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O-PRIMITIVE ORDERED PERMUTATION GROUPS

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Let G be a transitive l-subgroup of the lattice-ordered group $A(\Omega)$ of all order-preserving permutations of a chain Ω. (In fact, many of the results are generalized to partially ordered sets Ω and transitive groups G such that $\beta < \gamma$ implies $\beta g = \gamma$ for some positive $g \in G$, thus encompassing some results on non-ordered permutation groups.) The orbits of any stabilizer subgroup G_α, $\alpha \in \Omega$, are convex and thus can be totally ordered in a natural way. The usual pairing $\Delta \longleftrightarrow \Delta' = \{\alpha g \mid \alpha \in \Delta g\}$ establishes an o-anti-isomorphism between the set of "positive" orbits and the set of "negative" orbits. If Δ is an o-block (convex block) of G for which $\Delta G_\alpha = \Delta$, then Δ' is also an o-block. If G_α has a greatest orbit Γ, then $\{\beta \in \Omega \mid \Gamma' < \beta < \Gamma\}$ constitutes an o-block of G. A correspondence is established between the centralizer $Z_{A(\Omega)}G$ and a certain subset of the fixed points of G_α.

The main theorem states that every o-primitive group (G, Ω) which is not o-2-transitive or regular looks strikingly like the only previously known example, in which Ω is the reals and $G = \{f \in A(\Omega) \mid (\beta + 1)f = \beta f + 1 \text{ for all } \beta \in \Omega\}$. The "configuration" of orbits of G_α must consist of a set o-isomorphic to the integers of "long" (infinite) orbits with some fixed points interspersed; and there must be a "period" $z \in Z_{A(\bar{\Omega})}G$ ($\bar{\Omega}$ the Dedekind completion of Ω) analogous to the map $\beta z = \beta + 1$ in the example. Periodic groups are shown to be l-simple, and more examples of them are constructed.

Transitivity guarantees that the "configuration" of orbits of G_α is independent of α, so that we may speak of the *configuration* of G (defined more precisely later). There is appreciable interplay between this configuration and other properties of G. For example, o-2-transitive groups are characterized by having only one positive orbit, and regular groups by having configurations consisting entirely of fixed points.

For periodically o-primitive groups, the period z is the unique o-permutation of $\bar{\Omega}$ such that for every $\beta \in \Omega$, βz is the sup of the first positive orbit of G_β. $(\beta z)g = (\beta g)z$ for all $\beta \in \Omega$, $g \in G$, and in fact z generates $Z_{A(\bar{\Omega})}G$. This periodicity is of paramount importance. For example, it guarantees that the action of $g \in G$ on any long orbit of G_α determines its action on all of Ω.

Transitive l-subgroups of $A(\Omega)$ have been studied from a lattice-ordered group (l-group) orientation by Holland [5, 6, 7], Lloyd [10, 11], Sik [15], and McCleary [12, 13]. Holland showed that every l-group is l-isomorphic to a subdirect product of transitive l-permutation groups [5]. A nonlattice point of view has beeen taken by Holland and McCleary [8, 14], where it was shown that every transitive ordered permutation group can be embedded in the generalized ordered wreath product of its o-primitive "components" (an important motivation for the present paper); and by G. Higman [4] and Wielandt [17, §6]. The concept of configuration is a refinement of the concept of rank in [3].

The generalization to partially ordered Ω requires very little additional work, but it is less intuitive than the totally ordered case and the reader will not lose much if he assumes that Ω is totally ordered, or even that G is an l-permutation group, as we have done in this introduction.

2. Coherent o-permutation groups. Let Ω be a partially ordered set (po-set) containing more than one point. Points of Ω will be denoted by lower case Greek letters; subsets, by upper case Greek letters; and permutations, by lower case Roman letters. The image of $\beta \in \Omega$ under the permutation f will be denoted by βf, so that if g is also a permutation, $\beta(fg) = (\beta f)g$.

An *order-preserving permutation* (*o-permutation, automorphism*) of Ω is a permutation f such that for $\beta, \gamma \in \Omega$, $\beta < \gamma$ iff $\beta f < \gamma f$. We define $f \leqq g$ iff $\beta f \leqq \beta g$ for all $\beta \in \Omega$, making the group $A(\Omega)$ of all permutations of Ω into a partially ordered group (po-group). If Ω is totally ordered, f is an o-permutation provided only that $\beta < \gamma$ implies $\beta f < \gamma f$. In this case $A(\Omega)$ is an l-group, with $\beta(f \vee g) = \max\{\beta f, \beta g\}$ and $\beta(f \wedge g) = \min\{\beta f, \beta g\}$; and G is said to be an *l-permutation group* if it is an l-subgroup of $A(\Omega)$, i.e., a subgroup which is also a sublattice. Standard results about po-groups and l-groups can be found in [2], but we shall make minimal use of them.

