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CHARLES IRVIN VINSONHALER

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# A NOTE ON TWO GENERALIZATIONS OF $QF$-3

CHARLES VINSONHALER

**If $M$ is an $R$-module, then the dual of $M$ is defined to be $\mathrm{Hom}_R(M, R)$. Artinian $QF$-3 rings $R$ have been characterized by the following two properties:**

**(1) The class of $R$-modules with zero duals is closed under taking submodules.**

**(2) The class of torsionless $R$-modules is closed under extension.**

**These properties are independent and, in the present paper, we study the two classes of rings $R$ which satisfy each of these conditions separately.**

Let $R$ be a ring with identity. $R$ is said to be (left) $QF$–3 provided there is an idempotent $e$ in $R$ such that $Re$ is faithful and injective as a (left) $R$-module. The notion of $QF$–3 rings is derived from the definition of $QF$–3 algebras introduced by Thrall in [4].

If $M$ is a left $R$-module, let $M^* = \mathrm{Hom}(M, R)$ denote the "dual" of $M$, with the usual right module structure. For left Artinian rings $R$, Wu, Mochizuki and Jans [5] have given the following two properties characterizing those which are $QF$–3.

(1) If $M_1 \subseteqq M_2$ are $R$-modules, then $M_2^* = (0)$ implies $M_1^* = (0)$.

(2) The class of torsionless $R$-modules is closed under extension.

That is, if $A$ and $C$ are torsionless $R$-modules, and $0 \to A \to B \to C \to 0$ is an exact sequence of $R$-modules, then $B$ is torsionless.

In this note, rings satisfying (1) or (2) separately are studied. Those satisfying (1) are called SZD and those satisfying (2), TCE. For (left) $R$-modules $M$, the following notation is used,

$Z(M) = \{m \in M \mid Em = 0$ for some essential left ideal $E \subseteqq R\}$ (the singular submodule of $M$)

$S(M) =$ the sum of all simple submodules of $M$ (the socle of $M$)

$E(M) =$ injective hull of $M$

## SZD and TCE Rings

PROPOSITION 1. *A ring $R$ is SZD if and only if the following are equivalent for every $R$-module $M$.*

(1) $\mathrm{Hom}(M, R) = (0)$ (2) $\mathrm{Hom}(M, E(R)) = (0)$

*Proof.* Assume $R$ is SZD. Condition (2) implies (1) trivially. To show (1) implies (2), assume $M^* = (0)$ and let $f \neq 0$ in $\mathrm{Hom}(M, E(R))$. Set $L = f(M) \cap R$ and $M_0 = f^{-1}(L)$. Then $M_0 \neq (0)$ and $f|_{M_0}: M_0 \to R$ is nonzero, so that $M_0^* \neq (0)$. Since $R$ is SZD, this implies $M^* \neq (0)$, a contradiction.

Conversely, if (1) $\Leftrightarrow$ (2), let $M^* = 0$. If $M_0$ is a submodule of $M$, we have $\mathrm{Hom}(M, R) = (0) \Rightarrow \mathrm{Hom}(M, E(R)) = (0) \Rightarrow \mathrm{Hom}(M_0, E(R)) = (0) \Rightarrow \mathrm{Hom}(M_0, R) = (0)$.

PROPOSITION 2. *If $R$ is* SZD *and $Z(R) = (0)$, then $E(R)$ is torsionless.*

*Proof.* Let $K = \bigcap_{f \in \mathrm{Hom}(E(R), R)} \mathrm{Ker} f$, and assume $K \neq (0)$. Then $\mathrm{Hom}(E(K), R) \neq (0)$ since $R$ is SZD. Choose $f \neq 0$ in $\mathrm{Hom}(E(K), R)$ and pick $x \in E(K)$ such that $f(x) \neq 0$. Set $A = \{r \in R \mid rf(x) = 0\}$. Because $Z(R) = (0)$, $A$ is not essential in $R$, and there is a left ideal $L \neq (0)$ such that $L \cap A = (0)$. Then $Lx \cap K \neq (0)$, so there is a $r \in L$ such that $rx \in K$ and $f(rx) \neq 0$. But $E(R) = E(K) \oplus Y$ for some $Y \subseteqq E(R)$, and $f$ can be extended to $\bar{f}: E(R) \to R$, contradicting the definition of $K$.

