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**ON THE SPECTRAL RADIUS FORMULA IN BANACH
ALGEBRAS**

JAN-ERIK BJÖRK

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B will always denote a commutative semi-simple Banach algebra with a unit element. If $f \in B$ then $r(f)$ denotes its spectral radius. A sequence $F = (f_j)_{j=1}^{\infty}$ is called a spectral null sequence if $\|f_j\| \leq 1$ for each j , while $\lim_{j \rightarrow \infty} r(f_j) = 0$. If $F = (f_j)$ is a spectral null sequence we put $r_N(F) = \limsup_{j \rightarrow \infty} \|f_j^N\|^{1/N}$ for each $N \geq 1$. Finally we define the complex number $r_N(B) = \sup\{r_N(F) : F \text{ is a spectral null sequence in } B\}$. In general $r_N(B) = 1$ for all $N \geq 1$ and the aim of this paper is to study the case when $r_N(B) < 1$ for some N .

We say that B satisfies a bounded inverse formula if there exists some $0 < \varepsilon < 1$ and a constant K_0 such that for all f in B satisfying $\|f\| \leq 1$ and $r(f) \leq \varepsilon$, it follows that $\|(e - f)^{-1}\| \leq K_0$. In Theorem 3.1. we prove that B satisfies a bounded inverse formula if and only if $r_N(B) < 1$ for some N .

In §1 we give a criterion which implies that B is a sup-norm algebra. In §2 we introduce the so called infinite product of B which will enable us to study spectral null sequences in §3.

1. Sup-norm algebras. Recall that B is a sup-norm algebra if there exists a constant K such that $\|f\| \leq Kr(f)$ for all f in B . Clearly this happens if and only if $r_1(B) = 0$. Next we give an example where $r_1(B) = 1$ while $r_2(B) = 0$.

Let $B = C^1[0, 1]$ be the algebra of all continuously differentiable functions on the closed unit interval. If $f \in B$ we put $\|f\| = \sup\{|f(x)| + |f'(y)| : 0 \leq x, y \leq 1\}$. The maximal ideal space M_B can be identified with $[0, 1]$, so the spectral radius formula shows that $r(f) = \sup\{|f(x)| : 0 \leq x \leq 1\}$. From this we easily deduce that $r_2(B) = 0$. In fact we also notice that $\|f^n\| \leq n \|f\| (r(f))^{n-1}$ holds for all $n \geq 2$. We will now prove that this estimate is sharp.

THEOREM 1.1. *Let the norm in B satisfy $\|f^n\| \leq qn \|f\| r(f)^{n-1}$ for some $q < 1$ and some $n \geq 2$. Then B is a sup-norm algebra and there is a constant $K(n, q)$ such that $\|f\| \leq K(n, q)r(f)$ for all $f \in B$.*

LEMMA 1.2. *Let $n \geq 3$ and suppose that $\|f^n\| \leq K \|f\| r(f)^{n-1}$ for all f in B and some constant K . Then there is a constant $K(n)$ such that $\|f^2\| \leq K(n)K \|f\| r(f)$.*

Proof. Notice that all the inequalities above are homogeneous. Hence it is sufficient to consider the case when $\|f\| = 1$. If now $r(f) = \varepsilon$, then we must prove that $\|f^2\| < K(n)K\varepsilon$ for some $K(n)$.

Under the hypothesis, we note that

$$\|(k\varepsilon + f)^n\| \leq K \|k\varepsilon + f\|(\varepsilon + k\varepsilon)^{n-1} \leq K\varepsilon^{n-1}(1 + n)^n$$

for all $0 \leq k \leq n$.

Now consider the inhomogeneous system of equations

$$\sum_{j=0}^n \binom{n}{j} (k\varepsilon)^{n-j} f^j = (k\varepsilon + f)^n, \quad 0 \leq k \leq n,$$

which we wish to solve for the f^j . The determinant of the system is $\varepsilon^n \dots \varepsilon^n K_0(n)$, and the determinants of the minors can be expressed similarly. Using Cramer's rule to solve this system for f^2 , we obtain the estimate $\|f^2\| \leq K(n)K\varepsilon$, as required.