Our o-permutation group G will always be assumed to be a transitive subgroup of $A(\Omega)$ (i.e., $\beta, \gamma \in \Omega$ implies $\beta g = \gamma$ for some $g \in G$). Thus Ω must be homogeneous; and if ordered nontrivially ($\beta < \gamma$ for some $\beta, \gamma \in \Omega$), it must be infinite. Furthermore, we shall always assume that if $\beta < \gamma \in \Omega$, there exists $1 < g \in G$ such that $\beta g = \gamma$. (This property implies its dual, which states that if $\beta > \gamma$, there exists $1 > g \in G$ such that $\beta g = \gamma$; and implies that if $\beta f < \gamma$, $f \in G$, then there exists $g \in G$ such that $\beta g = \gamma$ and $g > f$). Transitive groups that satisfy this property will be called *coherent*. Of course, if Ω is totally ordered, transitivity need not be separately assumed. Transitive l-permutation groups are coherent, for if $\beta < \gamma$ and $\beta g = \gamma$, then also $\beta(g \vee 1) = \gamma$. However, the group in Example 7 is not coherent. If Ω is trivially ordered, $A(\Omega)$ is just the symmetric group $S(\Omega)$, and is itself trivially ordered; and its transitive subgroups are automatically coherent.

B is a *convex subset* (*segment*) of a *po*-set A if $b_1 \leqq a \leqq b_2$, b_1, $b_2 \in B$, $a \in A$ implies $a \in B$. If C and D are any subsets of A, we define $C \leqq D$ iff $c \leqq d$ for some $c \in C$, $d \in D$. If A is totally ordered, and C and D are nonvoid disjoint segments of Ω, then $C < D$ iff $c < d$ for all $c \in C$, $d \in D$.

If (G, Ω) is a transitive (but not necessarily coherent) *o*-permutation group, let $R(G_\alpha)$ designate $\{G_\alpha g \mid g \in G\}$, ordered as above to give the usual partial ordering on the collection of right cosets of a convex subgroup of a *po*-group. As with nonordered transitive permutation groups, we make G act faithfully on $R(G_\alpha)$ by defining $(G_\alpha g) = G_\alpha(gk)$, $g, k \in G$. Here we obtain an *o*-permutation group.

An *o-isomorphism* from one *o*-permutation group (G, Ω) onto another (K, Σ) consists of a *po*-set isomorphism θ_Ω from Ω onto Σ and a *po*-group isomorphism θ_G from G onto K such that for all $\omega \in \Omega$, $g \in G$, $(\omega g)\theta_\Omega = (\omega\theta_\Omega)(g\theta_G)$. The importance of coherence is explained by

THEOREM 1. *Let (G, Ω) be a transitive o-permutation group and let $\alpha \in \Omega$. Then G is coherent if and only if the correspondence $\alpha g \leftrightarrow G_\alpha g$ between Ω and $R(G_\alpha)$ and the identity map on G furnish an o-isomorphism between (G, Ω) and $(G, R(G_\alpha))$.*

Proof. Suppose that G is coherent. $\alpha g_1 = \alpha g_2$ iff $g_1 g_2^{-1} \in G_\alpha$ iff $G_\alpha g_1 = G_\alpha g_2$, so we have a one-to-one correspondence between Ω and $R(G_\alpha)$. $\alpha g_1 \leqq \alpha g_2$ iff $\alpha g_1 k = \alpha g_2$ for some $1 \leqq k \in G$ (by coherence) iff $G_\alpha g_1 k = G_\alpha g_2$ (for some $1 \leqq k \in G$) iff $G_\alpha g_1 \leqq G_\alpha g_2$, so the correspondence is an *o*-isomorphism. For $h \in G$, $(\alpha g)h = \alpha(gh) \leftrightarrow G_\alpha(gh) = (G_\alpha g)h$. This establishes the *o*-permutation group isomorphism. The converse is clear.

G is *regular* if it is transitive and $G_\alpha = \{1\}$.

