0-PRIMITIVE ORDERED PERMUTATION GROUPS

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Let $G$ be a transitive $l$-subgroup of the lattice-ordered group $A(\Omega)$ of all order-preserving permutations of a chain $\Omega$. (In fact, many of the results are generalized to partially ordered sets $\Omega$ and transitive groups $G$ such that $\beta < \gamma$ implies $\beta g = \gamma$ for some positive $g \in G$, thus encompassing some results on non-ordered permutation groups.) The orbits of any stabilizer subgroup $G_\alpha, \alpha \in \Omega$, are convex and thus can be totally ordered in a natural way. The usual pairing $\Delta \mapsto \Delta' = \{ag \mid \alpha \in \Delta g\}$ establishes an o-anti-isomorphism between the set of "positive" orbits and the set of "negative" orbits. If $\Delta$ is an o-block (convex block) of $G$ for which $\Delta G_\alpha = \Delta$, then $\Delta'$ is also an o-block. If $G_\alpha$ has a greatest orbit $\Gamma$, then $\{\beta \in \Omega \mid \Gamma' < \beta < \Gamma\}$ constitutes an o-block of $G$. A correspondence is established between the centralizer $Z_{A(\Omega)}G$ and a certain subset of the fixed points of $G_\alpha$.

The main theorem states that every o-primitive group $(G, \Omega)$ which is not o-2-transitive or regular looks strikingly like the only previously known example, in which $\Omega$ is the reals and $G = \{f \in A(\Omega) \mid (\beta + 1)f = \beta f + 1 \text{ for all } \beta \in \Omega\}$. The "configuration" of orbits of $G_\alpha$ must consist of a set o-isomorphic to the integers of "long" (infinite) orbits with some fixed points interspersed; and there must be a "period" $z \in Z_{A(\overline{\Omega})}G$ ($\overline{\Omega}$ the Dedekind completion of $\Omega$) analogous to the map $\beta z = \beta + 1$ in the example. Periodic groups are shown to be $l$-simple, and more examples of them are constructed.

Transitivity guarantees that the "configuration" of orbits of $G_\alpha$ is independent of $\alpha$, so that we may speak of the configuration of $G$ (defined more precisely later). There is appreciable interplay between this configuration and other properties of $G$. For example, o-2-transitive groups are characterized by having only one positive orbit, and regular groups by having configurations consisting entirely of fixed points.

For periodically o-primitive groups, the period $z$ is the unique o-permutation of $\overline{\Omega}$ such that for every $\beta \in \Omega$, $\beta z$ is the sup of the first positive orbit of $G_\beta$. $(\beta z)g = (\beta g)z$ for all $\beta \in \Omega, g \in G$, and in fact $z$ generates $Z_{A(\overline{\Omega})}G$. This periodicity is of paramount importance. For example, it guarantees that the action of $g \in G$ on any long orbit of $G_\alpha$ determines its action on all of $\Omega$.

Transitive $l$-subgroups of $A(\Omega)$ have been studied from a lattice-ordered group ($l$-group) orientation by Holland [5, 6, 7], Lloyd [10, 11], Sik [15], and McCleary [12, 13]. Holland showed that every $l$-group
is \(l\)-isomorphic to a subdirect product of transitive \(l\)-permutation groups [5]. A nonlattice point of view has been taken by Holland and McCleary [8, 14], where it was shown that every transitive ordered permutation group can be embedded in the generalized ordered wreath product of its \(o\)-primitive "components" (an important motivation for the present paper); and by G. Higman [4] and Wielandt [17, §6]. The concept of configuration is a refinement of the concept of rank in [3].

The generalization to partially ordered \(\Omega\) requires very little additional work, but it is less intuitive than the totally ordered case and the reader will not lose much if he assumes that \(\Omega\) is totally ordered, or even that \(G\) is an \(l\)-permutation group, as we have done in this introduction.

2. Coherent \(o\)-permutation groups. Let \(\Omega\) be a partially ordered set (po-set) containing more than one point. Points of \(\Omega\) will be denoted by lower case Greek letters; subsets, by upper case Greek letters; and permutations, by lower case Roman letters. The image of \(\beta \in \Omega\) under the permutation \(f\) will be denoted by \(\beta f\), so that if \(g\) is also a permutation, \(\beta (fg) = (\beta f)g\).

An order-preserving permutation (o-permutation, automorphism) of \(\Omega\) is a permutation \(f\) such that for \(\beta, \gamma \in \Omega\), \(\beta < \gamma\) iff \(\beta f < \gamma f\). We define \(f \leq g\) iff \(\beta f \leq \beta g\) for all \(\beta \in \Omega\), making the group \(A(\Omega)\) of all permutations of \(\Omega\) into a partially ordered group (po-group). If \(\Omega\) is totally ordered, \(f\) is an o-permutation provided only that \(\beta < \gamma\) implies \(\beta f < \gamma f\). In this case \(A(\Omega)\) is an \(l\)-group, with \(\beta (f \lor g) = \max \{\beta f, \beta g\}\) and \(\beta (f \land g) = \min \{\beta f, \beta g\}\); and \(G\) is said to be an \(l\)-permutation group if it is an \(l\)-subgroup of \(A(\Omega)\), i.e., a subgroup which is also a sublattice. Standard results about po-groups and \(l\)-groups can be found in [2], but we shall make minimal use of them.

Our o-permutation group \(G\) will always be assumed to be a transitive subgroup of \(A(\Omega)\) (i.e., \(\beta, \gamma \in \Omega\) implies \(\beta g = \gamma\) for some \(g \in G\)). Thus \(\Omega\) must be homogeneous; and if ordered nontrivially (\(\beta < \gamma\) for some \(\beta, \gamma \in \Omega\)), it must be infinite. Furthermore, we shall always assume that if \(\beta < \gamma \in \Omega\), there exists \(1 < g \in G\) such that \(\beta g = \gamma\). (This property implies its dual, which states that if \(\beta > \gamma\), there exists \(1 > g \in G\) such that \(\beta g = \gamma\); and implies that if \(\beta f < \gamma, f \in G\), then there exists \(g \in G\) such that \(\beta g = \gamma\) and \(g > f\)). Transitive groups that satisfy this property will be called coherent. Of course, if \(\Omega\) is totally ordered, transitivity need not be separately assumed. Transitive \(l\)-permutation groups are coherent, for if \(\beta < \gamma\) and \(\beta g = \gamma\), then also \(\beta (g \lor 1) = \gamma\). However, the group in Example 7 is not coherent. If \(\Omega\) is trivially ordered, \(A(\Omega)\) is just the symmetric group \(S(\Omega)\), and is itself trivially ordered; and its transitive subgroups are automatically coherent.
Let $B$ be a convex subset (segment) of a po-set $A$ if $b_1 \leq a \leq b_2$, $b_1, b_2 \in B$, $a \in A$ implies $a \in B$. If $C$ and $D$ are any subsets of $A$, we define $C \subseteq D$ iff $c \leq d$ for some $c \in C$, $d \in D$. If $A$ is totally ordered, and $C$ and $D$ are nonvoid disjoint segments of $\Omega$, then $C < D$ iff $c < d$ for all $c \in C$, $d \in D$.

If $(G, \Omega)$ is a transitive (but not necessarily coherent) o-permutation group, let $R(G_a)$ designate $\{G_ag \mid g \in G\}$, ordered as above to give the usual partial ordering on the collection of right cosets of a convex subgroup of a po-group. As with nonordered transitive permutation groups, we make $G$ act faithfully on $R(G_a)$ by defining $(G_ag) = G_a(gk)$, $g, k \in G$. Here we obtain an o-permutation group.

An o-isomorphism from one o-permutation group $(G, \Omega)$ onto another $(K, \Sigma)$ consists of a po-set isomorphism $\Theta: \Omega \to \Sigma$ and a po-group isomorphism $\Theta: G \to K$ such that for all $\omega \in \Omega$, $g \in G$, $(\omega g)\theta_a = (\omega \theta_a)(g \theta_a)$. The importance of coherence is explained by

**Theorem 1.** Let $(G, \Omega)$ be a transitive o-permutation group and let $\alpha \in \Omega$. Then $G$ is coherent if and only if the correspondence $\alpha g \leftrightarrow G_a g$ between $\Omega$ and $R(G_a)$ and the identity map on $G$ furnish an o-isomorphism between $(G, \Omega)$ and $(G, R(G_a))$.

*Proof.* Suppose that $G$ is coherent. $\alpha g_1 = \alpha g_2$ iff $g_1g_2^{-1} \in G_a$ iff $G_a g_1 = G_a g_2$, so we have a one-to-one correspondence between $\Omega$ and $R(G_a)$. $\alpha g_1 \leq \alpha g_2$ iff $\alpha g_1 k = \alpha g_2$ for some $1 \leq k \in G$ (by coherence) iff $G_a g_1 k = G_a g_2$ (for some $1 \leq k \in G$) iff $G_a g_1 \leq G_a g_2$, so the correspondence is an o-isomorphism. For $h \in G$, $(\alpha g)h = \alpha(gh) \leftrightarrow G_a(gh) = (G_a g)h$. This establishes the o-permutation group isomorphism. The converse is clear.

$G$ is regular if it is transitive and $G_a = \{1\}$.

**Corollary 2.** Let $G$ be regular. Then $G$ is coherent if and only if $(G, \Omega)$ is o-isomorphic to the right regular representation of $G$. In particular, the right regular representation of $G$ is coherent.

3. The configuration of an o-permutation group. There will usually be one (arbitrary) point $\alpha$ in $\Omega$ on which our attention will be especially focused. The orbit of $G_\alpha$ which contains $\delta$ is $\delta G_\alpha = \{\delta h \mid h \in G_\alpha\}$. $\alpha G_\alpha = \{\alpha\}$. If $\delta G_\alpha$ is not trivially ordered, it is infinite. The orbits of $G_\alpha$ partition $\Omega$. In general, the orbits of $G_\alpha$ need not be convex (Examples 3 and 6), although of course they are convex if $\Omega$ is trivially ordered. We also have

**Proposition 3.** If $G$ is a transitive l-subgroup of $A(\Omega)$, $\Omega$ totally ordered, then the orbits of $G_\alpha$ are convex.
Proof. Suppose $\beta \leq \gamma \leq \delta$ and $\beta h = \delta$ for some $h \in G_a$. By transitivity, $\beta g = \gamma$ for some $g \in G$. Let $f = (h \vee 1) \wedge (g \vee 1)$. Then $\beta f = \gamma$. Since $1 \leq f \leq h \vee 1 \in G_a$, the convexity of $G_a$ implies that $f \in G_a$.