Proof of Theorem 1.1. Firstly we choose $\varepsilon > 0$ so small that $1 - \varepsilon^n > 2n\varepsilon^n + q$. Next we introduce the power series $\phi(z) = \varepsilon + a_1z + a_2z^2 + \dots$, which satisfies $(\phi(z))^n = \varepsilon^n + z$ for all $|z| < \varepsilon^n$. Notice that $na_1\varepsilon^{n-1} = 1$ holds. If $0 < x < \varepsilon^n$ we put

$$A_v(x) = x^v(|a_{vn}| + \dots + |a_{v(n-1)}|).$$

Then it is clear that the sum $U(x) = A_1(x) + A_2(x) + \dots$ is finite, while $\lim U(x) = 0$ as $x \rightarrow 0$.

Note that from Lemma 1.2. there is a constant $K(n)$ such that $\|f^k\| \leq K(n)r(f)$ for all $2 \leq k \leq n - 1$ and all f in B satisfying $\|f\| \leq 1$. It follows that there is a constant $K(n, \varepsilon)$ such that $\|a_2f^2 + \dots + a_{n-1}f^{n-1}\| \leq K(n, \varepsilon)r(f)$ for all f satisfying $\|f\| \leq 1$.

Now we choose $\delta > 0$ so small that $n\delta^{n-1} < \varepsilon^n$ and $U(n\delta^{n-1}) + K(n, \varepsilon)\delta < \varepsilon$ holds.

Suppose now that B is not a sup-norm algebra. Then we can choose f in B such that $\|f\| = 1$ while $r(f) < \delta$. The assumption shows that $\|f^n\| \leq qn\delta^{n-1} \leq n\delta^{n-1} \leq \varepsilon^n$. Hence $\|f^{vn+k}\| \leq \|f^n\|^v \|f^k\| \leq (n\delta^{n-1})^v \rightarrow$ for all $v \geq 1$ and all $k = 0 \dots (n - 1)$. It follows that we can define the element $g = \phi(f) = \varepsilon + a_1f + a_2f^2 + \dots$ in B .

We get $\|g\| \leq \varepsilon + |a_1| + \|a_2f^2 + \dots + a_{n-1}f^{n-1}\| + U(n\delta^{n-1}) \leq 2\varepsilon + |a_1|$. We also have $r(g) \leq (r(\varepsilon^n + f))^{1/n} \leq (\varepsilon^n + \delta)^{1/n}$.

It follows that $1 - \varepsilon^n \leq \|\varepsilon^n + f\| = \|g^n\| \leq qn \|g\| r(g)^{n-1} \leq qn(2\varepsilon + (n\varepsilon^{n-1})^{-1})(\varepsilon^n + \delta)^{1-1/n} = Z(\delta)$.

Clearly $Z(\delta)$ tends to $2qn\varepsilon^n + q$ as $\delta \rightarrow 0$. The original choice of ε shows that $1 - \varepsilon^n \leq Z(\delta)$ cannot hold for sufficiently small values of δ . This proves that B must be a sup-norm algebra and the proof gives a lower bound for δ , once we have fixed ε .

2. The infinite product of a Banach algebra. Firstly we introduce the infinite product.

DEFINITION 2.1. Put $B_\infty = \{(f_j)_1^\infty: (f_j) \text{ is a sequence in } B \text{ such that } \sup_j \|f_j\| < \infty \text{ while } \lim_{j \rightarrow \infty} r(we - f_j) = 0 \text{ for some } w \in C^1\}$.

Clearly B_∞ is a Banach algebra if to each $F = (f_j)$ we define $\|F\| = \sup_j \|f_j\|$. If $F = (f_j)$ and if $N \geq 1$, then we put $\pi_N(F) = (g_j)$, where $g_j = 0$ for $j \leq N$ and $g_j = f_j$ for $j > N$.

A complex-valued homomorphism H on B_∞ is free if $H(F) = H(\pi_N(F))$ for all $N \geq 1$ and each $F \in B_\infty$. The part at infinity in M_{B_∞} consists of the points determined by free homomorphisms. We denote this set by M_∞ .

To each $N \geq 1$ we have an idempotent e_N in B_∞ , whose N th component is e while all the other components are zero. The set $\Delta_N = \{x \in M_{B_\infty}: \hat{e}_N(x) = 1\}$ is a clopen (closed and open) subset of M_{B_∞} . We can identify Δ_N with M_B . For if $x \in M_B$ we get a point $T_N(x)$ in Δ_N satisfying $\hat{F}(T_N(x)) = \hat{f}_N(x)$ for all $F = (f_j)$. It is easily seen that T_N is a homeomorphism from M_B onto Δ_N .