COROLLARY 2. *Let G be regular. Then G is coherent if and only if (G, Ω) is o-isomorphic to the right regular representation of G. In particular, the right regular representation of G is coherent.*

3. The configuration of an o-permutation group. There will usually be one (arbitrary) point α in Ω on which our attention will be especially focused. The *orbit* of G_α which contains δ is $\delta G_\alpha = \{\delta h \mid h \in G_\alpha\}$. $\alpha G_\alpha = \{\alpha\}$. If δG_α is not trivially ordered, it is infinite. The orbits of G_α partition Ω. In general, the orbits of G_α need not be convex (Examples 3 and 6), although of course they are convex if Ω is trivially ordered. We also have

PROPOSITION 3. *If G is a transitive l-subgroup of $A(\Omega)$, Ω totally ordered, then the orbits of G_α are convex.*

Proof. Suppose $\beta \leqq \gamma \leqq \delta$ and $\beta h = \delta$ for some $h \in G_\alpha$. By transitivity, $\beta g = \gamma$ for some $g \in G$. Let $f = (h \vee 1) \wedge (g \vee 1)$. Then $\beta f = \gamma$. Since $1 \leqq f \leqq h \vee 1 \in G_\alpha$, the convexity of G_α implies that $f \in G_\alpha$.

To escape having to assume that the orbits of G_α are convex, we shall "enlarge" them to convex sets. The *conexification* Conv (Δ) of $\Delta \subseteqq \Omega$ is $\{\xi \in \Omega \mid \delta_1 \leqq \xi \leqq \delta_2$ for some $\delta_1, \delta_2 \in \Delta\}$. If Δ is an orbit of G_α, we shall call Conv (Δ) an *orbital* of G_α. Of course, if the orbits of G_α are convex, the concepts of "orbital" and "orbit" coincide. If Γ is an orbital of G and $\gamma \in \Gamma$, then the orbital Conv (γG_α) of G_α determined by γ is Γ. The orbitals of G_α partition Ω into convex subsets. The set of orbitals of G_α is partially ordered; and is totally ordered if Ω is totally ordered. Two orbits in different orbitals are related as are their orbitals; and two orbits in the same orbital are of course each less than or equal to the other.

Those orbitals of G_α which are strictly greater than $\{\alpha\}$ will be called *positive*; those strictly less than $\{\alpha\}$, *negative*. All points in a positive (negative) orbital are strictly greater than (less than) α. No orbital is both positive and negative; and if Ω is totally ordered, every orbital except $\{\alpha\}$ is one or the other. These remarks apply also to orbits of G_α.

We define for each orbit Δ a *paired orbit* $\Delta' = \Delta'^\alpha = \{\alpha g \mid \alpha \in \Delta g\}$. (The notation Δ' will always refer to pairings with respect to the point denoted by the letter α). It is shown in [18, § 16] that Δ' is indeed an orbit of G_α; that the map $\Delta \to \Delta'$ is one-to-one from the set of orbits of G_α onto itself; and that $\Delta'' = \Delta$. $\alpha g \in \Delta'$ iff $\alpha \in \Delta g$, and if $\alpha \in \Delta g$, then $\Delta' = (\alpha g) G_\alpha$.

PROPOSITION 4. *Let (G, Ω) be a coherent o-permutation group. The map $\Delta \to \Delta'$ is an o-anti-automorphism of the set of orbits of G_α. Since $\{\alpha\}$ is self-paired, the appropriate restriction provides an o-anti-isomorphism from the set of positive orbits of G_α onto the set of negative orbits. If Ω is totally ordered, only $\{\alpha\}$ is self-paired.*

Proof. Use coherence.

A subset Δ of Ω will be called α-full if it contains each orbit of G_α that it meets, i.e., if it is a union of orbits of G_α. Thus the α-full sets are precisely those sets Δ such that $\Delta h = \Delta$ for each $h \in G_\alpha$. We obtain a canonical correspondence between the α-full subsets of Ω and the subsets of the set of orbits of G_α by letting the α-full set Δ correspond to the set of orbits contained in Δ. We shall frequently make the tempting identification and refer to α-full sets as being subsets of the set of orbits of G_α. A convex α-full set Δ is a union of orbitals and is a convex subset of the *po*-set of orbitals of G_α.

