

Pacific Journal of Mathematics

**AN ALGEBRAIC PROPERTY OF THE TOTALLY SYMMETRIC
LOOPS ASSOCIATED WITH KIRKMAN-STEINER TRIPLE
SYSTEMS**

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The concept of an x -root of degree r in a loop of order n is introduced. It is shown that the totally symmetric loop of order $n + 1$ derived from any Kirkman-Steiner triple system of order n admits a maximal identity-root. A statistical-combinatorial application of this algebraic property is then indicated. Finally, two open problems are also given.

A mathematical system consisting of an n -set Ω and a binary operation $*$ is said to form a loop of order n if the following axioms are satisfied :

(1) Ω contains an identity element e such that $x * e = e * x = x$ for every x in Ω .

(2) Any two of the elements in the equation $x * y = z$ uniquely determine the third.

Since the notation $x * y$ is too bulky we shall use, hereafter, the notation xy instead. A loop is said to be a totally symmetric loop if it also satisfies

(3) $xy = yx$ and $x(xy) = y$ for all x and y in Ω .

In this paper, we shall introduce and study an algebraic property of totally symmetric loops of order $n \equiv 3(\text{mod } 6)$. In the final part of this paper we shall indicate, briefly, a statistical-combinatorial application of this study. A few open questions are also stated.

We begin by introducing and reviewing certain concepts and results that will be relevant to our forthcoming results.

DEFINITION 1. We say a loop \mathcal{L} of order n accepts a

$$(k_1, k_2, \dots, k_r)$$

orthogonal partition if the n^2 cells in the Cayley table of \mathcal{L} can be divided into r mutually disjoint exhaustive sets S_1, S_2, \dots, S_r ; in such a way that (1) S_i has k_i cells from each row and each column, (2) each element of \mathcal{L} appears k_i times in the cells of S_i ,

$$(3) \quad k_1 + k_2 + \cdots + k_r = n .$$

In particular a set S_i is called a transversal of \mathcal{L} if $k_i = 1$. If two transversals have no cells in common, they are said to be parallel; if they have exactly one cell in common, they are called orthogonal.

A set $\{t_1, t_2, \dots, t_r\}$ mutually orthogonal transversals of \mathcal{L} is said to be an x -root of degree r if these transversals are all sharing a unique cell containing the element x . Clearly any x -root of degree r occupies $r(n-1) + 1$ cells of the Cayley table of a loop of order n . An x -root of degree r in the Cayley table of a loop of order n is said to be a maximal x -root if $r = n - 2$. The following lemma justifies this terminology.

LEMMA 1. *For any x -root of degree r in a loop of order n , $r \leq n - 2$.*

Proof. Let the cell in the given x -root that contains the element x occur in row i and column j . Then the remaining $2n - 2$ cells of row i and column j , together with the $n - 1$ other cells containing the element x , cannot be in the x -root. Thus there remains only $n^2 - 3n + 3$ cells to accommodate the given x -root. However, as pointed out before, this x -root must occupy $r(n - 1) + 1$ cells. Hence $r \leq n - 2$.

DEFINITION 2. Let Σ be an n -set, $n \equiv 1, 3 \pmod{6}$. Then a Steiner triple system of order n on Σ is a collection of $n(n-1)/6$ unordered triples (x, y, z) with x, y, z in Σ , such that every pair of distinct elements of Σ belongs to exactly one triple. A triple system of order $n \equiv 3 \pmod{6}$ is said to be a Kirkman-Steiner triple system of order n if it is a Steiner triple system with the following additional stipulation: the set of triples can be partitioned into $r = (n-1)/2$ disjoint classes such that the totality of elements in each class exhaust the set on which the system is defined.

While Reiss [9] has shown the sufficiency of $n \equiv 1, 3 \pmod{6}$ for the existence of a Steiner triple system of order n , Ray-Chaudhuri and Wilson [8] have proved the sufficiency of $n \equiv 3 \pmod{6}$ for the existence of a Kirkman-Steiner triple system of order n .

The coextensiveness of totally symmetric loops of order $n + 1$ with Steiner triple systems of order n has been shown by Bruck [2] who proved the following theorem:

THEOREM 1. *A totally symmetric loop of order $n + 1$ exists if and only if there exists a Steiner triple system of order n .*

For the sake of clarity of later arguments, we shall sketch a proof of this theorem here.