If we put $\Delta = \bigcup \Delta_N: N \geq 1$, then it is easily seen that $\Delta = M_{B_\infty} \setminus M_\infty$. Here Δ is open and hence M_∞ is closed. The set M_∞ contains a distinguished point m_∞ , determined by the complex-valued homomorphism which sends $F = (f_j)$ into the complex number w satisfying $\lim_{j \rightarrow \infty} r(we - f_j) = 0$.

With the notations above the following result is evident.

LEMMA 2.2. *Let V be an open neighborhood of m_∞ in M_{B_∞} . Then there is an integer N such that $\Delta_j \subset V$ for all $j > N$.*

LEMMA 2.3. *Let $b\Delta$ be the topological boundary of Δ in M_{B_∞} . Then $b\Delta = \{m_\infty\}$.*

Proof. Lemma 2.2. means that the clopen sets Δ_N converge to $\{m_\infty\}$. Then it is a trivial topological fact that m_∞ is the only boundary point of Δ .

The result below was motivated by Theorem 2 in [2].

THEOREM 2.4. *The set M_∞ is a closed and connected subset of M_{B_∞} .*

Proof. We already know that M_∞ is closed. Suppose next that S and T are disjoint closed subsets whose union is M_∞ , such that $m_\infty \in S$. Then Lemma 2.3. implies that $S \cup \Delta$ is clopen in M_{B_∞} . By Shilov's idempotent Theorem there is $E \in B_\infty$ such that $\hat{E} = 0$ on $S \cup \Delta$ while $\hat{E} = 1$ on T . In particular $\hat{E} = 0$ on each Δ_j , which

implies that the j th component is zero. Since this holds for all j we conclude that $E = 0$, and T is empty. Hence M_∞ is connected.

The next result gives a useful characterization of M_∞ . This result is due to the referee.

THEOREM 2.5. *Let I be the closed ideal of all F in B_∞ for which $\lim \|\pi_N(F)\| = 0$ as $N \rightarrow \infty$. Then M_∞ is the maximal ideal space of B_∞/I .*

Proof. A point m in M_{B_∞} induces a complex-valued homomorphism on B_∞/I if and only if $\hat{F}(m) = 0$ for all $F \in I$. Clearly each idempotent e_N belongs to I . This proves that if m annihilates I , then m must belong to M_∞ . Conversely, if $m \in M_\infty$ then $\hat{F}(m) = \pi_N(F) \wedge (m)$ for all $N \geq 1$. Hence $|\hat{F}(m)| \leq \lim_{N \rightarrow \infty} \|\pi_N(F)\| = 0$ follows if $F \in I$. This proves that every point in M_∞ annihilates I .

If $F = (f_j)$ is in B_∞ we put $r_N(F) = \limsup_{j \rightarrow \infty} \|f_j^N\|^{1/N}$ for each $N \geq 1$. Let us also put $|F|_\infty = \sup \{|\hat{F}(m)| : m \in M_\infty\}$. With these notations the following result is a direct consequence of Theorem 2.5.

PROPOSITION 2.6. *If $F \in B_\infty$, then $|F|_\infty = \lim_{N \rightarrow \infty} r_N(F)$.*

3. Spectral null sequences.

THEOREM 3.1. *The following conditions on B are equivalent:*

- (a) $r_N(B) < 1$ for some $N \geq 1$.
- (b) B satisfies a bounded inverse formula.
- (c) There is a constant K_q such that if $f \in B$ satisfies $\|f\| \leq 1$ and $r(f) = q < 1$, then $\|(e - f)^{-1}\| \leq K_q(1 - q)^{-1}$.

Proof. Since (c) \rightarrow (b) we only prove that (a) \rightarrow (b) and (b) \rightarrow (a). Firstly we assume that $r_N(B) < 1$ for some $N \geq 1$. Then we get some $\varepsilon > 0$ and $a < 1$ such that $\|f^N\| \leq a^N$ for all f satisfying $\|f\| \leq 1$ and $r(f) \leq \varepsilon$.

Let then $\|f\| \leq 1$ while $r(f) \leq q < 1$. Let s be the positive integer satisfying $q^s < \varepsilon \leq q^{s-1}$. It follows that $\|f^{Ns}\| \leq a^N$ and hence $\|f^{kNs}\| \leq a^{kN}$ for all $k \geq 1$. Using this fact we see that if $R = \sum f^j : j \geq sN$, then $\|R\| \leq sNa^N(1 - a^N)^{-1}$.

We have $(e - f)^{-1} = e + f + \dots + f^{N^{s-1}} + R$. Since $\|f\| \leq 1$ we get $\|(e - f)^{-1}\| \leq sN + \|R\| \leq K_0s$. Finally $\varepsilon \leq q^{s-1}$ which implies that $s \leq K_1(1 - q)^{-1}$. Hence (c) follows with $K_q = K_0K_1$.

