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**A NOTE ON TWO GENERALIZATIONS OF QF – 3**

CHARLES IRVIN VINSONHALER

## A NOTE ON TWO GENERALIZATIONS OF $QF$ -3

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If  $M$  is an  $R$ -module, then the dual of  $M$  is defined to be  $\text{Hom}_R(M, R)$ . Artinian  $QF$ -3 rings  $R$  have been characterized by the following two properties:

(1) The class of  $R$ -modules with zero duals is closed under taking submodules.

(2) The class of torsionless  $R$ -modules is closed under extension.

These properties are independent and, in the present paper, we study the two classes of rings  $R$  which satisfy each of these conditions separately.

Let  $R$  be a ring with identity.  $R$  is said to be (left)  $QF$ -3 provided there is an idempotent  $e$  in  $R$  such that  $Re$  is faithful and injective as a (left)  $R$ -module. The notion of  $QF$ -3 rings is derived from the definition of  $QF$ -3 algebras introduced by Thrall in [4].

If  $M$  is a left  $R$ -module, let  $M^* = \text{Hom}(M, R)$  denote the "dual" of  $M$ , with the usual right module structure. For left Artinian rings  $R$ , Wu, Mochizuki and Jans [5] have given the following two properties characterizing those which are  $QF$ -3.

(1) If  $M_1 \subseteq M_2$  are  $R$ -modules, then  $M_2^* = (0)$  implies  $M_1^* = (0)$ .

(2) The class of torsionless  $R$ -modules is closed under extension.

That is, if  $A$  and  $C$  are torsionless  $R$ -modules, and  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is an exact sequence of  $R$ -modules, then  $B$  is torsionless.

In this note, rings satisfying (1) or (2) separately are studied. Those satisfying (1) are called SZD and those satisfying (2), TCE. For (left)  $R$ -modules  $M$ , the following notation is used,

$Z(M) = \{m \in M \mid Em = 0 \text{ for some essential left ideal } E \subseteq R\}$  (the singular submodule of  $M$ )

$S(M)$  = the sum of all simple submodules of  $M$  (the socle of  $M$ )

$E(M)$  = injective hull of  $M$

### SZD and TCE Rings

PROPOSITION 1. A ring  $R$  is SZD if and only if the following are equivalent for every  $R$ -module  $M$ .

(1)  $\text{Hom}(M, R) = (0)$                       (2)  $\text{Hom}(M, E(R)) = (0)$

*Proof.* Assume  $R$  is SZD. Condition (2) implies (1) trivially. To show (1) implies (2), assume  $M^* = (0)$  and let  $f \neq 0$  in  $\text{Hom}(M, E(R))$ . Set  $L = f(M) \cap R$  and  $M_0 = f^{-1}(L)$ . Then  $M_0 \neq (0)$  and  $f|_{M_0}: M_0 \rightarrow R$  is nonzero, so that  $M_0^* \neq (0)$ . Since  $R$  is SZD, this implies  $M^* \neq (0)$ ,

a contradiction.

Conversely, if (1)  $\Leftrightarrow$  (2), let  $M^* = 0$ . If  $M_0$  is a submodule of  $M$ , we have  $\text{Hom}(M, R) = (0) \Rightarrow \text{Hom}(M, E(R)) = (0) \Rightarrow \text{Hom}(M_0, E(R)) = (0) \Rightarrow \text{Hom}(M_0, R) = (0)$ .

**PROPOSITION 2.** *If  $R$  is SZD and  $Z(R) = (0)$ , then  $E(R)$  is torsionless.*

*Proof.* Let  $K = \bigcap_{f \in \text{Hom}(E(R), R)} \text{Ker } f$ , and assume  $K \neq (0)$ . Then  $\text{Hom}(E(K), R) \neq (0)$  since  $R$  is SZD. Choose  $f \neq 0$  in  $\text{Hom}(E(K), R)$  and pick  $x \in E(K)$  such that  $f(x) \neq 0$ . Set  $A = \{r \in R \mid rf(x) = 0\}$ . Because  $Z(R) = (0)$ ,  $A$  is not essential in  $R$ , and there is a left ideal  $L \neq (0)$  such that  $L \cap A = (0)$ . Then  $Lx \cap K \neq (0)$ , so there is a  $r \in L$  such that  $rx \in K$  and  $f(rx) \neq 0$ . But  $E(R) = E(K) \oplus Y$  for some  $Y \subseteq E(R)$ , and  $f$  can be extended to  $\tilde{f}: E(R) \rightarrow R$ , contradicting the definition of  $K$ .

