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Let $\phi(x, y)$ be a solution to the equation:

$$(1) \quad (1 + \phi_y^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (1 + \phi_x^2)\phi_{yy} = 2H(1 + \phi_x^2 + \phi_y^2)^{3/2} .$$

The quantity H in equation (1) represents the mean curvature of the surface $z = \phi(x, y)$. In case $H = 0$, (1) is the minimal surface equation. For minimal surfaces, the well-known three point condition may be stated as follows:

THEOREM 1. Let $\phi(x, y)$ be a solution to the Dirichlet problem for the minimal surface equation in some bounded region R. Let T be the continuous space curve defined by the values of $\phi(x, y)$ over ∂R, the boundary of R. Then, if P is a plane tangent to the surface $z = \phi(x, y)$ for (x, y) in R, P will have at least 4 points in common with T.

The objective of this paper is to establish a natural analogue of the three-point condition for surfaces of positive, constant mean curvature.

It will be shown that certain "interior tangent spheres" of radius $1/H$ play the same roles, for surfaces of constant mean curvature $H > 0$ defined on a disk of radius $\rho < 1/H$, that tangent planes do for minimal surfaces.

(Rado's statement of the three-point condition for minimal surfaces appears in [3], pg. 34, et. seq.)

DEFINITION. Let S be a surface of constant mean curvature $H > 0$, defined by $z = \phi(x, y)$, where $\phi(x, y)$ is a solution to equation (1) in some region R. At the point $\bar{x}_0 = (x_0, y_0, \phi(x_0, y_0))$, let the normal line to the surface be drawn. Let a sphere, P_0, of radius $1/H$ be constructed, whose center lies a distance $1/H$ from $\bar{x}_0$, on the normal line through $\bar{x}_0$, in the direction specified by the normal vector to S at $\bar{x}_0$. P_0 will be called the "interior tangent sphere to S at $\bar{x}_0$".

In case R is a disk of radius $\rho < 1/H$, and T is a space curve consisting of the (continuous) boundary values of $\phi(x, y)$ on ∂R, it will be shown that P_0 has at least four points in common with T:

THEOREM II. *Let P_0 be the interior tangent sphere to the surface $z = \phi(x, y)$ at a non-umbilical point, x_0, where $\phi(x, y)$ is a solution to a given Dirichlet problem for equation* (1) *in the disk $x^2 + y^2 = \rho^2 < 1/H^2$, $H > 0$. Let T be the continuous space curve defined by the values of $\phi(x, y)$ over the boundary of the disk. Then P_0 must have at least*

4 points in common with T.

The assumption that $\bar{x}_0$ is a non-umbilical point is not severe, since as in the case of minimal surfaces, the umbilical points of a surface of constant mean curvature are isolated points, except for the sphere, for which every point is umbilical. The proof of Theorem II will depend on the observation that the difference of two solutions to equation (1) must satisfy the strong maximum principle. (Ref. R. Courant and D. Hilbert [1] and on a comparison lemma due to R. Finn (Ref. [2].)

As $H \to 0$, the radius of the "interior tangent sphere" of Theorem II tends towards infinity, and Theorem II becomes a statement of the three point condition for minimal surfaces.

Much of the material in this paper was contained in the author's doctoral thesis, written at Stanford University under the direction of Robert Finn and Newton Hawley. The author wishes to thank Professors Finn and Hawley for their advice and encouragement.

2. Comparison theorems for quasi-linear elliptic equations. In this section we will state, without proof, several theorems for quasi-linear elliptic equations which are essential for the proof of the three-point condition for surfaces of constant mean curvature.

The first is the maximum principle for quasi-linear equations having the form of equation (1).

$$\text{Let } M(\phi) = (1 + \phi_y^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (1 + \phi_x^2)\phi_{yy};$$

THEOREM 2.1. (Courant-Hilbert, [1], pg. 321, et. seq.). *If $\phi(x, y)$ is C^2 in a domain R, and if $\phi(x, y)$ has a maximum at an interior point of R, then $M(\phi) \leqq 0$ at this point.*

Theorem 1 implies that solutions of equation (1) in R will not have interior maxima. Interior minima may exist, however, as with the hemispherical solution

$$\phi(x, y) = -\sqrt{1/H^2 - (x^2 + y^2)}\,.$$

Different solutions of the same quasi-linear equation may be compared, as stated in the following comparison theorem (Courant-Hilbert, [1], pg. 322 et seq.):

THEOREM 2.2. *Let ϕ and Ψ be solutions of equation* (1) *in some region R. Denote by ω the difference of these solutions, $\omega = \phi - \psi$. Then ω has neither a maximum nor a minimum in the interior of R.*

A theorem concerning the comparison of solutions of quasi-linear elliptic equations, which is of importance in the study of minimal surfaces and surfaces of constant mean curvature has been proved by R. Finn (R. Finn, [2]).