To escape having to assume that the orbits of $G_a$ are convex, we shall "enlarge" them to convex sets. The *conexification* $\text{Conv}(A)$ of $A \subseteq \Omega$ is $\{\xi \in \Omega | \delta_1 \leq \xi \leq \delta_2 \text{ for some } \delta_1, \delta_2 \in A\}$. If $A$ is an orbit of $G_a$, we shall call $\text{Conv}(A)$ an *orbital* of $G_a$. Of course, if the orbits of $G_a$ are convex, the concepts of "orbital" and "orbit" coincide. If $\Gamma$ is an orbital of $G$ and $\gamma \in \Gamma'$, then the orbital $\text{Conv}(\gamma G_a)$ of $G_a$ determined by $\gamma$ is $\Gamma$. The orbitals of $G_a$ partition $\Omega$ into convex subsets. The set of orbitals of $G_a$ is partially ordered; and is totally ordered if $\Omega$ is totally ordered. Two orbits in different orbitals are related as are their orbitals; and two orbits in the same orbital are of course each less than or equal to the other.

Those orbitals of $G_a$ which are strictly greater than $\{a\}$ will be called *positive*; those strictly less than $\{a\}$, *negative*. All points in a positive (negative) orbital are strictly greater than (less than) $\alpha$. No orbital is both positive and negative; and if $\Omega$ is totally ordered, every orbital except $\{a\}$ is one or the other. These remarks apply also to orbits of $G_a$.

We define for each orbit $A$ a *paired orbit* $A' = A' = \{ag | a \in Ag\}$. (The notation $A'$ will always refer to pairings with respect to the point denoted by the letter $a$). It is shown in [18, §16] that $A'$ is indeed an orbit of $G_a$; that the map $A \rightarrow A'$ is one-to-one from the set of orbits of $G_a$ onto itself; and that $A'' = A$. $ag \in A'$ iff $a \in Ag$, and if $a \in Ag$, then $A' = (ag)G_a$.

**Proposition 4.** Let $(G, \Omega)$ be a coherent $o$-permutation group. The map $A \rightarrow A'$ is an $o$-anti-automorphism of the set of orbits of $G_a$. Since $\{a\}$ is self-paired, the appropriate restriction provides an $o$-anti-isomorphism from the set of positive orbits of $G_a$ onto the set of negative orbits. If $\Omega$ is totally ordered, only $\{a\}$ is self-paired.

**Proof.** Use coherence.

A subset $A$ of $\Omega$ will be called $\alpha$-full if it contains each orbit of $G_a$ that it meets, i.e., if it is a union of orbits of $G_a$. Thus the $\alpha$-full sets are precisely those sets $A$ such that $Ah = A$ for each $h \in G_a$. We obtain a canonical correspondence between the $\alpha$-full subsets of $\Omega$ and the subsets of the set of orbits of $G_a$ by letting the $\alpha$-full set $A$ correspond to the set of orbits contained in $A$. We shall frequently make the tempting identification and refer to $\alpha$-full sets as being subsets of the set of orbits of $G_a$. A convex $\alpha$-full set $A$ is a union of orbitals and is a convex subset of the $po$-set of orbitals of $G_a$. 


Now we extend the concept of pairings to $\alpha$-full sets. If $\Delta$ is $\alpha$-full, we define $\Delta'$ to be $\{ag | \alpha \in \Delta g\} = \bigcup \{\Gamma' | \Gamma$ is an orbit of $G_\alpha$ and $\Gamma' \subseteq \Delta\}$. If $\{\Delta_i | i \in I\}$ is any family of $\alpha$-full sets, then $\bigcup \{\Delta_i | i \in I\}$ is $\alpha$-full and is paired with $\bigcup \{\Delta'_i | i \in I\}$; and similarly for intersections. If $\Delta^\alpha = \Delta$, we say $\Delta$ is symmetric with respect to $\alpha$.

**Proposition 5.** If $\Delta$ is an $\alpha$-full set, then $\text{Conv}(\Delta)$ is $\alpha$-full and $[\text{Conv}(\Delta)]' = \text{Conv}(\Delta')$. If $\Delta$ is already convex, so is $\Delta'$. If $\Delta$ is symmetric with respect to $\alpha$, so is $\text{Conv}(\Delta)$.

**Proof.** $\Delta \rightarrow \Delta'$ is an $\alpha$-anti-automorphism.

Since an orbital $\Delta$ of $G_\alpha$ is always $\alpha$-full, the last proposition implies that $\Delta'$ is also an orbital, and that it contains precisely those orbits paired with orbits contained in $\Delta$.

**Theorem 6.** Proposition 4 holds for orbitals of $G_\alpha$.

If $\beta G_\alpha = \{\beta\}$, $\beta$ is said to be a fixed point of $G_\alpha$. If not, $\beta G_\alpha$ is a long orbit of $G_\alpha$ and $\text{Conv}(\beta G_\alpha)$ a long orbital. Unless it is trivially ordered, a long orbit(al) must be infinite. We make six definitions:

- $\text{FxG}_\alpha = \{\beta \in \Omega | \beta$ is a fixed point of $G_\alpha\}$.
- $\text{SFxG}_\alpha = \{\beta \in \Omega | \beta, \beta' \in \text{FxG}_\alpha\}$.
- $\text{WFxG}_\alpha = \{\beta \in \Omega | \beta \in \text{FxG}_\alpha$, but $\beta'$ is a long orbit\}$.
- $\text{LnG}_\alpha = \{\Delta \subseteq \Omega | \Delta$ is a long orbit of $G_\alpha\}$.
- $\text{SLnG}_\alpha = \{\Delta \subseteq \Omega | \Delta, \Delta' \in \text{LnG}_\alpha\}$.
- $\text{WLnG}_\alpha = \{\Delta \subseteq \Omega | \Delta \in \text{LnG}_\alpha$, but $\Delta'$ is a fixed point\}$.

Points in $\text{FxG}_\alpha$ will be called strongly fixed; points in $\text{WFxG}_\alpha$, weakly fixed; orbits in $\text{SLnG}_\alpha$, strongly long; and orbits in $\text{WLnG}_\alpha$, weakly long. $XG_\alpha$ will be a variable which can take on as values each of these six sets. Each $XG_\alpha$ is $\alpha$-full and thus may be thought of either as a subset of the set of orbits of $G_\alpha$ or as a subset of $\Omega$. Clearly $\Omega$ is partitioned by $\text{FxG}_\alpha$ and $\text{LnG}_\alpha$. In turn, $\text{FxG}_\alpha$ is partitioned by $\text{SFxG}_\alpha$ and $\text{WFxG}_\alpha$; and $\text{LnG}_\alpha$, by $\text{SLnG}_\alpha$ and $\text{WLnG}_\alpha$. $\text{SFxG}_\alpha$ and $\text{SLnG}_\alpha$ are self-paired; and $\text{WFxG}_\alpha$ is paired with $\text{WLnG}_\alpha$.

**Proposition 7.** $\beta \in \text{SFG}_\alpha$ iff $G_\beta = G_\alpha$.

$\beta \in \text{WFxG}_\alpha$ iff $G_\beta \supset G_\alpha$.

$\beta \in \text{WLnG}_\alpha$ iff $G_\beta \subset G_\alpha$.

$\beta \in \text{SLnG}_\alpha$ iff $G_\beta \not\subset G_\alpha$ and $G_\beta \not\supset G_\alpha$.

**Proof.** Clearly $\beta \in \text{FxG}_\alpha$ iff $G_\beta \supset G_\alpha$. Pick $g \in G$ such that $\beta g = \alpha$ and thus $\alpha g \in (\beta G_\alpha)'$. Then $G_\beta \subset G_\alpha$ iff $\alpha \in \text{FxG}_\beta$ iff $\alpha g \in \text{FxG}_\alpha$ iff $(\beta G_\alpha)'$ is a fixed point of $G_\alpha$. The proposition follows.
We shall say that $G$ is balanced if $WFxG_a$ is the empty set • (iff $WLnG_a = \emptyset$ iff $SFxG_a = FxG_a$ iff $SLnG_a = LnG_a$). By Proposition 7, $G$ fails to be balanced iff $G_a$ is properly contained in one of its conjugates. It follows that finite groups are balanced; in fact, paired orbits have equal cardinalities [18, Theorem 16.3]. Examples can be constructed of $l$-permutation groups $(G, \Omega)$, $\Omega$ totally ordered, which are not balanced.

Proposition 5 yields

**PROPOSITION 8.** Any orbit of $G_a$ which is not strongly long is convex. Hence if two different orbits of $G_a$ lie in the same orbital of $G_a$, both are strongly long.

We now apply the $XG_a$ terminology to orbitals of $G_a$, being assured that an orbital Conv$(\Delta)$ is contained in that $XG_a$ containing the orbit $\Delta$.

The $\alpha$-configuration of $G$ is defined to be the po-set (o-set if $\Omega$ is totally ordered) of orbitals of $G_a$, partitioned into $SFxG_a$, $WFxG_a$, $SLnG_a$, and $WLnG_a$, with the point $\alpha$ distinguished; together with the involution $\Delta \rightarrow \Delta'$. $\alpha$ is called the origin. (Actually, the $\alpha$-configuration is completely determined by the po-set of orbitals, the subset of fixed points, the origin, and the involution.) We want to show that this configuration is actually independent of $\alpha$. By an o-isomorphism from the $\alpha$-configuration of $(G, \Omega)$ onto the $\beta$-configuration of $(K, \Sigma)$, we mean a po-set isomorphism $\psi$ from the po-set orbitals of $G_a$ onto that of $K$, such that $(XG_a)\psi = XK_\beta$ for each $XG_a$, $\{\alpha\}\psi = \{\beta\}$, and $(\Delta\psi)^\beta = (\Delta')^\beta \psi$ for all orbitals $\Delta$ of $G_a$. When there is such an o-isomorphism, we shall say that the two configurations are "the same configuration".