**COROLLARY.** *If  $R$  is SZD and  $Z(R) = (0)$ , then an  $R$ -module  $M$  is torsionless if and only if  $E(M)$  is torsionless.*

*Proof.* If  $E(M)$  is torsionless,  $M$  is a torsionless submodule. If  $M$  is torsionless,  $M$  can be embedded in a product,  $\pi R$ , of copies of  $R$ . Then  $E(M)$  can be embedded in a product,  $\pi E(R)$ , of copies of  $E(R)$ . Since  $E(R)$  is torsionless by Prop. 2, so is  $\pi E(R)$ , and hence  $E(M)$  is torsionless.

**COROLLARY.** *If  $R$  is SZD and  $Z(R) = 0$ , then  $R$  is TCE.*

*Proof.* Kato [2] has observed that the proof in [5] can be modified slightly to show that in any ring, SZD and TCE are equivalent to  $E(R)$  being torsionless. Hence, by Prop. 2,  $R$  must be TCE.

**THEOREM 3.** *If  $R$  is right perfect and  $Z(R) = (0)$ , then SZD implies QF-3.*

*Proof.* Tachikawa [3] has shown that in a right perfect ring,  $E(R)$  torsionless implies  $R$  is QF-3.

We continue with some results on TCE rings. For an  $R$ -module  $M$  we let  $j_M$  denote the natural map from  $M$  to its double dual  $M^{**}$ .

**THEOREM 4.** *If  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$  is a short exact sequence of  $R$ -modules with  $A$  and  $C$  torsionless, then  $B$  is torsionless if and only*

if  $\text{Im } j_A \cap \text{Ker } \alpha^{**} = (0)$ , where  $\alpha^{**}$  is the induced map from  $A^{**}$  to  $B^{**}$ .

*Proof.* Apply the exact sequence in Ext to  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ , to obtain an exact sequence  $0 \rightarrow C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^* \xrightarrow{\delta} X \rightarrow 0$ , where  $X \subseteq \text{Ext}_R^1(C, R)$  is the image of  $A^*$  under the connecting map  $\delta$  (see [1]). Take the dual of the latter sequence to obtain

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X^* & \longrightarrow & A^{**} & \xrightarrow{\alpha^{**}} & B^{**} \\
 & & & & \uparrow j_A & & \uparrow j_B \\
 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \longrightarrow 0
 \end{array}$$

a row exact, commuting diagram. Now  $B$  torsionless implies  $j_B \alpha$  is monic, so that  $\alpha^{**} j_A$  is monic, and  $\text{Ker } \alpha^{**} \cap \text{Im } j_A = (0)$ . Conversely, if  $\text{Ker } \alpha^{**} \cap \text{Im } j_A = (0)$ , then  $\text{Ker } j_B \cap \text{Im } \alpha = (0)$ . Thus if  $0 \neq b \in \text{Ker } j_B$ ,  $\beta(b) \neq 0$ . Since  $C$  is torsionless, there is a map  $f: C \rightarrow R$  such that  $f(\beta(b)) \neq 0$ . But then  $f\beta: B \rightarrow R$  satisfies  $(f\beta)(b) = 0$ , contradicting  $b \in \text{Ker } j_B$  (see [1]). Therefore  $\text{Ker } j_B = (0)$  and  $B$  is torsionless.

Theorem 4 says  $R$  is TCE if and only if  $\text{Im } j_A \cap \text{Ker } \alpha^{**} = (0)$  for every short exact sequence  $0 \rightarrow A \xrightarrow{\alpha} B \rightarrow C \rightarrow 0$  with  $A$  and  $C$  torsionless.

We now define a special type of torsionless  $R$ -module for use in further investigation of TCE rings.

**DEFINITION.** An  $R$ -module  $M \neq (0)$  is completely torsionless (c.t.) provided  $M$  is torsionless and has no nontrivial torsionless factors.

It is immediate that a c.t. module  $M$  must be isomorphic to a left ideal of  $R$ , for there must be a nonzero map  $f: M \rightarrow R$  since  $M$  is torsionless, and  $\text{Ker } f = (0)$  since  $M$  has no torsionless factors.

**LEMMA 5.** *If  $R$  is left Noetherian, every left ideal has a completely torsionless factor.*

*Proof.* Let  $L$  be a left ideal in  $R$ . If  $L$  is not c.t.,  $L$  has a torsionless factor  $L/L_1$ . If  $L/L_1$  is not c.t., there is left ideal  $L_2 \supseteq L_1$  such that  $L/L_2$  is torsionless. Continuing in this fashion we obtain an ascending chain  $\{L_i\}$  of left ideals which must terminate. That is,  $L/L_n$  is c.t. for some  $n$ .