Now we assume that (b) holds in B . Suppose that $r_N(B) = 1$ for all N . To each $j \geq 1$ we can choose f_j such that $\|f_j\| = 1$ and $r(f_j) < (j + 1)^{-1}$, while $\|f_j^j\|^{1/j} > 1 - 1/j$.

Let us consider $F = (f_j)$ in B_∞ . Since $\lim_{j \rightarrow \infty} \|F^j\|^{1/j} = 1$, it

follows that there is some $w \in C^1$ satisfying $|w| = 1$ while $we - F'$ is not invertible in B_∞ .

Consider the elements $g_j = (e - f_j/w)^{-1}$ which exist for all $j \geq 1$. Clearly (b) implies that $\|g_j\| \leq K$ for some fixed constant K . Since $\lim_{j \rightarrow \infty} r(f_j) = 0$ it follows that the element $G = (g_j)$ exists in B_∞ . Now $(we - F')Gw^{-1} = e$ in B_∞ which shows that $we - F'$ is invertible, a contradiction. Hence $r_N(B) < 1$ must hold for some N .

Let us observe that a spectral null sequence $F' = (f_j)$ simply is an element of B_∞ for which $\|F'\| \leq 1$ and $\hat{F}'(m_\infty) = 0$. The following result is a direct consequence of Proposition 2.6.

THEOREM 3.2. *The following two conditions on B are equivalent:*

- (a) $\lim r_N(B) = 0$ as $N \rightarrow \infty$.
- (b) $M_\infty = \{m_\infty\}$.

Finally we study spectral null sequences satisfying polynomial conditions.

THEOREM 3.3. *Let p be a polynomial of the form $z^s (1 + a_1z + \dots + a_tz^t)$, with $s > 1$. Then there exist constants K and c with the following property: If $f \in B$ satisfies $\|f\| \leq 1$, $\|p(f)\| \leq \varepsilon$ and $r(f) \leq \varepsilon$, where $\varepsilon \leq c$, then $\|f^s\| \leq K\varepsilon$.*

Proof. For each $\varepsilon > 0$ we put $S(\varepsilon) = \{f \in B : \|f\| \leq 1, \|p(f)\| \leq \varepsilon \text{ and } r(f) \leq \varepsilon\}$. Suppose the constants c and K do not exist. Then there is a decreasing sequence (ε_j) , with $\lim_{j \rightarrow \infty} \varepsilon_j = 0$, while $S(\varepsilon_j)$ contains an element f_j for which $\|f_j^s\| > j\varepsilon_j$.

We may assume that $1 > |a_1|\varepsilon_1 + \dots + |a_t|\varepsilon_1^t$ holds. This implies that the elements $u_j = e + a_1f_j + \dots + a_t f_j^t$ are invertible in B .

Now $p(f_j) = f_j^s u_j$ and hence $j\varepsilon_j < \|f_j^s\| \leq \|p(f_j)\| \|u_j^{-1}\| \leq \varepsilon_j \|u_j^{-1}\|$. This means that $\|u_j^{-1}\| > j$ for all j , so the element $G = (u_j)$ is not invertible in B_∞ .

Now we obtain a contradiction by proving that G must be invertible in B_∞ . Since $\lim_{j \rightarrow \infty} \|p(f_j)\| = 0$ it follows that $\lim \|p(\pi_N(G))\| = 0$ as $N \rightarrow \infty$. Then Proposition 2.6. shows that $p(G)$ must vanish on M_∞ .

Hence the set $\hat{G}(M_\infty)$ is contained in the finite set of zeros of p . Using Theorem 2.4. we see that $\hat{G}(M_\infty)$ is connected. It follows that $\hat{G}(M_\infty) = \{\hat{G}(m_\infty)\}$. Clearly $\hat{G}(m_\infty) = 1$ holds and hence \hat{G} does not vanish on M_∞ . The choice of ε_1 shows that $\hat{G} \neq 0$ on Δ too. This proves that G is invertible in B_∞ which gives the desired contradiction.

Finally we raise some problems. We do not know if the condition that $r_N(B) < 1$ for some $N > 2$ implies that $r_2(B) < 1$. We

also ask if the condition that $r_N(B) < 1$ for some $N \geq 2$ implies that $\lim r_J(B) = 0$ as $J \rightarrow \infty$.

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