For any $f \in G$, an o-automorphism of $(G, \Omega)$ is provided by $\theta_a$, defined by $\omega \theta_a = \omega f$, and $\theta_a$, defined by $g \theta_a = f^{-1} g f$. Hence the map $\Delta \rightarrow \Delta f$ is an o-isomorphism from the $\alpha$-configuration onto the $\beta$-configuration. Moreover, if $\alpha f_1 = \alpha f_2$, with $f_1, f_2 \in G$, then $f_1 f_2^{-1} \in G_a$, so for each $\alpha$-full set $\Delta$, $\Delta f_1 f_2^{-1} = \Delta$ and thus $\Delta f_1 = \Delta f_2$. This proves the fundamental

**THEOREM 9.** Let $G$ be a coherent subgroup of $A(\Omega)$. Let $\alpha, \beta \in \Omega$ and pick $f \in G$ such that $\alpha f = \beta$. Then $\Delta \rightarrow \Delta f$ furnishes a canonical o-isomorphism (independent of the choice of $f$) from the $\alpha$-configuration onto the $\beta$-configuration. The canonical o-isomorphism from the $\alpha$-configuration onto the $\beta$-configuration, followed by that from the $\beta$-configuration onto the $\gamma$-configuration, yields the canonical o-isomorphism from the $\alpha$-configuration onto the $\gamma$-configuration.
Hence we may speak of the configuration of $G$ without reference to a particular point of $\Omega$. Obviously if two $o$-permutation groups are $o$-isomorphic, they have the same configuration. Of course we can state a similar definition of configuration in terms of orbits rather than orbitals. Two groups having the same orbit configurations necessarily have the same orbital configurations; but not conversely (Examples 2 and 3). However, the orbit configuration is determined by the orbital configuration together with the number of orbits in each orbital. When we speak of configurations, we shall mean orbital configurations unless specified otherwise.

Two distinct points $\beta < \gamma$ of $\Omega$ have three possible relationships: $\beta < \gamma$, $\beta > \gamma$, and $\beta$ incomparable with $\gamma$. $G$ is $o$-2-transitive if for any $\beta, \gamma, \sigma, \tau \in \Omega$ such that $\beta$ and $\gamma$ are related in the same way as are $\sigma$ and $\tau$, there exists $g \in G$ such that $\beta g = \sigma$ and $\gamma g = \tau$. If $G$ is $o$-2-transitive, $G$ must have precisely one positive orbit and precisely one negative orbit (unless $\Omega$ is trivially ordered); and precisely one incomparable orbit (unless $\Omega$ is totally ordered). Conversely, it is easy to see that if $G$ has such a configuration, $G$ is $o$-2-transitive. Thus $o$-2-transitive groups can be characterized in terms of orbit configurations; though not in terms of orbital configurations (Example 3), except among the class of $l$-permutation groups.

We shall be interested also in those groups whose orbital configurations are the same as the orbit configurations described above for $o$-2-transitive groups. These groups are characterized by the property that for any $\beta, \gamma, \sigma, \tau \in \Omega$ such that $\beta$ and $\gamma$ are related as are $\sigma$ and $\tau$, there exists $g_1 \in G$ such that $\beta g_1 = \sigma$ and $\gamma g_1 \leq \tau$; and $g_2 \in G$ such that $\beta g_2 = \sigma$ and $\gamma g_2 \geq \tau$. Such groups will be called $o$-2-semitransitive. An $o$-2-semitransitive $l$-permutation group is automatically $o$-2-transitive.

The regular groups can of course be characterized as those whose configurations consist entirely of (strongly) fixed points.

Groups lying between the extremes of $o$-2-transitivity and regularity can be found among the examples at the end of the paper. See especially Examples 5 and 8. When $\Omega$ is totally ordered, the $o$-anti-isomorphism $\Delta \to \Delta'$ reduce the problem of determining the $o$-set of all orbitals to that of determining the $o$-set of positive orbitals. It can be shown that every $o$-set occurs as the $o$-set of positive orbitals for some transitive $(A(\Omega), \Omega)$.

If $\Delta$ is an orbit of $G_{\alpha}$, the canonically corresponding orbit of $G_{\beta}$ will be denoted by $\Delta_{\beta}$. In particular, $\Delta_{\alpha} = \Delta$. $\Delta_{\beta}$ is to be thought of as “the $\Delta$ orbit of $G_{\beta}$”. Of course, $(\Delta_{\alpha})f = \Delta_{\alpha f}$. Since $\Delta \to \Delta f$ also yields a a canonical isomorphism from the set of $\alpha$-full sets onto the set of $(\alpha f)$-full sets, we may apply the same notation to $\alpha$-full sets $\Delta_{\alpha}$, and in particular to orbitals of $G_{\alpha}$. 
PROPOSITION 10. If $\alpha g \in SFxG_\alpha$, $g \in G$, then for each orbit(al) $\Delta$ of $G_\alpha$, $\Delta g$ is another orbit(al) of $G_\alpha$, and it lies in the same $XG_\alpha$ as $\Delta$.


4. O-blocks. By o-block of an o-permutation group $(G, \Omega)$, we mean a convex subset $\square \neq \Delta \subseteq \Omega$ having the property that for any $g \in G$, $\Delta g = \Delta$ or $\Delta g \cap \Delta = \square$. If the convexity requirement is removed, one has simply a block as defined in [18, §6]. Of course, these two concepts coincide when $\Omega$ is trivially ordered. The intersection of any collection of o-blocks is an o-block (provided it is not empty) and the union of any tower of o-blocks is an o-block. If $\Delta$ is an o-block, the o-block system $\vec{\Delta}$ is the po-set (o-set if $\Omega$ is totally ordered) of translates $\Delta g (g \in G)$ of $\Delta$. Since $G$ is transitive, the o-block systems of $G$ correspond to the convex $G$-congruences, where a $G$-congruence is said to be convex if its congruence classes are convex.

We partially order the blocks containing $\alpha$ by inclusion, obtaining a complete lattice, of which the o-blocks containing $\alpha$ form a complete sublattice; and similarly for the subgroups of $G$ containing $G_\alpha$.

THEOREM 11. Let $(G, \Omega)$ be a coherent o-permutation group. In the well known o-correspondence $\Delta \rightarrow \{g \in G\mid \Delta g = \Delta\}$ and $C \rightarrow \alpha C$ between the lattice of blocks containing $\alpha$ and the lattice of subgroups containing $G_\alpha$, the convex subgroups $C$ correspond precisely to the o-blocks $\Delta$.

Proof. Clearly if $\Delta$ is convex, $\{g \in G\mid \Delta g = \Delta\}$ is convex. Now assume that $C$ is convex. Suppose $\alpha c \leq \beta \leq \alpha d$, $c, d \in C$. Pick $f \in G$ such that $\alpha f = \beta$. Use coherence to pick $s \in G$ such that $\alpha s = \alpha d$ and $f \leq s$. Since $d \in C$ and $sd^{-1} \in G_\alpha \subseteq C$, $s \in C$. Similarly, pick $t \in C$ such that $t \leq f$. Since $C$ is convex, $t \leq f \leq s$ implies $f \in C$, so that $\beta = \alpha s = \alpha f \in \alpha C$. Therefore $\alpha C$ is convex. This result fails without coherence (Example 7).

We may make a complete lattice of the set of block systems of $G$ by defining $\vec{\Gamma} \leq \vec{\Delta}$ iff $\Gamma \subseteq \Delta$, where $\Gamma$ and $\Delta$ are the blocks in $\vec{\Gamma}$ and $\vec{\Delta}$ which contain $\alpha$. Obviously the definition is independent of the choice of $\alpha$. The set of o-block systems forms a complete sublattice. It is proved in [8, Theorem 3] that if $\Omega$ is totally ordered, the lattice of o-block systems is also totally ordered. Thus Theorem 11 gives us

COROLLARY 12. The convex subgroups of $G$ which contain $G_\alpha$ are totally ordered under inclusion.

For the special case of $l$-permutation groups, this was proved by Holland [5]. His result mentioned only the convex prime $l$-subgroups
containing \( G_\alpha \), but since \( G_\alpha \) is prime, every subgroup containing it must automatically be a prime \( l \)-subgroup, and thus the two results coincide.

**Proposition 13.** A block \( \Delta \) of \( G \) which contains \( \alpha \) must be \( \alpha \)-full and symmetric with respect to \( \alpha \).

**Theorem 14.** Let \( G \) be a coherent subgroup of \( A(\Omega) \), and let \( \Delta = \Delta_\alpha \) be a convex \( \alpha \)-full set. Then \( \Gamma = \{ \beta \in \Omega \mid \Delta_\beta = \Delta_\alpha \} \) is a (symmetric) \( o \)-block of \( G \).

**Proof.** \( C = \{ g \in G \mid \Delta g = \Delta \} \) is a convex subgroup of \( G \) containing \( G_\alpha \). But \( \Gamma = \alpha C \), which is an \( o \)-block of \( G \) by Theorem 11.

It is immediate from the proof of Theorem 14 that even if \( \Delta \) is not convex, \( \Gamma \) is still a block of \( G \). This can also be deduced from the statement of the theorem. For if we throw away the order on \( \Omega \), leaving \( \Omega \) trivially ordered and \( G \) coherent, then \( \Delta \) becomes convex, so by the theorem, \( \Gamma \) is a block of \( G \). Similar remarks apply to many of the theorems to come.

**Theorem 15.** Let \( G \) be a coherent subgroup of \( A(\Omega) \). If \( \Delta \) is an \( \alpha \)-full \( o \)-block of \( G \), then \( \Delta' \) is also an \( (\alpha \)-full) \( o \)-block of \( G \), and \( \{ \beta \in \Omega \mid \Delta_\beta = \Delta_\alpha \} \) is the translate of \( \Delta' \) which contains \( \alpha \).

**Proof.** Let \( \Gamma \) be the \( o \)-block \( \{ \beta \in \Omega \mid \Delta_\beta = \Delta_\alpha \} \). Pick \( f \in G \) such that \( \alpha \in \Delta f \). Then \( \Gamma f \), also an \( o \)-block, is equal to \( \{ \eta \in \Omega \mid \Delta_\eta = \Delta f \} = \{ \eta \in \Omega \mid \alpha \in \Delta_\eta \} \) (because \( \Delta \) is a block) = \( \{ \alpha g \mid \alpha \in \Delta_\alpha g = \Delta g \} = \Delta' \).

**Corollary 16.** Let \( \Delta \) be a weakly long orbit of \( G_\alpha \). Then \( \Delta \) is an \( o \)-block of \( G \). Indeed, if \( \alpha g \neq \alpha, g \in G \), then \( \Delta g \cap \Delta = \emptyset \).

**Proof.** Theorems 15 and 14. Thus for an \( \alpha \)-full \( o \)-block \( \Delta \), \( \Delta' \) need not lie in the same \( o \)-block system as \( \Delta \).