THEOREM 2.3.

Hypotheses:

(i) *Let* $Q(\phi)$ *denote the expression*:

$$\sum_{i,j=1}^{2} a_{ij}(x_1, x_2, \phi_{x_1}, \phi_{x_2})\phi_{x_i x_j} + b(x_1, x_2, \phi_{x_1}, \phi_{x_2}),\ a_{12} = a_{21}\ .$$

Assume that (a_{ij}) *is positive definite for* (x, y) *in* R *and for all* $\phi(x, y)$ *to be considered.*

(ii) *Let* $\Gamma = \partial R$ *be the union of two closed sets,* Γ_α *and* Γ_β. *Let every interior point* x_α *of* Γ_α *be the end point of a line segment entering* R, *and let* $(\partial\phi/\partial s)$ *denote the derivative of* ϕ *along this line segment, in the direction approaching* x_α.

(iii) *Let* $\phi^{(1)}$ *be a function defined in* R, *such that* $\phi^{(1)}$ *tends to a finite or infinite limit at each point of* Γ_α, *and such that*

$$\lim_{x\to\Gamma_\alpha}\left(\frac{\partial}{\partial s}\ \phi^{(1)}\right) = \infty$$

on each of the indicated line segments.

Conclusion. Let $\phi^{(2)}$ *be any function defined and continuously differentiable in* $R + \Gamma$, *such that*

$$\lim_{x\to\Gamma_\beta}[\phi^{(1)} - \phi^{(2)}] \geqq 0\ .$$

If $Q(\phi^{(1)}) \leqq Q(\phi^{(2)})$ *in* R, *then*

$$\lim_{x\to\Gamma_\alpha}(\phi^{(1)} - \phi^{(2)} \geqq 0\ ,$$

and the strict inequality holds for any approach to an interior point of Γ_α.

Theorem 2.3 has been used by R. Finn to provide a simple proof of the fact that isolated singularities of solutions to the minimal surface equation are removable.

3. *Proof of Theorem II.* Let $\phi(x, y)$ be a solution to equation (1) in a region R, and let P_0 be the interior tangent sphere to the surface $z = \phi(x, y)$ at the point $\vec{x}_0 = (x_0, y_0, \phi(x_0, y_0))$, where (x_0, y_0) is

an interior point of R. We shall assume that $\vec{x}_0$ is not an umbilical point of the surface.

Define

$$p_0 = \left.\frac{\partial\phi}{\partial x}\right|_{x_0, y_0}, \quad q_0 = \left.\frac{\partial\phi}{\partial y}\right|_{x_0, y_0},$$

$$w_0 = 1 + p_0^2 + q_0^2 .$$

$$\vec{n}_0 = (-p_0/\sqrt{w_0}, -q_0/\sqrt{w_0}, 1/\sqrt{w_0}) .$$

The center of the interior tangent sphere P_0 is located at the point

$$\vec{x}_0 + \frac{1}{H}\vec{n}_0 = (x_0', y_0', z_0') .$$

The lower hemisphere of P_0 has the equation

$$z_{0L}(x, y) = z_0' - \sqrt{1/H^2 - (x - x_0')^2 - (y - y_0')^2} .$$

We define the difference function

$$\Psi_0(x, y) = \phi(x, y) - z_{0L}(x, y) .$$

Let K denote the Gauss curvature of the surface $z = \phi(x, y)$.

LEMMA 3.1.

$$(\Psi_{0xy})^2 - \Psi_{0xx}\Psi_{0yy} = w_0^2(H^2 - K) \text{ at the point } (x_0, y_0) .$$

Proof. The lemma is established by an easy calculation, together with the observation that $\Psi_0(x_0, y_0) = 0 = \Psi_{0x}|_{x_0\, y_0} = \Psi_{0y}|_{x_0\, y_0}$, since P_0 is tangent to the surface $z = \phi(x, y)$ at the point $\vec{x}_0$.

