

# Pacific Journal of Mathematics

**APPROXIMATION OF CURVES**

HEINRICH W. GUGGENHEIMER

## APPROXIMATION OF CURVES

H. GUGGENHEIMER

**Generalizing recent results of J. M. Sloss we show: A curve in  $n$ -space that admits a continuously differentiable first order frame can be  $C^1$  approximated to any desired accuracy by a continuous, piecewise  $C^{r+2}$  curve for which the curvature functions are prescribed  $C^r$  ( $r = 0, 1, \dots, \infty, \omega$ ) functions of the arc length. The result can be extended to riemannian geometry.**

The theorem published by James M. Sloss [4] in this journal to the effect that a regular  $C^3$  curve can be approximated by a piecewise helix that either is circular or whose curvature or torsion are those of the given curve, can be generalized. The proof can be simplified by an explicit use of Gronwall's inequality [2] which is one of the strongest tools available in the treatment of ordinary differential equations. Gronwall's inequality says that

$$u(t) \leq C + \int_a^t u(\tau)v(\tau)d\tau, \quad t \geq a, \quad u(t) \geq 0, \quad v(t) \geq 0,$$

implies

$$u(t) \leq C \exp \int_a^t v(\tau)d\tau.$$

We fix a coordinate system in  $E^n$ . Let  $a_i$  be the unit vector on the  $i$ th coordinate axis and

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

the frame of these vectors. We are given a curve  $X(s)$ ,  $0 \leq s \leq L$ , as a function of its arc length. We say that  $X$  admits a continuously differentiable frame of first order [1] if there exists a continuously differentiable orthogonal matrix function  $\Theta(s)$  so that the unit vector  $e_1(s) = X'(s)$  is the first vector of the frame  $e = \Theta a$ . Then  $X$  must be a  $C^2$  curve. Conversely, if  $X$  is  $C^2$ , we may define  $e$  as the frame on  $S^{n-1}$  for which  $e_1$  is the unit tangent  $e_1(s) \in S^{n-1}$   $e_k$  ( $1 \leq k \leq n-1$ ) is tangent to the equatorial  $S^{k-1}$  in the  $S^k$  defined by  $a_1, \dots, a_{k+1}$  and  $e_n$  is tangent to the meridian. Naturally, if  $X$  is  $C^{n+1}$ , the Frenet frame may be taken as  $e$ . We put  $A(s) = \Theta'(s)\Theta(s)^{-1}$ .

For matrices we use the sup-norm of the corresponding linear

transformation. The norm of the frame  $e = \Theta a$  shall be the norm of  $\Theta$ .

**THEOREM.** *A  $C^2$  curve of finite length can be approximated in the  $C^1$  topology by a continuous, piecewise differentiable curve for which the curvature functions are prescribed  $C^r$  functions of the arc length,  $r \geq 0$ .*

The structure equations of the curve  $X$  and the frame  $e$  are

$$\begin{aligned} X'(s) &= e_1(s), \\ e'(s) &= A(s)e(s), \end{aligned}$$

with a continuous matrix  $A(s)$ ,  $0 \leq s \leq L$ . We want to approximate  $X(s)$  by a continuous curve  $Y(s)$  so that

$$\begin{aligned} Y'(s) &= e_1^*(s), \\ e^{*'}(s) &= A^*(s)e^*(s), \end{aligned}$$

except at finitely many points where the equations hold only in the sense of forward one-sided derivatives and where  $A^*(s)$  is a given (continuous) matrix function of  $s$ . We shall not use the fact that  $A^*(s)$  is skew-symmetric and has nonzero entries  $a_{ij}^*$ , only for  $j = i \pm 1$  [3].

We put

$$M = \max \|A(s)\|, N = \max \|A^*(s)\|,$$

and choose  $\Delta s$  subject to

$$\Delta s \leq \min(N^{-1}, \varepsilon(e(1+L)(M+N))^{-1}), \frac{L}{\Delta s} \text{ an integer.}$$

For  $k\Delta s \leq s \leq (k+1)\Delta s$ ,  $0 \leq k < L/\Delta s$ , define  $Y(s)$  by

$$e^*(s) = e(k\Delta s) + \int_{k\Delta s}^s A^*(\sigma)e^*(\sigma)d\sigma,$$

$$Y(s) = Y(k\Delta s) + \int_{k\Delta s}^s e_1^*(\sigma)d\sigma,$$

$$Y(0) = X(0).$$

Then

$$\begin{aligned} \|e^*(s) - e(s)\| &= \left\| \int_{k\Delta s}^s (A^*e^* - Ae)d\sigma \right\| \\ &\leq \int_{k\Delta s}^s \|A^* - A\| \|e\| d\sigma + \int_{k\Delta s}^s \|A^*\| \|e^* - e\| d\sigma \end{aligned}$$

$$\leq (M + N)\Delta s + N \int_{k\Delta s}^s \|e^* - e\| d\sigma .$$

By Gronwall's inequality,

$$\begin{aligned} \|e^*(s) - e(s)\| &\leq (M + N)\Delta s \exp(N(s - k\Delta s)) \\ &\leq (M + N)e\Delta s \leq \varepsilon/(1 + L). \end{aligned}$$