Proof. Let A be a totally symmetric loop of order $n + 1$ and let $H = A - \{e, \text{ the identity element in } A\}$. Then the collection of all unordered triples (x, y, z) with x, y, z in H , such that $xy = z$, forms a Steiner triple system on H . Conversely, given a Steiner triple system of order n on an n -set W , we can then form a totally symmetric loop of order $n + 1$ from these triples as follows: Define an operation \circ on the set $\mathcal{L}^* = WU\{e\}$ by: (1) $a \circ b = c$ if and only if (a, b, c) is in \mathcal{L}^* , (2) $e \circ a = a \circ e = a$, and (3) $a^2 = e^2 = e$ for all a in \mathcal{L}^* . Then \mathcal{L}^* together with the binary operation \circ forms a totally symmetric loop of order $n + 1$.

Let Σ be an n -set, $n \equiv 3(\text{mod } 6)$ and let \mathcal{K} be a Kirkman-Steiner triple system on Σ . Let also \mathcal{L}^* be the totally symmetric loop of order $n + 1$ derived from \mathcal{K} . Denote the identity element in \mathcal{L}^* by e . Partition \mathcal{L}^* into $r = (n - 1)/2$ disjoint classes $C_i, i = 1, 2, \dots, r$ as described in Definition 2. Then we have the following lemma.

LEMMA 2. *C_i determines an e -root of degree 2 in the Cayley table of \mathcal{L}^* .*

Proof. Denote an arbitrary triple in C_i by

$$(a_{ij}, b_{ij}, c_{ij}), j = 1, 2, \dots, n/3 .$$

Identify three cells in the Cayley table of \mathcal{L}^* by the 2-tuples $(a_{ij}, b_{ij}), (b_{ij}, c_{ij})$ and (c_{ij}, a_{ij}) , the components of each 2-tuple being the row and column indicies respectively. Now let j run through all the $n/3$ triples in C_i . Then the corresponding $3 \times n/3 = n$ cells determined by the preceding rule, together with the cell corresponding to row and column indices (e, e) , form a transversal for \mathcal{L}^* . Denote this transversal by t_{i_1} . Another transversal t_{i_2} is obtained by considering the cell (e, e) and the three cells in the Cayley table described by the 2-tuples $(b_{ij}, a_{ij}), (c_{ij}, b_{ij})$ and (a_{ij}, c_{ij}) , where we let j run through the values $1, 2, \dots, n/3$. These exhibition rules clearly guarantee that t_{i_1} is orthogonal to t_{i_2} and that the point of intersection is the cell (e, e) .

We shall now prove the following:

THEOREM 2. *The totally symmetric loop \mathcal{L}^* derived from any Kirkman-Steiner triple system contains a maximal identity-root.*

Proof. By Lemma 2 every class in the given Kirkman-Steiner triple system determines an e -root of degree 2 in the Cayley table of \mathcal{L}^* , where e is the identity in \mathcal{L}^* . The method of exhibition in the lemma together with the fact that every pair of distinct elements in the triple system appears exactly once reveals that the transversal $t_{i,k}(k = 1, 2)$ is orthogonal to $t_{i',k}(k = 1, 2)$ if $i \neq i'$ with cell (e, e) as the intersection point. Since there are $(n - 1)/2$ classes, we have $2(n - 1)/2 = n - 1$ pairwise orthogonal transversals sharing the cell (e, e) , i.e., an identity-root of degree $n - 1$. Since the order of \mathcal{L}^* is $n + 1$, the proof is complete.

As an immediate application we have

COROLLARY. *Every totally symmetric loop of order $n + 1$ derived from a Kirkman-Steiner triple system of order n implies the existence of a set consisting of at least a pair of mutually orthogonal Latin squares of order n .*

A proof of this corollary, together with some additional results, will be given in another paper. However, we should remark that, in particular, for $n = 15$, the corresponding pair of orthogonal Latin squares can be embedded in a set of three mutually orthogonal Latin squares of order 15, thus disproving MacNeish's [5] conjecture for order 15.