When \( \Omega \) is totally ordered, we may complete \( \Omega \) by Dedekind cuts and consider \( \Omega \) to be a subset of its Dedekind completion \( \Omega \tilde{} \) (without end points). Each \( f \in A(\Omega) \) can be extended to \( f \in A(\Omega \tilde{}) \) by defining \( \tilde{} f \) to be \( \text{sup} \{ \beta f \mid \beta \in \Omega, \beta \leq \tilde{} \omega \} \). \( A(\Omega) \) is an \( l \)-subgroup of \( A(\Omega \tilde{}) \), but in general is not transitive even on \( \Omega \setminus \Omega \). A point \( \tilde{} \omega \in \Omega \tilde{} \) is \( \alpha \)-full if it is fixed by \( G_\alpha \). Equivalently, \( \tilde{} \omega \) is \( \alpha \)-full if it is the sup (inf) of an \( \alpha \)-full segment of \( \Omega \). If \( \tilde{} \omega \in \Omega \), then \( \tilde{} \omega \) is \( \alpha \)-full iff \( \tilde{} \omega \in FxG_\alpha \). For any \( \alpha \)-full point \( \tilde{} \omega_\alpha \), and for any \( g \in G, \tilde{} \omega_\alpha g = \tilde{} \omega_\alpha g \) is the (\( \alpha g \))-full point canonically corresponding to \( \tilde{} \omega_\alpha \).

**Proposition 17.** Suppose that \( \Omega \) is totally ordered and that \( \tilde{} \omega_\alpha \) is
an $\alpha$-full point. Then $\{\beta \in \Omega | \bar{\omega}_\beta = \bar{\omega}_\alpha\}$ is an $\omega$-block of $G$.

Proof. $\{\eta \in \Omega | \eta \leq \bar{\omega}_\alpha\}$ is an $\alpha$-full segment of $\Omega$. Apply Theorem 14.

**Lemma 18.** Suppose $\Omega$ is totally ordered. Let $A$ be an $\alpha$-full set. If $\alpha g \geq \alpha$, then $(\inf A)g \geq \inf A$ and $(\sup A)g \geq \sup A$.

Proof. Pick $1 \leq k \in G$ such that $\alpha k = \alpha g$. Since $A$ is $\alpha$-full, $\Delta g = \Delta k$.

It is easily checked that

**Lemma 19 ([7, Lemma 3]).** Let $\alpha \in A \subseteq \Omega$. Suppose that $\Delta g = A$ for each $g \in G$ such that $\alpha g \in \Delta$. Then $\Delta$ is a block of $G$.

**Lemma 20.** Suppose that $\alpha \in A \subseteq \Omega$, $\Omega$ totally ordered, and that $A$ is convex, $\alpha$-full, and symmetric with respect to $\alpha$. Let $\Pi$ be any cofinal subset of $A$. Then $\Delta$ is an $\omega$-block of $G$ provided only that $\alpha g \in \Pi$, $g \in G$, implies $\inf \Delta g \geq \inf A$ and $\sup \Delta g \geq \sup A$.

Proof. By the first lemma, we see first that $\Delta g = \Delta$ when $\alpha \leq \alpha g \in \Pi$; and next that $\Delta g = \Delta$ when $\alpha \leq \alpha g \in \Delta$. In view of the second lemma, the conclusion follows from the symmetry of $\Delta$.

**Theorem 21.** Let $G$ be a coherent subgroup of $A(\Omega)$, $\Omega$ totally ordered. Suppose $G$ has a (long) orbital $\Delta$ cofinal with $\Omega$, so that $\Delta'$ is a (long) orbital coinitial with $\Omega$. Then $\{\beta \in \Omega | \Delta' < \beta < \Delta\}$ is an $\omega$-block of $G$.

Proof. By transitivity, terminal orbitals must be long. Now let $\Pi$ be the $\alpha$-full set $\Gamma = \{\beta \in \Omega | \Delta' < \beta < \Delta\}$ and let $\bar{\alpha} = \sup \Gamma$. We show first that if $\alpha < \alpha g \in \Gamma$, $g \in G$, then $\bar{\alpha} g > \bar{\alpha}$. For suppose $\bar{\alpha} g > \bar{\alpha}$. Pick $h \in G$ such that $\bar{\alpha} h < \alpha$. Since $\Delta$ is cofinal with $\Omega$, we can pick $\delta \in \Delta$ such that $\delta h > \bar{\alpha}$. Now pick $k \in G_\alpha$ such that $(\bar{\alpha} g) k > \delta$. Since $k \in G_\alpha$ and $\Gamma$ is $\alpha$-full, $(\alpha g) k \in \Gamma$, so that $(\alpha g) k \leq \bar{\alpha}$. Since $(\alpha g) h \leq \bar{\alpha} h < \alpha$, we can use coherence to pick $h \leq f \in G$ such that $(\alpha g) f = \alpha$. But $\alpha g k f \geq \bar{\alpha} g k h > \delta h > \bar{\alpha}$, contradicting the fact that $\bar{\alpha}$ is $\alpha$-full. Therefore $\bar{\alpha} g > \bar{\alpha}$ when $\alpha < \alpha g \in \Gamma$. Similarly, $(\inf \Gamma) f < \inf \Gamma$ when $\alpha > \alpha f \in \Gamma$, and thus since $\Gamma$ is symmetric, $(\inf \Gamma) g \geq \inf \Gamma$ when $\alpha < \alpha g \in \Gamma$. By the last lemma, $\Gamma$ is an $\omega$-block of $G$.

In generalizations of theorems about finite permutation groups, $F \times G_\alpha$ often must be expressed as $SF \times G_\alpha (= F \times G_\alpha$ if $G$ is finite). For example:
THEOREM 22. Let \((G, \Omega)\) be a coherent o-permutation group. Then \(SFxG_a\) is a block of \(G\).

Proof. \(SFxG_a\) is \(\alpha\)-full, so \((SFxG_a)g = SFxG_{ag}\). In view of Proposition 7, this says that \(\{\beta \in \Omega | G_\beta = G_a\}g = \{\gamma \in \Omega | G_\gamma = G_ag\}\), which is equal to \(SFxG_a\) if \(G_ag = G_a\), and does not meet \(SFxG_a\) otherwise.

5. O-primitive groups. Following Holland's definition for \(l\)-groups [7], we define a coherent subgroup \(G\) of \(A(\Omega)\), \(\Omega\) partially ordered, to be o-primitive if \(G\) has no o-blocks except \(\Omega\) and the singletons \(\{\omega\}\). Theorem 11 establishes Holland's result (obtained in essentially the same way) that \(G\) is o-primitive if and only if \(G_a\) is a maximal proper convex subgroup of \(G\). O-permutation groups which are primitive are a fortiori o-primitive. On the other hand, \(A(I)\), \(I\) the integers, is o-primitive, but not primitive.

PROPOSITION 23. Let \((G, \Omega)\) be a coherent o-permutation group, \(\Omega\) totally ordered. If \(G\) is o-2-semitransitive, it is o-primitive. If \(G\) is o-2-transitive, it is primitive.

An o-group \(K\) is Archimedean if for any \(1 < k, f \in K, f < k^n\) for some positive integer \(n\); i.e., if \(K\) contains no proper convex subgroups. \(K\) is Archimedean iff \(K\) is isomorphic as an o-group to an o-subgroup of the additive reals [2, p. 45].

PROPOSITION 24. Suppose that \((G, \Omega)\) is regular, with \(\Omega\) totally ordered. Then \((G, \Omega)\) is o-primitive iff \(G\) is Archimedean.

Proof. By Theorem 11, since \(G_a = \{1\}\).

This proposition almost characterizes the o-primitive regular groups in terms of their configurations. Unfortunately, it is possible for an Archimedean o-group (the rationals) to be isomorphic as an o-set to a non-Archipedean o-group \((Q \times I, Q\) the rationals, \(I\) the integers). This is the reason for the word "almost".

Among o-primitive groups on totally ordered sets \(\Omega\), there are thus two classes which lie at opposite extremes in terms of the amount of movement possible within \(G_a\): the Archimedean regular groups, which we have almost characterized in terms of their configurations; and the o-2-semitransitive groups, which we have completely characterized in terms of their configurations. The remaining o-primitive groups will be discussed in detail in §7. For now, we apply §4 to o-primitive groups in general.

If \(A \subseteq \Omega\) and \(\beta, \gamma \in \Omega\), we say that \(\beta\) and \(\gamma\) can be separated by
A if some translate \(\Delta g (g \in G)\) of \(A\) contains precisely one of \(\beta\) and \(\gamma\). An orbit \(\bar{\omega} G\) of \(G\) is dense in \(\bar{\Omega}\) if it meets every nontrivial segment of \(\bar{\Omega}\). Of course, \(\bar{\omega} G = \Omega\) if \(\bar{\omega} \in \Omega\), and \(\bar{\omega} G \cap \Omega = \emptyset\) if \(\bar{\omega} \in \bar{\Omega} \setminus \Omega\).

**Theorem 25.** Let \((G, \Omega)\) be a coherent \(o\)-permutation group. The following are equivalent (except that if \(\Omega\) is not totally ordered, only the first three make sense):

(i) \(G\) is \(o\)-primitive.

(ii) For every segment \(\emptyset \neq A \subset \Omega\), any \(\beta \neq \gamma \in A\) can be separated by \(A\).

(iii) For every \(\alpha\)-full segment \(\emptyset \neq A_a \subset \Omega\), \(A_b \neq A_t\) for \(\beta \neq \gamma (\alpha, \beta, \gamma \in \Omega)\).

(iv) For every \(\alpha\)-full point \(\bar{\omega}_a \in \bar{\Omega}\), \(\bar{\omega}_b \neq \bar{\omega}_t\) for \(\beta \neq \gamma (\alpha, \beta, \gamma \in \Omega)\).

(v) For every \(\bar{\omega} \in \bar{\Omega}\), \(\bar{\omega} G\) is dense in \(\bar{\Omega}\).

Proof. It is clear that each of these conditions implies (i). Now suppose that \(G\) is \(o\)-primitive. If \(A\) is a segment, \(\emptyset \neq A \subset \Omega\), then a convex \(G\)-congruence is given by the relation \(\beta \equiv \gamma\) iff \(\beta\) and \(\gamma\) cannot be separated by \(A\); and since some pairs \(\beta \neq \gamma \in \Omega\) can be separated by \(A\), every pair can, so that (ii) holds. For (v), if \(\bar{\Gamma}\) were a nontrivial segment of \(\bar{\Omega}\) which did not meet \(\bar{\omega} G\), then for \(\beta \neq \gamma \in \bar{\Gamma} \cap \Omega\) and \(A = \{\omega \in \Omega| \omega < \bar{\omega}\}\), \(\beta\) and \(\gamma\) could not be separated by \(A\). For (iii), we use Theorem 14; and for (iv), Proposition 17. For \(\Omega\) totally ordered and \(G\) an \(l\)-subgroup of \(A(\Omega)\), the equivalence of (i), (ii), and (v) was shown by Holland [7, Theorem 2]. For \(\Omega\) trivially ordered, the equivalence of (i) and (ii) was shown by Wielandt [17, Theorem 7.12].

