) if and only if, there exists \( n + 2 \) points,

\[
\{x_i\}_{i=1}^{n+2}, a \leq x_1 < x_2 < \cdots < x_{n+2} \leq b, \text{ with the property that } |f(x) - p_n(x)| \text{ attains its maximum on } [a, b] \text{ at these points and } f(x_i) - p_n(x_i) = -[f(x_{i+1}) - p_n(x_{i+1})] \text{ for } i = 1, 2, \ldots, n+1.
\]

The sets we consider here are ellipses which are of course a generalization of intervals. So, for \( a \geq 0 \), let \( E_a = \{z + a/z: |z| = 1\} \). Now let \( \mathcal{F}_n(E_a) \) denote those complex valued functions \( f \), not themselves polynomials of degree \( n \), continuous on \( E_a \), having the property that there exists \( n + 2 \) points \( \{z_k\}_{k=1}^{n+2} \) in \( E_a \), such that \( p_n(f, E_a) = p_n(f, \{z_k\}_{k=1}^{n+2}) \). It is known [1, p. 22] that there always exists a set \( D \subset E_a \), consisting of \( n + k \) points, \( 2 \leq k \leq n + 3 \), such that \( p_n(f, E_a) = p_n(f, D) \). Furthermore, to this author's knowledge, every example of best uniform approximation to rational functions on infinite sets in the plane (e.g., [3], [4] and [5]) is one in which such a set consisting of \( n + 2 \) points exists or, can be shown equivalent to such an example.

2. Main theorem. Given \( n + 2 \) points \( \{z_k\}_{k=1}^{n+2} \) in \( E_a \) let \( z_k \) be such that \( z_k = z_k + a/z_k, \quad |z_k| = 1 \) and if \( a = 1 \), \( 0 \leq \text{Arg } z_k \leq \pi \) for \( k = 1, 2, \ldots, n + 2 \). The \( z_k \)'s are uniquely determined. Now let

\[
\Phi_k = z_k^{-n/2} \prod_{\substack{j=1 \atop j \neq k}}^{n+2} \frac{|(z_k z_j - a)|}{|z_k z_j - a|} \text{ for } k = 1, 2, \ldots, n+2 \text{ where } 0 \leq \text{arg } z^{i/2} < \pi.
\]
THEOREM 1. If \( f \) is continuous on \( E_a \) and \( p_n \) is a polynomial of degree \( n \), \( n \in \mathbb{Z}^+ \), then \( f \in \mathcal{F}_a(E_a) \) and \( p_n = p_n(f, E_a) \) if and only if there exists \( n + 2 \) points \( \{\xi_k\}_{k=1}^{n+1} \) in \( E_a \), with \( 0 \leq \text{Arg } \xi_i < \text{Arg } \xi_{i+1} < \cdots < \text{Arg } \xi_{n+2} < 2\pi \) if \( a \neq 1 \) or \(-2 < \xi_1 < \cdots < \xi_{n+2} < 2 \) if \( a = 1 \), where \( |f(\xi) - p_n(\xi)| \) attains its maximum on \( E_a \) and, \( (f(\xi_i) - p_n(\xi_i))/\Phi_i = -[(f(\xi_{i+1}) - p_n(\xi_{i+1}))/\Phi_{i+1}] \) for \( i = 1, 2, \cdots, n + 1 \) where the \( \Phi_i \)'s are defined in terms of the \( \xi_i \)'s as above.

Proof. In order to prove our theorem we make use of a lemma which is a reformulation of a result [2] due to T. S. Motzkin and J. L. Walsh.