Before finishing, let us mention a few open problems.

(1) Prove or disprove that the totally symmetric loop of order $n + 1$ derived from any arbitrary Steiner triple system of order n admits a maximal x -root.

(2) Characterize those loops whose Cayley tables admit a $(1, 1, \dots, 1)$ orthogonal partition.

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REFERENCES

1. R. C. Bose, S. S. Shrikhande and E. T. Parker, *Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture*, *Canad. J. Math.*, **12** (1960), 189-203.
2. R. H. Bruck, *What is a loop?* (in *Studies in Modern Algebra*, (1963) edited by A. A.

Albert. Prentice Hall, Englewood Cliffs.

3. M. Hall, Jr., *Combinatorial theory*, (1967). Blaisdell Pub. Co., Mass.
4. A. Hedayat and E. Seiden, *On a method of sum composition of orthogonal Latin squares*, Research paper RM-238 (1969). Dept. of Statistics and Probability, Michigan State University. See also Atti del Convegno di Geometria Combinatoria e sue Applicazioni, Perugia, 11-17 settembre (1970), 239-256.
5. H. F. MacNeish, *Euler's squares*, Ann. of Math., **23** (1922), 221-227.
6. E. T. Parker, *Construction of some sets of mutually orthogonal Latin squares*, Proc. Amer. Math. Soc., **10** (1959), 946-949.
7. ———, Personal communication.
8. D. K. Ray-Chaudhuri and R. M. Wilson, *Solution of Kirkman's school girl problem*, Proc. Amer. Math. Soc. Sympos. in Pure Math. (Combinatorics), **19** (1971), 187-203.
9. M. Reiss, *Über eine Steinersche Combinatorische Aufgabe welche in 45 sten Bande dieses Journals*, Seite 181, gestellt worden ist, J. Reine U. Angew. Math. 56 (1859), 326-344.
10. J. R. Ryser, *Combinatorial Mathematics*, 1963. Wiley, New York.

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Louis I. Alpert and L. V. Toralballa, <i>An elementary definition of surface area in E^{n+1} for smooth surfaces</i>	261
Eamon Boyd Barrett, <i>A three point condition for surfaces of constant mean curvature</i>	269
Jan-Erik Björk, <i>On the spectral radius formula in Banach algebras</i>	279
Peter Botta, <i>Matrix inequalities and kernels of linear transformations</i>	285
Bennett Eisenberg, <i>Baxter's theorem and Varberg's conjecture</i>	291
Heinrich W. Guggenheimer, <i>Approximation of curves</i>	301
A. Hedayat, <i>An algebraic property of the totally symmetric loops associated with Kirkman-Steiner triple systems</i>	305
Richard Howard Herman and Michael Charles Reed, <i>Covariant representations of infinite tensor product algebras</i>	311
Domingo Antonio Herrero, <i>Analytic continuation of inner function-operators</i>	327
Franklin Lowenthal, <i>Uniform finite generation of the affine group</i>	341
Stephen H. McCleary, <i>0-primitive ordered permutation groups</i>	349
Malcolm Jay Sherman, <i>Disjoint maximal invariant subspaces</i>	373
Mitsuru Nakai, <i>Radon-Nikodým densities and Jacobians</i>	375
Mitsuru Nakai, <i>Royden algebras and quasi-isometries of Riemannian manifolds</i>	397
Russell Daniel Rupp, Jr., <i>A new type of variational theory sufficiency theorem</i>	415
Helga Schirmer, <i>Fixed point and coincidence sets of biconnected multifunctions on trees</i>	445
Murray Silver, <i>On extremal figures admissible relative to rectangular lattices</i>	451
James DeWitt Stein, <i>The open mapping theorem for spaces with unique segments</i>	459
Arne Stray, <i>Approximation and interpolation</i>	463
Donald Curtis Taylor, <i>A general Phillips theorem for C^*-algebras and some applications</i>	477
Florian Vasilescu, <i>On the operator $M(Y) = TYS^{-1}$ in locally convex algebras</i>	489
Philip William Walker, <i>Asymptotics for a class of weighted eigenvalue problems</i>	501
Kenneth S. Williams, <i>Exponential sums over $GF(2^n)$</i>	511