**Theorem 26.** Let \((G, \Omega)\) be \(o\)-primitive. Then \(G\) is balanced and \(F x G_a\) is a block of \(G\).

Proof. Since weakly long orbits are \(o\)-blocks, \(G\) is balanced, so \(F x G_a = SF x G_a\) is a block.

6. Centralizers. In Example 8, the map \(z: \Omega \rightarrow \Omega\) given by \(\beta z = \beta + 1\) lies in the centralizer \(Z_{A(\Omega)} G\) of \(G\) in \(A(\Omega)\). This phenomenon will be of paramount importance in the study of \(o\)-primitive groups. Accordingly, we devote this section to the study of centralizers.

When \(\Omega\) is totally ordered, we shall be interested also in the centralizer of \(G\) in \(A(\bar{\Omega})\). We define \(\bar{F} x G_a = \{\bar{\omega} \in \bar{\Omega}| \bar{\omega} G_a = \bar{\omega}\} = \{\bar{\omega} \in \bar{\Omega}| G_a \supseteq G_a\}\) and \(\bar{S} F x G_a = \{\bar{\omega} \in \bar{\Omega}| \bar{\omega} G_a = \bar{\omega}\) and \(\alpha G_a = \alpha\} = \{\bar{\omega} \in \bar{\Omega}| G_a = G_a\}\). Points in these two sets are \(\alpha\)-full. By Proposition 7, \(\bar{F} x G_a \cap \Omega = F x G_a\) and \(\bar{S} F x G_a \cap \Omega = SF x G_a\). In the two lemmas which follow, if \(\Omega\) is not totally ordered, one replaces \(\bar{\Omega}\) by \(\Omega\), \(\bar{F} x F_a\) by \(F x G_a\),
and $\tilde{SF}xG_\alpha$ by $SFxG_\alpha$.

**Lemma 27.** Let $z: \Omega \rightarrow \bar{\Omega}$ be a function which centralizes $G$, and let $\bar{\omega}_\alpha = az$. Then $\bar{\omega}_\alpha \in FxG_\alpha$, and for all $\beta \in \Omega$, $\beta z = \bar{\omega}_\beta$. If $z$ is one-to-one, $\bar{\omega}_\alpha \in \tilde{SF}xG_\alpha$.

Proof. For any $g \in G$, $\alpha z g = \alpha g z$; so that $\alpha z \in FxG_\alpha$, and $\alpha z \in \tilde{SF}xG_\alpha$ if $z$ is one-to-one. Now let $\beta \in \Omega$ and pick $k \in G$ such that $\alpha k = \beta$. Then $\beta z = \alpha z k = \omega_\alpha k = \omega_k = \omega_\beta$.

**Corollary 28.** $Z_{S(\Omega)} G = Z_{A(\Omega)} G$, where $S(\Omega)$ is the symmetric group on $\Omega$.

Proof. If $\bar{\omega}_\alpha \in \tilde{F}xG_\alpha$, then for any $\alpha \leq \beta \in \Omega$, $\bar{\omega}_\alpha \leq \bar{\omega}_\beta$ by coherence.

**Lemma 29.** Let $\bar{\omega}_\alpha \in \tilde{F}xG_\alpha$. Define $z: \bar{\Omega} \rightarrow \bar{\Omega}$ by setting $\beta z = \bar{\omega}_\beta$ for $\beta \in \bar{\Omega}$, and $\bar{\gamma} z = \sup \{ \beta z | \beta \leq \bar{\gamma} \}$ for $\bar{\gamma} \in \bar{\Omega}$. Then $z$ centralizes $G$. If $\bar{\omega}_\alpha \in \tilde{SF}xG_\alpha$, $z$ is one-to-one.

Proof. For $g \in G$, $\beta \in \Omega$, $\beta g z = \bar{\omega}_\beta g = \bar{\omega}_\beta g = \beta z g$. It follows that $\bar{\gamma} z g = \bar{\gamma} z g$ for $\bar{\gamma} \in \bar{\Omega}$. If $\bar{\omega}_\alpha \in \tilde{SF}xG_\alpha$, $z$ is one-to-one on $\Omega$ and hence on $\bar{\Omega}$.

For finite permutation groups, Kuhn [9] established a correspondence between $Z_{S(\Omega)} G$ and $FxG_\alpha$. Again $FxG_\alpha$ must be expressed as $SFxG_\alpha$.

**Theorem 30.** Let $G$ be a coherent subgroup of $A(\Omega)$ and let $Z = Z_{A(\Omega)} G = Z_{S(\Omega)} G$. If $z \in Z$ and if $\omega_\alpha = \alpha z \in SFxG_\alpha$, then $\beta z = \omega_\beta$ for all $\beta \in \Omega$. Conversely, if $\omega_\alpha \in SFxG_\alpha$ and if $z: \Omega \rightarrow \Omega$ is defined by setting $\beta z = \omega_\beta$ for $\beta \in \Omega$, then $z \in Z$. $Z$ is a po-group and $z \leftrightarrow \alpha z$ gives an o-isomorphism between the po-set $Z$ and the po-set $SFxG_\alpha$.

**Corollary 31.** The po-sets which occur as $SFxG_\alpha$ for coherent o-permutation groups $(G, \Omega)$ are precisely those po-sets which are carriers of po-groups. The o-sets which occur in this way with $\Omega$ totally ordered are those which are carriers of o-groups.

Proof. Theorem 30 and Corollary 2.

**Theorem 32.** Let $G$ be a coherent subgroup of $A(\Omega)$, $\Omega$ totally ordered. Let $\alpha < \omega_\alpha \in SFxG_\alpha$ and let $z \in Z_{A(\Omega)} G$ be defined by $\beta z = \omega_\beta$, $\beta \in \Omega$. For $\gamma \in \Omega$, $B(\gamma, \omega_\gamma) = \text{Conv} \{ \gamma z^i | i \in I \}$, $I$ the integers, is the smallest o-block of $G$ containing $\gamma$ and $\omega_\gamma$, and the collection of $B(\gamma, \omega_\gamma)$'s forms an o-block system of $G$. Since $(\delta z)g = (\delta g)z$ for
$g \in G, \delta \in \Omega$, the action of $g$ on $B(\gamma, \omega_i)$ is determined by its action on $(\gamma, \omega_i)$, and we shall say that $z$ is a period of $G$.

**Proof.** If $g \in G$ is such that $\gamma g = \gamma z^i$ for some $i$, then for any $j$, $(\gamma z^j)g = \gamma g z^j = \gamma z^{i+j}$. Apply Lemma 20 to show that $B(\gamma, \omega_i)$ is an $o$-block of $G$. The rest is clear.

**THEOREM 33.** Let $(G, \Omega)$ be $o$-primitive, $\Omega$ totally ordered, and let $Z = Z_{A(\Omega)}G$. Let $z \in Z$ and let $\bar{\omega}_a = \alpha z \in FxG_a = SFxG_a$. Then for $\beta \in \Omega$, $\beta z = \bar{\omega}_\beta$; and for $\bar{\gamma} \in \bar{\Omega}$, $\bar{\gamma}z = \sup \{\beta z | \beta \in \Omega, \beta \leq \bar{\gamma}\}$. Conversely, if $\bar{\omega}_a \in FxG_a$ and if $z$ is defined by $\beta z = \bar{\omega}_\beta$ for $\beta \in \Omega$ and $\bar{\gamma}z = \sup \{\beta z | \beta \in \Omega, \beta \leq \bar{\gamma}\}$ for $\gamma \in \bar{\Omega}$, then $z \in Z$. $Z$ is an $o$-group and $z \leftrightarrow az$ gives an $o$-isomorphism between the $o$-set $Z$ and the $o$-set $FxG_a$.

**Proof.** $FxG_a = SFxG_a$ because $G_a$ is a maximal proper convex subgroup of $G$. If $z \in Z$, then $\Omega z$ is a dense subset of $\bar{\Omega}$ by Theorem 25, so since $z$ preserves order, $\bar{\gamma}z = \sup \{\beta z | \beta \in \Omega, \beta \leq \bar{\gamma}\}$ for $\bar{\gamma} \in \bar{\Omega}$. Conversely, $\beta z = \bar{\omega}_\beta$ maps $\Omega$ one-to-one onto a dense subset of $\bar{\Omega}$, so $\bar{\gamma}z = \sup \{\beta z | \beta \in \Omega, \beta \leq \bar{\gamma}\}$ extends $z$ to an $o$-permutation of $\bar{\Omega}$.

**COROLLARY 34.** If $G$ is $o$-2-semitransitive, $Z_{A(\bar{\Omega})}G$ is trivial. If $G$ is $o$-primitive and regular, $Z_{A(\bar{\Omega})}G$ is isomorphic as an $o$-group to the integers or the reals.

**Proof.** Use the theorem. In the regular case, $G$ is the regular representation of a subgroup of the reals, and every proper Dedekind complete subgroup of the reals is discrete. In the next section we shall deal with the remaining $o$-primitive groups.

**PROPOSITION 35.** For any totally ordered $\Omega$ and any subset $F$ of $A(\Omega)$, $Z_{A(\bar{\Omega})}F$ is a (not necessarily transitive) $l$-subgroup of $A(\Omega)$.

**Proof.** Since an $l$-group is a distributive lattice, if $z_1$ and $z_2$ commute with $f \in F$, then $(z_1 \vee z_2)f = z_1f \vee z_2f = fz_1 \vee fz_2 = f(z_1 \vee z_2)$.

7. Periodically $o$-primitive groups. We assume from now on that $\Omega$ is totally ordered. Earlier we noted that $o$-2-semitransitive groups and Archimedean regular groups are $o$-primitive. Now we assume that $G$ is one of the remaining $o$-primitive groups and prove that it looks strikingly like the group in Example 8.