Hence,

$$\begin{aligned} |Y(s) - X(s)| &\leq |Y(k\Delta s) - X(k\Delta s)| + \int_{k\Delta s}^s |e_1^*(\sigma) - e_1(\sigma)| d\sigma \\ &\leq |Y(k\Delta s) - X(k\Delta s)| + \int_{k\Delta s}^s \|e^*(\sigma) - e(\sigma)\| d\sigma \\ &\leq |Y(k\Delta s) - X(k\Delta s)| + \varepsilon\Delta s(L + 1)^{-1} . \end{aligned}$$

By induction,  $|Y(k\Delta s) - X(k\Delta s)| \leq \varepsilon k\Delta s(1 + L)^{-1}$  and

$$\begin{aligned} |Y(s) - X(s)| &\leq \varepsilon L(1 + L)^{-1} , \\ |Y(s) - X(s)| + |Y'(s) - X'(s)| &\leq \varepsilon \frac{L}{1 + L} + \varepsilon \frac{1}{1 + L} = \varepsilon . \end{aligned}$$

The same argument would hold for a matrix  $A^*(s)$  with bounded, integrable entries if the differential equations are supposed to hold a.e., in the sense of Caratheodory. For  $C^r$  entries,  $Y$  is a  $C^{r+2}$  curve. Since the structure of the matrices was not used, the same argument holds for riemannian or Minkowski geometry.

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# Pacific Journal of Mathematics

Vol. 40, No. 2

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Louis I. Alpert and L. V. Toralballa, <i>An elementary definition of surface area in <math>E^{n+1}</math> for smooth surfaces</i> . . . . .	261
Eamon Boyd Barrett, <i>A three point condition for surfaces of constant mean curvature</i> . . . . .	269
Jan-Erik Björk, <i>On the spectral radius formula in Banach algebras</i> . . . . .	279
Peter Botta, <i>Matrix inequalities and kernels of linear transformations</i> . . . . .	285
Bennett Eisenberg, <i>Baxter's theorem and Varberg's conjecture</i> . . . . .	291
Heinrich W. Guggenheimer, <i>Approximation of curves</i> . . . . .	301
A. Hedayat, <i>An algebraic property of the totally symmetric loops associated with Kirkman-Steiner triple systems</i> . . . . .	305
Richard Howard Herman and Michael Charles Reed, <i>Covariant representations of infinite tensor product algebras</i> . . . . .	311
Domingo Antonio Herrero, <i>Analytic continuation of inner function-operators</i> . . . . .	327
Franklin Lowenthal, <i>Uniform finite generation of the affine group</i> . . . . .	341
Stephen H. McCleary, <i>0-primitive ordered permutation groups</i> . . . . .	349
Malcolm Jay Sherman, <i>Disjoint maximal invariant subspaces</i> . . . . .	373
Mitsuru Nakai, <i>Radon-Nikodým densities and Jacobians</i> . . . . .	375
Mitsuru Nakai, <i>Royden algebras and quasi-isometries of Riemannian manifolds</i> . . . . .	397
Russell Daniel Rupp, Jr., <i>A new type of variational theory sufficiency theorem</i> . . . . .	415
Helga Schirmer, <i>Fixed point and coincidence sets of biconnected multifunctions on trees</i> . . . . .	445
Murray Silver, <i>On extremal figures admissible relative to rectangular lattices</i> . . . . .	451
James DeWitt Stein, <i>The open mapping theorem for spaces with unique segments</i> . . . . .	459
Arne Stray, <i>Approximation and interpolation</i> . . . . .	463
Donald Curtis Taylor, <i>A general Phillips theorem for <math>C^*</math>-algebras and some applications</i> . . . . .	477
Florian Vasilescu, <i>On the operator <math>M(Y) = TYS^{-1}</math> in locally convex algebras</i> . . . . .	489
Philip William Walker, <i>Asymptotics for a class of weighted eigenvalue problems</i> . . . . .	501
Kenneth S. Williams, <i>Exponential sums over <math>GF(2^n)</math></i> . . . . .	511