Since $\vec{x}_0$ is not an umbilical point, $H^2 - K > 0$, and $(x_0, y_0, \Psi_0(x_0, y_0))$ is therefore a hyperbolic point for the surface $z = \Psi_0(x, y)$. We shall use the familiar properties of hyperbolic points on C^2 surfaces. Let L_1 and L_2 be lines in the (x, y) plane, passing through (x_0, y_0) in the (orthogonal) directions of principal curvature of the surface $z = \Psi_0(x, y)$. Inside a sufficiently small neighborhood, N, of (x_0, y_0), $\Psi_0(x, y) < 0$ at all points of $L_1 \cap N$, and $\Psi_0(x, y) > 0$ at all points of $L_2 \cap N$, except for the point (x_0, y_0), where $\Psi_0(x, y) = 0$.

The lines L_1, L_2, define a right-handed coordinate system. We denote by l_1, l_2, l_3, and l_4, the intersections of N with the half-lines resulting from the deletion of (x_0, y_0) from L_1 and L_2, as shown in Figure 1.

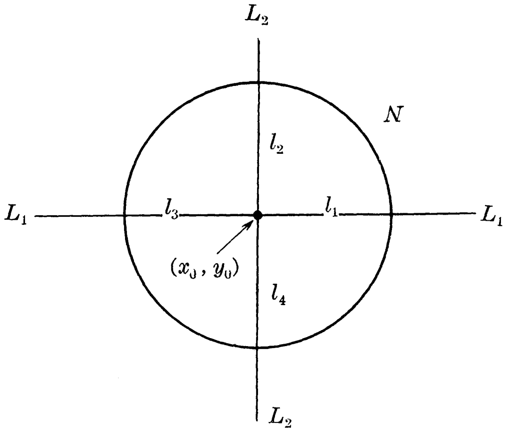

FIGURE 1

Thus, $\Psi_0(x, y) < 0$ on l_1, l_3, and > 0 on l_2, l_4, respectively.

Let R_0 denote the domain of definition of $\Psi_0(x, y)$. R_0 is the intersection of R with the projection of P_0 on the (x, y) plane, as illustrated by Figure 4. Since R is a disk of radius $\rho < 1/H$, it follows that ∂R_0 is either a circle of radius ρ, or consists in part of a circle of radius ρ, and part of a circle of radius $1/H$.

The open sets G_j, $j = 1, \cdots 4$, are defined as follows:

G_j is the largest connected open subset of R_0 containing l_j, and such that the sign of $\Psi_0(x, y)$ is everywhere the same in G_j.

LEMMA 3.2.

$$G_1 \cap G_3 \text{ and } G_2 \cap G_4 \text{ are empty .}$$

Proof. Let $\Gamma_j = \partial G_j$, $j = 1, \cdots 4$.

Each set Γ_j consists only of points where $\Psi_0(x, y) = 0$, or of points of ∂R_0, where $\Psi_0(x, y)$ may or may not equal zero.

Since $\Psi_0(x, y)$ is the difference of two solutions of equation (1), it follows from Theorem 2.2 that $\Psi_0(x, y)$ cannot vanish identically on any of the boundary sets Γ_j, unless $\phi(x, y)$ is part of a hemisphere of radius $1/H$. Since $\vec{x}_0$ is not an umbilical point, this is not the case.

If $\Psi_0 \neq 0$ at some point p_j of Γ_j, then p_j cannot be an interior point of R_0, by the maximality of G_j. Therefore there exist points $p_1 \cdots p_4$, such that

(i) $p_j \in \Gamma_j \cap \partial R_0$, $j = 1, \cdots 4$

(ii) $\Psi_0 < 0$ on p_1 and p_3, $\Psi_0 > 0$ on p_2, p_4.

Let $g_j \in l_j$, $j = 1, \cdots 4$, and *let* p_j and g_j be connected by the continuous curve C_j lying entirely within G_j, for $j = 1, \cdots 4$.