**LEMMA 36.** $G_a$ has a first positive long orbital $\Delta_\alpha$. $\alpha$ is the only point between $\Delta^\prime$ and $\Delta$. 
**Proof.** Since $G$ is not regular, $G_a$ has a long orbital $A$. Since $G$ is balanced, $A$ may be assumed negative and thus not cofinal with $\Omega$, so that $\bar{\mu} = \sup A \in \bar{\Omega}$. Pick $g \in G$ such that $\alpha \in Ag$ and let $A_1 = \text{Conv}((\bar{\mu}g)G_a)$. Pick an arbitrary $\beta \in \Omega$ such that $\alpha < \beta < \bar{\mu}g$. Since $\bar{\mu}G$ is dense in $\bar{\Omega}$ by Theorem 25, we may pick $h \in G$ such that $(\bar{\mu}^{-1}h)G \subseteq \alpha \wedge \beta$ and thus $\alpha h^{-1} \in A$. Since also $\alpha g^{-1} \in A$, we may pick $k \in G_a$ such that $(\alpha g^{-1}h)k \geq \alpha h^{-1}$. Now $\alpha (g^{-1}kh) \geq \alpha$, but $(\bar{\mu}g)^{-1}kh \leq \bar{\mu}kh = \bar{\mu}h$ (since $\bar{\mu}$ is $\alpha$-full) $\leq \beta$. Finally, we pick $\bar{\omega} = \sup A \in \Omega$, so that $\mu - \sup Ae\Omega$. Pick $g \in G$ such that $\alpha \in A_g$ and let $A_1 = \text{Conv}((\bar{\mu}g)G_a)$. Pick an arbitrary $\beta \in \Omega$ such that $\alpha < \beta < \bar{\mu}g$. Since $\bar{\mu}G$ is dense in $\bar{\Omega}$ by Theorem 25, we may pick $h \in G$ such that $(\bar{\mu}^{-1}h)G \subseteq \alpha \wedge \beta$ and thus $\alpha h^{-1} \in A$. Since also $\alpha g^{-1} \in A$, we may pick $k \in G_a$ such that $(\alpha g^{-1}h)k \geq \alpha h^{-1}$.

Let us define $\bar{\omega} = \bar{\omega}_a \in \bar{F}xG_a$ to be $\sup A_1$. ($A_1$ is bounded above in $\Omega$ because $G$ is not $o$-2-semitransitive.) Let $z \in Z A(\bar{\omega})G$ be the $o$-permutation of $\bar{\Omega}$ associated with $\bar{\omega}_a$ by Theorem 33. For each integer $k$, we define $\bar{\omega}_k$ to be $\alpha z^k$. In particular, $\bar{\omega}_0 = \alpha$ and $\bar{\omega}_1 = \bar{\omega}$. We define $A_k$ to be $(\bar{\omega}_{k-1},\bar{\omega}_k) \subseteq \Omega$, so that $\bar{A}_k = \bar{A}_k z^{k-1}$. ($\bar{A}_k$ does not include $\bar{\omega}_{k-1}$ or $\bar{\omega}_k$). The new definition of $A_1$ agrees with the old. Since $G$ has period $z$ and since the orbitals of $G_a$ are convex, the fact that $A_k$ is an orbital of $G_a$ implies that each $A_k$ is an orbital of $G_a$. Thus for $k > 0$, $A_k$ is the $k^{th}$ positive long orbital; and $A_{-1}$ is the $k + 1^{st}$ long orbital to the left of $\alpha$. Since $G$ is balanced, $A_k$ is paired with $A_{-1-k}$. Between $A_k$ and $A_{k+1}$ lies precisely one point of $\bar{\Omega}$, namely $\bar{\omega}_k$. If $\bar{\omega}_k \in \Omega$, then $\bar{\omega}_k \in FxG_a(= SFxG_a)$.

**Lemma 37.** For any integers $n$ and $k$ and any $g \in G$, $\alpha g \in A_n$ implies $\bar{\omega}_k g \in A_{k+n}$.

**Proof.** $\bar{\omega}_k g = \alpha z^k g = \alpha z^k \in A_n \subseteq A_{k+n}$.

**Corollary 38.** $\text{Conv} \{A_k \mid k \text{ an integer} \} = \Omega$.

**Proof.** By Lemma 20, this set is an $o$-block of the $o$-primitive group $G$.

**Lemma 39.** Suppose that some $\bar{\omega}_i \in \Omega(i \neq 0)$. Let $n$ be the least positive integer such that $\bar{\omega}_n \in \Omega$. Then $\bar{\omega}_k \in \Omega$ iff $k$ is a multiple of $n$.

**Proof.** $\bar{\omega}_n$ is the least positive point in the symmetric set $SFxG_a$. Proposition 10 guarantees first that if $k$ is a multiple of $n$, $\bar{\omega}_k \in \Omega$; and then the converse.

Recapitulating, the (strongly) long orbitals $A_k$ of $G_a$ form a set
o-isomorphic to the integers; and denoting sup $\Delta_k$ by $\omega_k$, so that $\omega_\infty = \alpha$, either the (strongly) fixed points of $G_\alpha$ are precisely those $\omega_k$'s such that $k$ is a multiple of some fixed positive integer $n$, in which case we say that $G$ has $\text{Config}(n)$, or $\alpha$ is the only fixed point of $G_\alpha$, in which case we say that $G$ has $\text{Config}(\infty)$.

\[\alpha\]

**Main Theorem 40.** Suppose that $G$ is a coherent subgroup of $A(\Omega)$, $\Omega$ totally ordered, and that $G$ is o-primitive, but not o-2-semitransitive or regular. Then for some $n = 1, 2, \cdots, \infty$, $G$ has $\text{Config}(n)$. $Z_{A(\overline{\Omega})}G$ is cyclic, having as a generator the o-permutation $z$ of $\overline{\Omega}$ defined by $\beta z = (\omega_\beta)^{\beta}$, for $\beta \in \Omega$ and $\overline{\tau} z = \sup\{\beta z \mid \beta \in \Omega, \beta \leq \overline{\tau}\}$ for $\overline{\tau} \in \overline{\Omega}$.

We shall say that $z$ is the period of $G$ and that $G$ is periodically o-primitive. $\Delta_{k+1}$ is “one period up” from $\Delta_k$ in the sense that $\Delta_k z = \Delta_{k+1}$. If $G$ has $\text{Config}(n)$ for some finite $n$, $Z_{A(\overline{\Omega})}G$ is cyclic, having as a generator the o-permutation $\tilde{z}$ of $\Omega$ defined by $\beta \tilde{z} = (\omega_\alpha)^{\beta}$, $\beta \in \Omega$; and if $G$ has $\text{Config}(\infty)$, $Z_{A(\overline{\Omega})}G$ is trivial.

A few comments on this theorem are in order. $z$ generates $Z_{A(\overline{\Omega})}G$ by Theorem 33. The fact that $(\tilde{\delta} g) z = (\tilde{\delta} g) z$ for $g \in G, \tilde{\delta} \in \overline{\Omega}$, means that the action of $G$ on $\Omega$ is determined by its action on any interval $(\overline{\tau}, \overline{\tau} z)$, and in particular on any $\Delta_k$. $z$ is analogous to the function $z: \beta \rightarrow \beta + 1$ of Example 8. If $G$ has $\text{Config}(n)$ for some finite $n$ and if $\tilde{z}$ is the period associated with $\omega_\alpha$, then $\tilde{z}$ is nicer than $z$ in that it is in $A(\Omega)$ rather than merely in $A(\overline{\Omega})$, but it suffers the disadvantage of being a larger and ultimately less useful period. In the next section, we shall construct examples of o-primitive groups having all of these configurations. Unfortunately, o-imprimitive groups can also have all of these configurations except $\text{Config}(1)$. What o-blocks might there be containing $\alpha$?

**Proposition 41.** If an o-imprimitive group $G$ has $\text{Config}(n)$, $n$ finite, then for some integer $p, 1 \leq p \leq n/2$, the nontrivial o-blocks of $G$ containing $\alpha$ are precisely the sets $\text{Conv} (\Delta'_k \cup \Delta_k), k = 1, \cdots, p$. If $G$ has $\text{Config}(\infty)$, this result holds for some $p \geq 1$; or else every $\text{Conv} (\Delta'_k \cup \Delta_k)$ is an o-block.

**Proof.** Every nontrivial o-block containing $\alpha$ is symmetric and thus must be of the form $\text{Conv} (\Delta'_k \cup \Delta_k)$ for some $k \geq 1$. If $\text{Conv} (\Delta'_p \cup \Delta_p)$ is an o-block, successive applications of Theorem 21 show that $\text{Conv} (\Delta'_k \cup \Delta_k)$ is an o-block for $k = p - 1, p - 2, \cdots, 1$. By Proposition 10, if $n$ is finite, $\text{Conv} (\Delta'_p \cup \Delta_p)$ cannot be an o-block unless
All of the possibilities not excluded in the proposition do in fact occur for \( o \)-imprimitive \( l \)-permutation groups \((G, \Omega)\).

**Corollary 42.** If \( G \) has Config \((1)\), \( G \) is \( o \)-primitive.

**Corollary 43.** Suppose \( G \) has Config\((n)\) for some \( n = 1, 2, \ldots, \infty \). Then \( G \) is \( o \)-imprimitive iff \( \text{Conv}(A_i \cup A_j) \) is an \( o \)-block of \( G \).

This corollary says that whether \( G \) is periodically \( o \)-primitive is determined by its configuration and knowledge of whether \( \text{Conv}(A_i \cup A_j) \) is an \( o \)-block.

We now investigate the consequences of periodicity. By the support of \( g \in \Omega \) we mean \( \{ \beta \in \Omega | \beta g \neq \beta \} \).

**Corollary 44.** (Holland, [7]). If \( G \) is \( o \)-primitive, but not \( o \)-2-semitransitive, then any \( 1 \neq g \in G \) has support bounded neither above nor below.

**Corollary 45.** (Lloyd, [10]). If \( A(\Omega) \) is \( o \)-primitive, then it is either \( o \)-2-transitive or the regular representation of an Archimedean \( o \)-group.

*Proof.* Clearly \( A(\Omega) \) is not periodic; and the orbits of \( A(\Omega) \) are automatically convex.

An \( l \)-group is \( l \)-simple if it has no proper \( l \)-ideals.

**Corollary 46.** An \( o \)-primitive \( l \)-subgroup \( G \) of \( A(\Omega) \) is \( l \)-simple unless it is \( o \)-2-transitive and contains elements of unbounded support.

*Proof.* Suppose \( G \) is periodically \( o \)-primitive. If \( 1 \neq g \in G \), then every \( \beta \in \overline{\Omega} \) is contained in the support of some conjugate of \( g \) by Theorem 25. Using periodicity, we apply the argument given at the end of [6] to show that \( G \) is \( l \)-simple. If \( G \) is regular, it is an Archimedean \( o \)-group, so it is \( l \)-simple. If \( G \) is \( o \)-2-transitive and contains only elements of bounded support, then \( G \) is \( l \)-simple by the proof of Theorem 6 of [5]. Note that if \( \Omega \) is the reals, \( A(\Omega) \) is \( o \)-2-transitive, but the elements of bounded support form a proper \( l \)-ideal.