LEMMA. A necessary and sufficient condition that the given numbers \( \{\sigma_k\}_{k=1}^{n+2} \) be the deviations of some function \( f \) defined on the \( n + 2 \) points \( \{\xi_k\}_{k=1}^{n+2} \) and its polynomial of degree \( n \) of best uniform approximation to \( f \) on these points is that for some \( \rho \geq 0 \);

1) \(|\sigma_k| = \rho \) for \( k = 1, 2, \cdots, n + 2 \) and,

2) \( \text{arg } \sigma_k = \text{arg } \omega'(\xi_k) + \theta_0 \) for \( k = 1, 2, \cdots, n + 2 \) if \( \rho > 0 \) where

\[
\omega(\xi) = \prod_{k=1}^{n+2} (\xi - \xi_k) \quad \text{and} \quad \theta_0 = \text{arg } \left[ \sum_{k=1}^{n+2} f(\xi_k)/\omega'(\xi_k) \right].
\]

The necessary portion of our theorem will then follow if it is shown that;

\[
(2.1) \quad \text{arg } \left[ (\omega'(\xi_i)/\Phi_i)/(\omega'(\xi_{i+1})/\Phi_{i+1}) \right] = \pi \quad \text{for} \quad i = 1, 2, \cdots, n + 1.
\]

Now substituting \( z_j + a/z_j \) for \( \xi_j \) and using the definition of the \( \Phi_j \)'s we can show the (2.1) is equivalent to;

\[
(2.2) \quad \text{arg } \left[ (z_{i+1}^{n/2}/z_i^{n/2}) \prod_{j=i+1}^{n+2} [(z_i - z_j)/(z_{i+1} - z_j)] \right] = 0.
\]

But, (2.2) follows since \( z_i \) and \( z_{i+1} \) are by virtue of their definition adjacent on the unit circle \( U \) (i.e., \( z_i \) and \( z_{i+1} \) are on a connected arc in \( U \) containing none of the other \( z_j \)'s) and since; \( \text{arg } (z_{i+1}/z_i) = -2 \text{ arg } (z_i - z_j)/(z_{i+1} - z_j) \) for \( j \neq i, i + 1 \).

In order to prove the converse of our theorem we simply work backwards and show that; \( \text{arg } [f(\xi_k) - P_n(\xi_k)] = \omega'(\xi_k) + \theta_0 \) for some \( \theta_0 \) and \( k = 1, 2, \cdots, n + 2 \) and apply the aforementioned result of Motzkin and Walsh.

3. Special cases and applications. Chebyshev's Alternation Theorem follows as a special case of Theorem 1, when \( a = 1 \), since it is known [1, p. 22] that all real functions, not themselves polynomials of degree \( n \), continuous on \([-2, 2]\) are in the class \( \mathcal{F}_n([-2, 2]) \).
Also of interest because of its simple form is the case where \( a = 0 \) or \( E_a = U \) is the unit circle and where \( n \) is even. In this case our main theorem appears to provide us with a valuable tool in determining if a given function \( f \) is in \( \mathcal{T}_m(U) \) and if it is, in finding \( p_{2m}(f, U) \).

**Corollary 1.** If \( f \) is continuous on \( U \) and \( p_{2m} \) is a polynomial of degree \( 2m, m \in \mathbb{Z}^+ \), then \( f \in \mathcal{T}_m(U) \) and \( p_{2m} = p_{2m}(f, U) \) if and only if there exists \( 2m + 2 \) points, \( \{z_k\}_{k=1}^{2m+2} \), with \( 0 \leq \text{Arg } z_1 < \cdots < \text{Arg } z_{2m+2} < 2\pi \), where \( |f(z) - p_{2m}(z)| \) attains its maximum on \( U \) and where \( \left| f(z_k) - p_{2m}(z_k) \right|/z_k^m = -\left| f(z_{k+1}) - p_{2m}(z_{k+1}) \right|/z_{k+1}^m, \) for \( k = 1, 2, \cdots, 2m + 1 \).

Corollary 1 can be used to obtain a recently discovered example of best approximation [3], namely, if \( f(z) = (\alpha z + \beta)/(z - a)(1 - \overline{a}z) \), \( |\alpha| > 1 \), then;

\[
p_{2m}(f, U)(z) = [\alpha z + \beta - K_1 z^m(1 - \overline{\alpha}z)^2 - K_2(z - a)^2]/(z - a)(1 - \overline{a}z),
\]

where

\[
K_1 = (\alpha a + \beta)/a^m(1 - |a|^2)^2
\]

and,

\[
K_2 = \overline{a}(\alpha + \overline{\beta}a)/(1 - |a|^2)^2.
\]

**References**


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