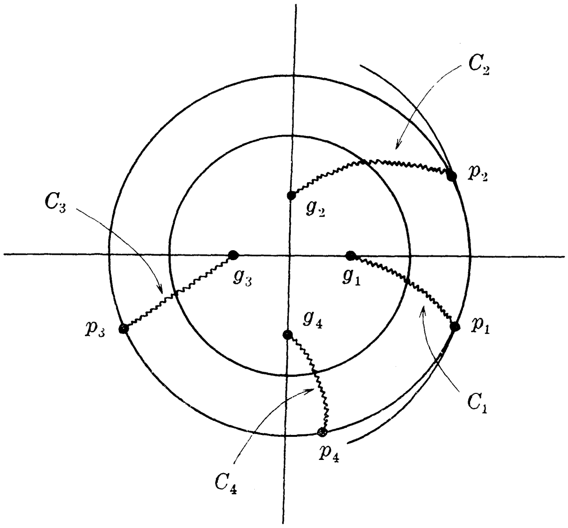

FIGURE 2

It follows that the sets $G_j, j = 1, \cdots, 4$, are disjoint. If, $G_1 \cap G_3$ is not empty, then the points g_1 and g_3 could be connected by a continuous curve lying entirely within the set $G_1 = G_1 \cap G_3 = G_3$. This curve must intersect either C_2 or C_4, where $\Psi_0(x, y) > 0$, and a contradiction results.

Let T denote the space curve determined by the values of $\phi(x, y)$ on ∂R, as illustrated by Figure 4. The proof of Theorem II is completed by considering two cases:

Case I. The projection of P_0 on the (x, y) plane contains R.

In this case, $R_0 = R$, and ∂R_0 is the circle of radius ρ. It follows from the continuity of $\Psi_0(x, y)$, together with the arguments presented in the proof of Lemma 3.2, that on any arc of ∂R_0 joining a pair of the points p_j, there must be a point on which $\Psi_0(x, y) = 0$. Therefore, the space curve T and the interior tangent sphere P_0 coincide at least four times.

Case II. $R_0 \neq R$. In this case, $\partial R_0 = S_H \cup S_\rho$, where S_H and S_ρ are arcs of circles of radius $1/H$ and ρ, respectively.

LEMMA 3.3. *If $R_0 \neq R$, then Γ_2 and Γ_4 contain points of S_ρ where $\Psi_0(x, y) > 0$.*

Proof. Suppose the contrary. Then Γ_2 consists of points where $\Psi_0(x, y) = 0$, and points on S_H where $\Psi_0(x, y) > 0$.

Let $\Gamma_2 = \Gamma_\alpha \cup \Gamma_\beta$, where $\Gamma_\alpha =$ the closure of the set of points of

$\Gamma_2 \cap S_H$ where $\Psi_0(x, y) > 0$, and $\Gamma_\beta =$ the closure of the remainder. In general, $\Gamma_\alpha \cap \Gamma_\beta$ will not be empty.

Let $\phi^1(x, y) = z_{0L}(x, y)$ and $\phi^2(x, y) = \phi(x, y)$. We will verify that Γ_α, Γ_β, ϕ^1 and ϕ^2 satisfy the hypotheses of Theorem 2.3 in G_2.

Hypothesis (i) is satisfied since both ϕ^1 and ϕ^2 are solutions of equation (1) in R_0, and therefore in G_2.

If x_α is an interior point of Γ_α, then $\Psi_0(x, y)$ is > 0 at x_α. By continuity, $\exists \varepsilon < 0 \ni \Psi_0(p) > 0$ for all points $p \varepsilon R_0 \ni \|p - x_\alpha\| < \varepsilon$, where $\|p - x_\alpha\|$ denotes the Euclidean distance of p from x_α. Therefore, $R_0 \cap \{p \mid \|p - x_\alpha\| < \varepsilon\} \subset G_2$, by virtue of the maximal character of the connected open set G_2 of which x_α is a boundary point. Therefore, as illustrated by Figure 3, x_α is the end point of a line segment S_α lying entirely within G_2, verifying hypothesis (ii).

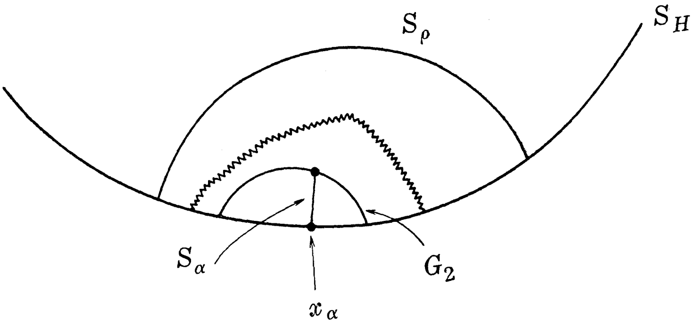

FIGURE 3

Since the gradient of $\phi^1(x, y)$ is infinite on S_H, hypothesis (iii) is verified.