An \( o \)-ideal of a \( po \)-group is a normal convex subgroup which is directed. The proof of Corollary 46 also yields

**Corollary 47.** Suppose that \( G \) is an \( o \)-primitive subgroup of \( A(\Omega), \Omega \) totally ordered. Then \( G \) lacks proper \( o \)-ideals unless it is \( o \)-2-semitransitive and contains elements of unbounded support.
Proposition 48. Suppose $G$ (not necessarily o-primitive) has $\text{Config}(n)$, $n$ finite. Then any two orbits $\Delta_j$ and $\Delta_k$ whose subscripts are equal modulo $n$ are o-isomorphic.


Proposition 49. Suppose $G$ is periodically o-primitive. Then all long orbitals of $G_a$ have the same cardinality.

Proof. Let $\Delta_k$ be any long orbital of $G_a$. All proper segments of $\Delta_k$ which are coinitial with $\Delta_k$ have the same cardinality $\aleph_1$; and all which are cofinal have the same cardinality $\aleph_\omega$. Furthermore, these cardinalities are independent of $k$. The proposition follows.

Corollary 50. Suppose that $G$ is periodically o-primitive and that some long orbital of $G_a$ is countable. Then all long orbitals of $G_a$ are o-isomorphic to the rationals and so is $\Omega$.

We can also deduce analogs of several theorems about nonordered permutation groups. For example, if $G$ is a primitive permutation group, $FxG_a = \{a\}$ unless $G$ is regular and $|\Omega|$ is prime [17, Theorem 7.14]. By Theorem 40, this is almost true if $G$ is an o-primitive o-permutation group. Wielandt [17, Theorem 10.13] shows that if a permutation group $G$ is primitive (and if $|\Omega| > \aleph_\omega$), then for every orbit $\Delta \neq \{a\}$ of $G_a$, $|\Delta| + |\Delta'| = |\Omega|$. The proof fails for o-primitive groups, but almost all of the conclusion is given by

Corollary 51. Let $G$ be an o-primitive group. Then for every long orbital $\Delta$ of $G_a$, $|\Delta| + |\Delta'| = |\Omega|$. Except when $G$ is o-2-semitransitive, we can strengthen this to $|\Delta| = |\Omega|$.

Proof. If $G$ is periodically o-primitive, use Proposition 48 and the fact that $G$ has $\text{Config}(n)$. If $G$ is o-2-semitransitive or regular, the conclusion is trivial. It is possible for an o-2-transitive group to have positive and negative orbits of different cardinalities (Example 4).

Wielandt [17, Theorem 10.15] also shows that under somewhat stronger hypotheses, $|\Delta'| = |\Delta|$. This conclusion is given by

Corollary 52. Let $G$ be o-primitive, but not o-2-semitransitive. Then for every orbital $\Delta$ of $G_a$, $|\Delta'| = |\Delta|$.

8. Full periodically o-primitive groups. For any periodically o-primitive group $G$, $G \subseteq \mathbb{Z}_{\Delta}\cap A(\Omega)$. We shall say that $G$ is full if equality obtains. By Proposition 35, a full periodically o-primitive
group $G$ is automatically an $l$-subgroup of $A(\Omega)$ and hence the orbits of $G_\alpha$ are convex.

**Proposition 53.** Every periodically o-primitive $(G, \Omega)$ is contained in a full group $(W, \Omega)$ having the same period $z$.

**Proof.** Take $W = Z_{A(\Omega)} \cap A(\Omega)$.

In order to construct groups having Config($n$), we characterize those o-sets which occur as first positive orbits in periodically o-primitive groups $G$ for which the orbits of $G_\alpha$ are convex. Let $I_n = \{1, \ldots, n\}$ if $n$ is finite; and let $I_n$ be the integers if $n = \infty$. Let $\Sigma_i = A_i \theta^{(i-1)} \subseteq A_i, i \in I_n$. The $\Sigma_i$'s are pairwise disjoint because $\Omega \theta^k \cap \Omega = \emptyset$ for $k = 1, \ldots, n - 1$ (all $k$ if $n = \infty$). Thus

(a) $\bar{A}$ has a collection $\{\Sigma_i | i \in I_n\}$ of dense pairwise disjoint subsets, with $\Sigma_1 = \bar{A}$.

Since for any $h \in G_\alpha, i \in I_n, \Sigma_i h = A_i \theta^{(i-1)} h = A_i \theta^{(i-1)} = \Sigma_i$, we have

(b) $\{f \in A(\bar{A}) | \Sigma_i f = \Sigma_i$ for all $i \in I_n\}$ is transitive on $\bar{A}$.

For $f \in A(\bar{A})$, let $L(f) = \{\delta \in \bar{A} | \delta < f\}$ and $R(f) = \{\delta \in \bar{A} | \delta > f\}$. Suppose $ag \in A_k, g \in G, k \in I_n$. Let $\mu = ag \theta^{(k-1)} \subseteq \Sigma_k$. Let $\nu = \omega g^{-1} (= \omega \theta^{k-n} g^{-1} = \omega g^{-1} \theta^{k-n} \subseteq \Sigma_{k-n}(\theta^{(k-1)})$ if $n$ finite). Since $g \theta^{(k-1)}$ maps $L(\nu)$ onto $R(\mu)$ and $g \theta^{-k}$ maps $R(\nu)$ onto $L(\mu)$, we obtain

(c) For any $\mu$ in any $\Sigma_k, k \in I_n$, there exists $\nu \in \Sigma_{n-k-1}$ if $n$ finite, and $\nu \in \bar{A} \setminus \cup \{\Sigma_i\}$ if $n = \infty$) such that there exists an o-isomorphism $s(\mu, \nu)$ of $L(\nu)$ onto $R(\mu)$ with $(L(\nu) \cap \Sigma_i)s(\mu, \nu) = R(\mu) \cap \Sigma_i$, where $p = j + k - 1$ (mod $n$ if $n$ finite), and there exists an o-isomorphism $t(\mu, \nu)$ of $R(\nu)$ onto $L(\mu)$ with $(R(\nu) \cap \Sigma_i)t(\mu, \nu) = L(\mu) \cap \Sigma_i$, where $q = j + k$ (mod $n$ if $n$ finite).

Sets $A_i$ satisfying these conditions will be discussed in the corollaries of the following theorem. When $n = 1$, these conditions state simply that $A(\bar{A})$ is transitive and that for $\delta \in A_i, \{\beta \in A_i | \beta < \delta\}$ is o-isomorphic to $\{\beta \in A_i | \beta > \delta\}$; or equivalently, that $A_i$ is an open interval of some chain $\Omega$ for which $A(\Omega)$ is o-2-transitive.

**Theorem 54.** The o-sets which occur as first positive orbits in periodically o-primitive groups $G$ which have Config($n$) and for which the orbits of $G_\alpha$ are convex are precisely those o-sets $A_i$ satisfying conditions (a), (b), and (c).

**Proof.** We construct, for any o-set $A_i$ satisfying these conditions, a full periodically o-primitive group $(G, \Omega)$ having $A_i$ as the first positive orbit of $G_\alpha$. As the construction for $n = \infty$ is similar to and simpler than the construction for finite $n$, we shall assume that $n$ is
Let $\Delta_1(=\Sigma_1), \ldots, \Delta_n$ be pairwise disjoint copies of $\Sigma_1, \ldots, \Sigma_n$, and let $A$ be the ordinal sum $\Delta_1 + \cdots + \Delta_n$ with a point $\alpha$ adjoined at the bottom. Let $\Omega$ be $A \times I$, $I$ the integers. For each $i \in I$, let $\Delta_i = \{(\sigma, a) \mid \sigma \in \Delta_i\}$, where $i = an + b$ ($1 \leq b \leq n$). This identifies $A$ with $\{(\lambda, 0) \mid \lambda \in A\}$. Let $\omega_i = \sup \Delta_i$. $\omega_i \in \Omega$ iff $i$ is a multiple of $n$. Define $\hat{z} \in A(\Omega)$ by $(\lambda, a)\hat{z} = (\lambda, a + 1)$. Now pick an $o$-isomorphism $w_i$ from $\Sigma_i$ onto $\Delta_i$, $i = 1, \ldots, n$, with $w_1$ the identity map on $\Delta_i$. Since $\Sigma_i$ is a dense subset of $\Delta_i$, we can extend $w_i$ to an $o$-isomorphism of $\Delta_i$ onto $\bar{\Delta}_i$. We define $z \in A(\bar{\Omega})$ as follows: For $\beta \in \bar{\Delta}_i$, $i = 1, \ldots, n - 1$, $\beta z = \beta w_i^{-1}w_{i+1}$, and for $\beta \in \bar{\Delta}_n$, $\beta z = \omega_n^{-1}\hat{z}$. $\omega_i z = \omega_{i+1}$, $i = 0, \ldots, n - 1$. This defines $z$ on $\bar{\Delta} = [a, \omega_n]$, and we extend it to $\bar{\Omega}$ so that it has $\hat{z}$ as a period, i.e., we define $(\beta z_j)z = (\beta z)_j$ for all $\beta \in [a, \omega_n], j \in I$.

We define $G$ to be $Z_{A(\bar{\Omega})} \cap A(\bar{\Omega})$, an $l$-subgroup of $A(\bar{\Omega})$. First we show that $G$ is transitive on $\omega_i$. It suffices to show that for each $\alpha \neq \lambda \in A$, there exists $g \in G$ such that $ag = \lambda$. $\lambda \in \Delta_k$ for some $k \in \mathcal{P}_n$, so that $\mu = \lambda w_k^{-1} \in \Sigma_k$. Pick $\nu \in \Sigma_{n-(k-1)}$, $s(\mu, \nu)$, and $t(\mu, \nu)$ as in (c). Now we define $g \in G$ as follows: $ag = \lambda$ and $(\nu w_{n-(k-1)})g = \omega_n$. For $\beta \in (L(\bar{\nu}) \cap \Sigma_j)w_j$, $\beta g = \beta w_j^{-1}s(\mu, \nu)w_{j+(k-1)} \in \Delta_{j+(k-1)}$, where if $j + (k-1) > n$, $w_{j+(k-1)} = w_{j+(k-1)-n}$. For $\beta \in (R(\bar{\nu}) \cap \Sigma_j)w_j$, $\beta g = \beta w_j^{-1}t(\mu, \nu)w_{j+k} \in \Delta_{j+k}$. This defines $g$ on $\bar{\Delta} = [a, \omega_n]$, and we extend it to $\bar{\Omega}$ by defining $(\beta z_j)g = (\beta g)z_j$ for all $\beta \in [a, \omega_n], j \in I$. Since $w_i^{-1}w_{i+1} = z$ and $z^n = \hat{z}$, we have $g \in G$, establishing the transitivity of $G$.