COROLLARY. *If $R$ is* SZD *and $Z(R) = (0)$, then an $R$-module $M$ is torsionless if and only if $E(M)$ is torsionless.*

*Proof.* If $E(M)$ is torsionless, $M$ is a torsionless submodule. If $M$ is torsionless, $M$ can be embedded in a product, $\pi R$, of copies of $R$. Then $E(M)$ can be embedded in a product, $\pi E(R)$, of copies of $E(R)$. Since $E(R)$ is torsionless by Prop. 2, so is $\pi E(R)$, and hence $E(M)$ is torsionless.

COROLLARY. *If $R$ is* SZD *and $Z(R) = 0$, then $R$ is* TCE.

*Proof.* Kato [2] has observed that the proof in [5] can be modified slightly to show that in any ring, SZD and TCE are equivalent to $E(R)$ being torsionless. Hence, by Prop. 2, $R$ must be TCE.

THEOREM 3. *If $R$ is right perfect and $Z(R) = (0)$, then* SZD *implies $QF$-3.*

*Proof.* Tachikawa [3] has shown that in a right perfect ring, $E(R)$ torsionless implies $R$ is $QF$-3.

We continue with some results on TCE rings. For an $R$-module $M$ we let $j_M$ denote the natural map from $M$ to its double dual $M^{**}$.

THEOREM 4. *If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a short exact sequence of $R$-modules with $A$ and $C$ torsionless, then $B$ is torsionless if and only* *if* $\operatorname{Im} j_A \cap \operatorname{Ker} \alpha^{**} = (0)$, *where* $\alpha^{**}$ *is the induced map from* $A^{**}$ *to* $B^{**}$.

*Proof.* Apply the exact sequence in Ext to $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, to obtain an exact sequence $0 \to C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^* \xrightarrow{\delta} X \to 0$, where $X \subseteqq \operatorname{Ext}^1_R(C, R)$ is the image of $A^*$ under the connecting map $\delta$ (see [1]). Take the dual of the latter sequence to obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^* & \longrightarrow & A^{**} & \xrightarrow{\alpha^{**}} & B^{**} & & \\ & & & & \uparrow{\scriptstyle j_A} & & \uparrow{\scriptstyle j_B} & & \\ & & 0 & \longrightarrow & A & \underset{\alpha}{\longrightarrow} & B & \underset{\beta}{\longrightarrow} & C \longrightarrow 0 \end{array}$$

a row exact, commuting diagram. Now $B$ torsionless implies $j_B\alpha$ is monic, so that $\alpha^{**}j_A$ is monic, and $\operatorname{Ker} \alpha^{**} \cap \operatorname{Im} j_A = (0)$. Conversely, if $\operatorname{Ker} \alpha^{**} \cap \operatorname{Im} j_A = (0)$, then $\operatorname{Ker} j_B \cap \operatorname{Im} \alpha = (0)$. Thus if $0 \neq b \in \operatorname{Ker} j_B$, $\beta(b) \neq 0$. Since $C$ is torsionless, there is a map $f: C \to R$ such that $f(\beta(b)) \neq 0$. But then $f\beta: B \to R$ satisfies $(f\beta)(b) = 0$, contradicting $b \in \operatorname{Ker} j_B$ (see [1]). Therefore $\operatorname{Ker} j_B = (0)$ and $B$ is torsionless.

Theorem 4 says $R$ is TCE if and only if $\operatorname{Im} j_A \cap \operatorname{Ker} \alpha^{**} = (0)$ for every short exact sequence $0 \to A \xrightarrow{\alpha} B \to C \to 0$ with $A$ and $C$ torsionless.

We now define a special type of torsionless $R$-module for use in further investigation of TCE rings.

DEFINITION. An $R$-module $M \neq (0)$ is completely torsionless (c.t.) provided $M$ is torsionless and has no nontrivial torsionless factors.

It is immediate that a c.t. module $M$ must be isomorphic to a left ideal of $R$, for there must be a nonzero map $f: M \to R$ since $M$ is torsionless, and $\operatorname{Ker} f = (0)$ since $M$ has no torsionless factors.

LEMMA 5. *If* $R$ *is left Noetherian, every left ideal has a completely torsionless factor.*

*Proof.* Let $L$ be a left ideal in $R$. If $L$ is not c.t., $L$ has a torsionless factor $L/L_1$. If $L/L_1$ is not c.t., there is left ideal $L_2 \supseteqq L_1$ such that $L/L_2$ is torsionless. Continuing in this fashion we obtain an ascending chain $\{L_i\}$ of left ideals which must terminate. That is, $L/L_n$ is c.t. for some $n$.