By assumption, $\Psi_0(x, y) \equiv 0$ on Γ_β, so that $\phi^1 = \phi^2$ on Γ_β. $Q(\phi^1) = Q(\phi^2)$ in G_2, since ϕ^1 and ϕ^2 are solutions of equation (1). The conclusion of Theorem 2.3, that $\underline{\lim}_{x \to \Gamma\alpha} [\phi^1 - \phi_2] \geqq 0$, i.e., that $\Psi_0(x, y) \leqq 0$, completes the proof of Lemma 3.3 by contradiction. Thus, Γ_2 (and Γ_4) contain points of S_ρ where $\Psi_0(x, y) > 0$.

To complete the proof of Theorem II, let p_2 and p_4 be such points, i.e., $p_j \in \Gamma_j \cap S_\rho$, $\Psi_0(p_j) > 0$, $j = 2, 4$.

By a brief argument, we can verify that there must exist a point $p_3 \varepsilon \Gamma_3 \cap S_\rho$, such that $\Psi_0 < 0$ on p_3, where p_3 lies between p_2 and p_4 on the arc S_ρ.

Thus,

$$\phi(p_2) > z_{0L}(p_2)$$
$$\phi(p_3) < z_{0L}(p_3)$$
$$\phi(p_4) > z_{0L}(p_4)$$

Therefore, the space curve T lies below the lower hemisphere of P_0 at the point p_3, and above the lower hemisphere of P_0 at the points p_2 and p_4.

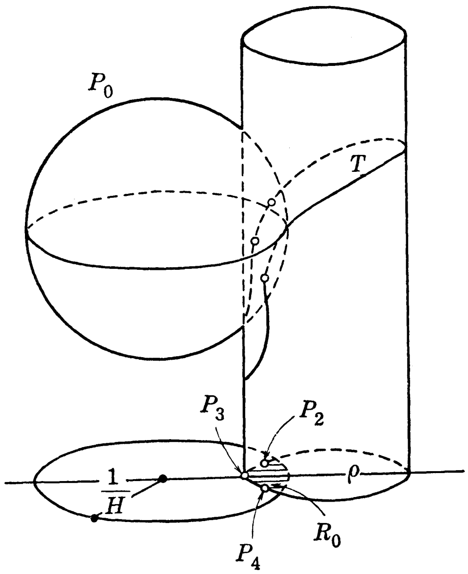

FIGURE 4

It follows that the space curve T must have passed from the outside to the inside of P_0 at least twice. The points where the space curve has pierced P_0 must project onto S_ρ at points which lie on S_ρ between p_2 and p_4. The space curve T cannot remain inside P_0, since there are points of T which project onto the (x, y) plane at points exterior to the projection on the (x, y) plane of P_0. Therefore T must eventually emerge from P_0, piercing P_0 twice more as it emerges.

We conclude that T has at least four points in common with P_0.

4. Discussion.

Theorem I can be used to derive a priori bounds on the gradient of solutions to the minimal surface equation. If T satisfies the condition that the inclinations of all planes having at least three points in common with T be uniformly bounded, then, an a priori bound on the inclinations of the tangent planes to the minimal surface $z = \phi(x, y)$ having T for its boundary data, is immediately known.

Theorem II provides analogous bounds on the gradient of solutions of equation (1) with H constant, > 0. For example, Let R be a disk of radius $\rho = (1 - \varepsilon)/H$, $0 < \varepsilon < 1$. Let T satisfy the condition that for every sphere P of radius $1/H$ having four or more points in common with T, the projection on the (x, y) plane of the center of P is within $\varepsilon/2H$ of the center of R. It is easy to verify that under this assumption

$$\sqrt{\phi_x^2 + \phi_y^2} \leqq \frac{1 - \varepsilon/2}{\sqrt{1 - (1 - \varepsilon/2)^2}} \cdot$$

It is interesting to note that as $H \to 0$, the surfaces and interior tangent spheres of Theorem II become minimal surfaces and tangent planes, respectively, producing Theorem I, the three-point condition for minimal surfaces, as a limiting case.

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