Each $\omega_j$ is fixed by $G_\alpha$ because for $h \in G_\alpha$, $\omega_j h = \alpha z^j h = \alpha h z^j = \alpha z^j = \omega_j$. By (b), the first positive orbit of $G_\alpha$ is $\Delta_1$, and since $G$ has period $z$, the $j$th positive long orbit of $G_\alpha$ is $\Delta_j$, so that $G$ has $\text{Config}(n)$. By periodicity, no Conv ($\Delta_j \cup \Delta_j$) is an $o$-block of $G$, so $G$ is $o$-primitive, and by construction, it is full.

**Corollary 55.** For each $n = 1, 2, \ldots, \infty$, there is a full periodically $o$-primitive group on the rationals (which are the only countable candidate) having $\text{Config}(n)$.

**Proof.** Let $\Delta_i$ be the rationals, which satisfy conditions (a), (b), and (c). (Take the $\Sigma_i$'s to be distinct cosets of the rationals in the reals). By Corollary 50, $\Omega$ is $o$-isomorphic to the rationals.

**Corollary 56.** Suppose that $\Omega$ is Dedekind complete and that $G$ is a coherent subgroup of $A(\Omega)$. (Do not assume that $G$ is $o$-primitive). Then

1. $G$ is the regular representation of the integers or the reals, or
2. $G$ is $o$-2-semitransitive and $|\Omega| = 2^{\aleph_0}$, or
3. $G$ is periodically $o$-primitive with $\text{Config}(1)$ and $|\Omega| = 2^{\aleph_0}$. $A(\Omega)$ is $o$-2-transitive for uncountably many nonisomorphic Dedekind
complete \( \Omega \)'s; and uncountably many nonisomorphic Dedekind complete \( \Omega \)'s support full periodically o-primitive groups having Config(1).

**Proof.** Since \( \Omega \) is Dedekind complete and nontrivial o-blocks of \( G \) have no sups in \( \Omega \), \( G \) must in fact be o-primitive. If \( g \) is regular, it is Archimedean, so since \( \Omega \) is Dedekind complete, \( G \) must be isomorphic as an o-permutation group to the regular representation of the integers or the reals. If \( G \) has Config(\( n \)) for some \( n \), then \( n = 1 \) because \( \Omega \) is Dedekind complete.

For the statements about cardinality, we appeal to some interesting results of Babcock [1]. Babcock's Theorem 22 states that a Dedekind complete chain, not the integers, which is homogeneous (and thus in its order topology satisfies the first countability axiom by [16, Theorem 1]) has cardinality \( 2^{\aleph_1} \). This finishes (2) and (3). When \( \Omega \) is Dedekind complete, the Config(1) conditions on \( \Delta \) state precisely that \( \Delta^* \) (\( \Delta \) with end points) is Dedekind complete and that any two nontrivial closed subintervals of \( \Delta^* \) are o-isomorphic. Babcock constructs uncountably many o-sets satisfying these conditions [1, p. 2]. Moreover, it can be verified that in this special case, \( \Delta \) is o-isomorphic to \( \Omega \), so we get uncountably many nonisomorphic Dedekind complete \( \Omega \)'s supporting full periodically o-primitive groups having Config(1). Of course, for each of these \( \Omega \)'s, \( A(\Omega) \) is o-2-transitive.

9. Locally o-primitive groups. Following Holland [7], we say when \( \Omega \) is totally ordered that \( G \) is **locally o-primitive** if in the totally ordered set (Theorem 12) of o-block systems of \( G \), there is a minimal nontrivial system \( \tilde{\Delta} \). Certainly o-primitive groups are locally o-primitive. The o-blocks in \( \tilde{\Delta} \) are called the **primitive segments** of \( G \). If \( \Gamma \) is a primitive segment, let \( G|\Gamma \) denote the restriction of \( G \) to \( \Gamma \), i.e., \( \{g|\Gamma : g \in G \text{ and } \Gamma g = \Gamma \} \). Then \( (G|\Gamma, \Gamma) \) is o-primitive. As noted in the introduction, every \( l \)-group can be embedded in a subdirect product of o-permutation groups \( (G_i, \Omega_i) \), with each \( \Omega_i \) totally ordered and \( G_i \) a transitive \( l \)-subgroup of \( A(\Omega_i) \). It can be further arranged that each \( G_i \) is locally o-primitive [7].

If for some (and hence each) primitive segment \( \Gamma \), \( G|\Gamma \) is o-2-semitranseitive (regular, periodically o-primitive), we shall say that \( G \) is **locally o-2-semitranseitive** (regular, periodically o-primitive). For example, the o-imprimitive groups of Proposition 41 are locally o-2-semitranseitive; and if \( \Omega \) is discrete, \( G \) is locally regular with primitive segments o-isomorphic to the integers.

**Theorem 57.** Every locally o-primitive group is locally o-2-semitranseitive, locally regular, or locally periodically o-primitive.
We almost characterize locally o-primitive groups by their configurations with

**THEOREM 58.** If \( G_\alpha \) has a first positive orbital, then \( G \) is locally o-primitive. Conversely, if \( G \) is locally o-primitive, then \( G_\alpha \) has a first positive orbital (unless \( G \) is locally regular and \( \Omega \) is not discrete).

**Proof.** Suppose that \( G_\alpha \) has a first positive orbital \( \Delta \). By Proposition 13, every o-block \( \neq \{ \alpha \} \) of \( G \) which contains \( \alpha \) must contain \( \Delta \). Let \( \Gamma \) be the intersection of all such o-blocks. Since \( \{ \alpha \} \neq \Gamma \), \( \Gamma \) must be a primitive segment of \( G \). Therefore \( G \) is locally o-primitive. The converse follows from the previous theorem.

10. Examples.

**EXAMPLE 1.** Let \( \Omega \) be the reals and let \( G \) be the set of o-permutations of \( \Omega \) having everywhere a strictly positive derivative. \( G \) is an o-2-transitive coherent subgroup of \( A(\Omega) \), but it is not an l-subgroup.

**EXAMPLE 2.** Let \( \Omega \) be the reals and let \( G \) be the linear group \( \{ ax + b | a, b \text{ real, } a > 0 \} \). \( ax + b \) is positive iff \( a = 1 \) and \( b \geq 0 \). Again \( G \) is coherent and o-2-transitive, but not an l-permutation group.

**EXAMPLE 3.** In Example 2, let \( H \) be the coherent subgroup of elements \( ax + b \) of \( G \) for which \( a \) is rational. \( H \) is not o-2-transitive, but is o-2-semitransitive. Although \( H \) is o-primitive, it is not primitive because the rationals form a block of \( H \).

**EXAMPLE 4.** Let \( \omega_1 \) be the first uncountable ordinal; let \( \Sigma \) be the rationals with the usual order; and let \( \Omega \) be the lexicographic product \( \Sigma \times \omega_1 \), ordered from the right, i.e., \( (\sigma_1, \gamma_1) \leq (\sigma_2, \gamma_2) \) iff \( \gamma_1 < \gamma_2 \), or \( \gamma_1 = \gamma_2 \) and \( \sigma_1 \leq \sigma_2 \). \( A(\Omega) \) is o-2-transitive. The negative orbit of \( A(\Omega)_\alpha \) is countable; the positive orbit is not.

**EXAMPLE 5.** Let \( I \) be the integers with the usual order. \( A(I) \) is isomorphic as an o-group to the integers. Let \( (G, \Omega) \) be the ordered wreath product of \( (A(I), I) \) with itself, i.e., \( \Omega = I \times I \) and each \( g \in G \) is given by \( (m, n)g = (m + k_n, n + k) \), where \( k \) depends only on \( g \), but \( k_n \) depends on \( n \) as well as \( g \). In fact, \( G = A(\Omega) \), and the configuration of \( G \) can be obtained by starting with \( I \), replacing one integer by a set of strongly fixed points o-isomorphic to \( I \), replacing each other integer by a strongly long orbit, and establishing the obvious pairings.

**EXAMPLE 6.** Let \( A(\Omega) \) be as in Example 5. Let \( G \) be the coherent subgroup of elements of \( A(\Omega) \) which satisfy
(1) \( k_n = k_p \) if \( n \equiv p \pmod{2} \)

and

(2) \( k_n \equiv k_p \pmod{2} \) even if \( n \equiv p \pmod{2} \).

None of the long orbits of \( G_\alpha \) is convex; indeed, each long orbital of \( G_\alpha \) contains precisely two long orbits. The configuration of \( G \) consists of alternating strongly long orbitals and o-blocks (each o-isomorphic to the integers) of strongly fixed points.

**Example 7.** In Example 6, replace (2) by (2') \( k_n = -k_p \) if \( n \equiv p \pmod{2} \). Then \( G \) is not coherent; indeed no point can be moved to its successor by a positive \( g \in G \). \((G, \Omega)\) is regular, but not o-isomorphic to the right regular representation of \( G \). \( \Delta \to \Delta' \) is not an o-anti-automorphism of the totally ordered set of orbit(al)s of \( G_\alpha \). \( \{(i, 0) | i \text{ even}\} \) is a block \( \Delta \) of \( G \) which is not convex; but \( \{g \in G | (0, 0)g \in \Delta\} \) is trivially ordered and hence is a convex subgroup of \( G \).

**Example 8 (Holland, [6]).** The only previously known example of an o-primitive group which is neither o-2-semitransitive nor regular was as follows: Let \( \Omega \) be the reals and let \( G = \{f \in A(\Omega) | f \text{ has period 1}, i.e., (\beta + 1)f = \beta f + 1 \text{ for all } \beta \in \Omega\} \). The map \( \beta z = \beta + 1 \), lies in the centralizer \( Z_{A(\Omega)} G \) of \( G \) in \( A(\Omega) \), and indeed \( G = \{f \in A(\Omega) | zf = fz\} \). \( G \) is a full periodically o-primitive group having \( \text{Config}(1) \). (See §7). It is shown in [6] that \( G \) is l-simple.

**Example 9.** Let \( G \) be the full periodically o-primitive group of Example 8. Let \( G^{(m)} \) consist of those elements of \( G \) which have \( m^{th} \) derivatives and whose first derivatives are positive everywhere. Then \( G \supset G^{(1)} \supset G^{(2)} \supset \cdots \). Each \( G^{(m)} \) is periodically o-primitive with period 1. The \( G^{(m)} \)'s are not l-subgroups of \( A(\Omega) \) and of course are not full.

**References**


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