**COROLLARY.** *If  $R$  is left Noetherian and  $M$  is an  $R$ -module with  $\text{Hom}(M, R) \neq (0)$ , then  $M$  has a completely torsionless factor.*

*Proof.* Pick  $f \neq 0$  in  $\text{Hom}(M, R)$ . Then  $f(M)$  has a c.t. factor, so  $M$  does also.

**THEOREM 6.** *If  $R$  is left Noetherian,  $R$  is TCE if and only if every short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  with  $C$  torsionless and  $A$  completely torsionless, must have  $B$  torsionless as well.*

*Proof.* The “only if” part follows from the definition of TCE. To show the “if” part, let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be exact with  $A$  and  $C$  torsionless. Set  $\bar{M} = \{M \mid M \text{ is a submodule of } A, B/M \text{ torsionless}\}$ . Then  $A \in \bar{M}$ , and if  $M_1 \supseteq M_2 \supseteq \dots$  is a descending chain in  $\bar{M}$ , define a map  $\phi: B/\bigcap_{i=1}^{\infty} M_i \rightarrow \prod_{i=1}^{\infty} B/M_i$  by  $\phi(b + \bigcap_{i=1}^{\infty} M_i) = \prod_{i=1}^{\infty} (b + M_i)$ . It is easy to check that  $\phi$  is an  $R$ -monomorphism. Thus,  $B/\bigcap_{i=1}^{\infty} M_i$  is isomorphic to a submodule of the torsionless module  $\prod_{i=1}^{\infty} B/M_i$ . This implies  $B/\bigcap_{i=1}^{\infty} M_i$  is torsionless, hence  $\bigcap_{i=1}^{\infty} M_i \in \bar{M}$ . Now apply Zorn’s Lemma to  $\bar{M}$  to obtain a minimal element  $M_0$ . If  $M_0$  is c.t., then  $0 \rightarrow M_0 \rightarrow B \rightarrow B/M_0 \rightarrow 0$  gives  $B$  torsionless by hypothesis. If  $M_0$  is not c.t., there is a completely torsionless factor  $M_0/N$  by Corollary to Lemma 5. But the exact sequence  $0 \rightarrow M_0/N \rightarrow B/N \rightarrow B/M_0 \rightarrow 0$  implies that  $B/N$  is torsionless, contradicting the minimality of  $M_0$  in  $\bar{M}$ . Thus  $M_0$  is in fact c.t., and  $B$  is torsionless.

We next consider short exact sequences where the factor module is c.t. The theorem in this case is only for finitely generated modules over left Artinian rings.

**THEOREM 7.** *Let  $R$  be left Artinian. The class of finitely generated torsionless modules is closed under extension if and only if every exact sequence of finitely generated modules,  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , with  $C$  completely torsionless and  $A$  torsionless, has  $B$  torsionless as well.*

*Proof.* Again the “only if” part is immediate. For the “if” part, let  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$  be exact with  $A$  and  $C$  finitely generated and torsionless. If  $C$  is not c.t., there is a c.t. factor  $C/D_1$ . Let  $A_1 = \beta^{-1}(D_1) \supseteq A$ . This gives  $0 \rightarrow A_1 \rightarrow B \rightarrow C/D_1 \rightarrow 0$  exact. If  $A_1$  is torsionless, then so is  $B$  by hypothesis. If not, consider  $0 \rightarrow A \rightarrow A_1 \rightarrow D_1 \rightarrow 0$ . Let  $D_1/D_2$  be a c.t. factor of  $D_1$ , and  $A_2 = \beta^{-1}(D_2) \supseteq A_1$ . This yields  $0 \rightarrow A_2 \rightarrow A_1 \rightarrow D_1/D_2 \rightarrow 0$  exact. If  $A_2$  is torsionless, then  $A_1$  is also, contradiction. The process may be continued inductively, obtaining at each stage  $D_{n-1}/D_n$  completely torsionless and  $A_n = \beta^{-1}(D_n)$ . The sequence  $C \cong B/A \supseteq A_1/A \supseteq A_2/A \supseteq \dots$  must terminate since  $C$  is finitely generated and  $R$  is left Artinian. By construction, the sequence stops at  $A_n/A$  if and only if  $A_n$  is torsionless. But  $A_n$  torsionless implies  $A_{n-1}$  torsionless, also by construction. We conclude that  $A_1$ , hence  $B$ , is torsionless.

Note that the above proof does not require that  $A$  be finitely generated, and the theorem can be generalized slightly. Consideration of short exact sequences with c.t. modules at both ends failed to yield any significant results.

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