COROLLARY. *If* $R$ *is left Noetherian and* $M$ *is an* $R$*-module with* $\operatorname{Hom}(M, R) \neq (0)$, *then* $M$ *has a completely torsionless factor.*

*Proof.* Pick $f \neq 0$ in Hom $(M, R)$. Then $f(M)$ has a c.t. factor, so $M$ does also.

THEOREM 6. *If $R$ is left Noetherian, $R$ is* TCE *if and only if every short exact sequence $0 \to A \to B \to C \to 0$ with $C$ torsionless and $A$ completely torsionless, must have $B$ torsionless as well.*

*Proof.* The "only if" part follows from the definition of TCE. To show the "if" part, let $0 \to A \to B \to C \to 0$ be exact with $A$ and $C$ torsionless. Set $\bar{M} = \{M \mid M$ is a submodule of $A$, $B/M$ torsionless$\}$. Then $A \in \bar{M}$, and if $M_1 \supseteqq M_2 \supseteqq \cdots$ is a descending chain in $\bar{M}$, define a map $\phi: B/\bigcap_{i=1}^{\infty} M_i \to \prod_{i=1}^{\infty} B/M_i$ by $\phi(b + \bigcap_{i=1}^{\infty} M_i) = \prod_{i=1}^{\infty} (b + M_i)$. It is easy to check that $\phi$ is an $R$-monomorphism. Thus, $B/\bigcap_{i=1}^{\infty} M_i$ is isomorphic to a submodule of the torsionless module $\prod_{i=1}^{\infty} B/M_i$. This implies $B/\bigcap_{i=1}^{\infty} M_i$ is torsionless, hence $\bigcap_{i=1}^{\infty} M_i \in \bar{M}$. Now apply Zorn's Lemma to $\bar{M}$ to obtain a minimal element $M_0$. If $M_0$ is c.t., then $0 \to M_0 \to B \to B/M_0 \to 0$ gives $B$ torsionless by hypothesis. If $M_0$ is not c.t., there is a completely torsionless factor $M_0/N$ by Corollary to Lemma 5. But the exact sequence $0 \to M_0/N \to B/N \to B/M_0 \to 0$ implies that $B/N$ is torsionless, contradicting the minimality of $M_0$ in $\bar{M}$. Thus $M_0$ is in fact c.t., and $B$ is torsionless.

We next consider short exact sequences where the factor module is c.t. The theorem in this case is only for finitely generated modules over left Artinian rings.

THEOREM 7. *Let $R$ be left Artinian. The class of finitely generated torsionless modules is closed under extension if and only if every exact sequence of finitely generated modules, $0 \to A \to B \to C \to 0$, with $C$ completely torsionless and $A$ torsionless, has $B$ torsionless as well.*

*Proof.* Again the "only if" part is immediate. For the "if" part, let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be exact with $A$ and $C$ finitely generated and torsionless. If $C$ is not c.t., there is a c.t. factor $C/D_1$. Let $A_1 = \beta^{-1}(D_1) \supseteqq A$. This gives $0 \to A_1 \to B \to C/D_1 \to 0$ exact. If $A_1$ is torsionless, then so is $B$ by hypothesis. If not, consider $0 \to A \to A_1 \to D_1 \to 0$. Let $D_1/D_2$ be a c.t. factor of $D_1$, and $A_2 = \beta^{-1}(D_2) \subseteqq A_1$. This yields $0 \to A_2 \to A_1 \to D_1/D_2 \to 0$ exact. If $A_2$ is torsionless, then $A_1$ is also, contradiction. The process may be continued inductively, obtaining at each stage $D_{n-1}/D_n$ completely torsionless and $A_n = \beta^{-1}(D_n)$. The sequence $C \cong B/A \supseteqq A_1/A \supseteqq A_2/A \supseteqq \cdots$ must terminate since $C$ is finitely generated and $R$ is left Artinian. By construction, the sequence stops at $A_n/A$ if and only if $A_n$ is torsionless. But $A_n$ torsionless implies $A_{n-1}$ torsionless, also by construction. We conclude that $A_1$, hence $B$, is torsionless.

Note that the above proof does not require that $A$ be finitely generated, and the theorem can be generalized slightly. Consideration of short exact sequences with c.t. modules at both ends failed to yield any significant results.

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