Now we extend the concept of pairings to α-full sets. If Δ is α-full, we define Δ' to be $\{\alpha g \mid \alpha \in \Delta g\} = \cup\{\Gamma' \mid \Gamma$ is an orbit of G_α and $\Gamma \subseteqq \Delta\}$. If $\{\Delta_i \mid i \in I\}$ is any family of α-full sets, then $\cup\{\Delta_i \mid i \in I\}$ is α-full and is paired with $\cup\{\Delta_i' \mid i \in I\}$; and similarly for intersections. If $\Delta'^\alpha = \Delta$, we say Δ is *symmetric with respect to* α.

PROPOSITION 5. *If Δ is an α-full set, then* Conv (Δ) *is α-full and* $[\mathrm{Conv}(\Delta)]' = \mathrm{Conv}(\Delta')$. *If Δ is already convex, so is Δ'. If Δ is symmetric with respect to α, so is* Conv (Δ).

Proof. $\Delta \to \Delta'$ is an o-anti-automorphism.

Since an orbital Δ of G_α is always α-full, the last proposition implies that Δ' is also an orbital, and that it contains precisely those orbits paired with orbits contained in Δ.

THEOREM 6. *Proposition 4 holds for orbitals of G_α.*

If $\beta G_\alpha = \{\beta\}$, β is said to be a *fixed point* of G_α. If not, βG_α is a *long orbit* of G_α and Conv (βG_α) a *long orbital.* Unless it is trivially ordered, a long orbit(al) must be infinite. We make six definitions:

$FxG_\alpha = \{\beta \in \Omega \mid \beta$ is a fixed point of $G_\alpha\}$.

$SFxG_\alpha = \{\beta \in \Omega \mid \beta, \beta' \in FxG_\alpha\}$.

$WFxG_\alpha = \{\beta \in \Omega \mid \beta \in FxG_\alpha$, but β' is a long orbit$\}$.

$LnG_\alpha = \{\Delta \subseteqq \Omega \mid \Delta$ is a long orbit of $G_\alpha\}$.

$SLnG_\alpha = \{\Delta \subseteqq \Omega \mid \Delta, \Delta' \in LnG_\alpha\}$.

$WLnG_\alpha = \{\Delta \subseteqq \Omega \mid \Delta \in LnG_\alpha$, but Δ' is a fixed point$\}$.

Points in $SFxG_\alpha$ will be called *strongly fixed*; points in $WFxG_\alpha$, *weakly fixed*; orbits in $SLnG_\alpha$, *strongly long*; and orbits in $WLnG_\alpha$, *weakly long*. XG_α will be a variable which can take on as values each of these six sets. Each XG_α is α-full and thus may be thought of either as a subset of the set of orbits of G_α or as a subset of Ω. Clearly Ω is partitioned by FxG_α and LnG_α. In turn, FxG_α is partitioned by $SFxG_\alpha$ and $WFxG_\alpha$; and LnG_α, by $SLnG_\alpha$ and $WLnG_\alpha$. $SFxG_\alpha$ and $SLnG_\alpha$ are self-paired; and $WFxG_\alpha$ is paired with $WLnG_\alpha$.

PROPOSITION 7.

$\beta \in SFG_\alpha$ *iff* $G_\beta = G_\alpha$.

$\beta \in WFxG_\alpha$ *iff* $G_\beta \supset G_\alpha$.

$\beta \in WLnG_\alpha$ *iff* $G_\beta \subset G_\alpha$.

$\beta \in SLnG_\alpha$ *iff* $G_\beta \not\supseteqq G_\alpha$ *and* $G_\beta \not\subseteqq G_\alpha$.

Proof. Clearly $\beta \in FxG_\alpha$ iff $G_\beta \supseteqq G_\alpha$. Pick $g \in G$ such that $\beta g = \alpha$ and thus $\alpha g \in (\beta G_\alpha)'$. Then $G_\beta \subseteqq G_\alpha$ iff $\alpha \in FxG_\beta$ iff $\alpha g \in FxG_\alpha$ iff $(\beta G_\alpha)'$ is a fixed point of G_α. The proposition follows.

We shall say that G is *balanced* if $WFxG_\alpha$ is the empty set $\square$ (iff $WLnG_\alpha = \square$ iff $SFxG_\alpha = FxG_\alpha$ iff $SLnG_\alpha = LnG_\alpha$). By Proposition 7, G fails to be balanced iff G_α is properly contained in one of its conjugates. It follows that finite groups are balanced; in fact, paired orbits have equal cardinalities [18, Theorem 16.3]. Examples can be constructed of l-permutation groups (G, Ω), Ω totally ordered, which are not balanced.

Proposition 5 yields

PROPOSITION 8. *Any orbit of G_α which is not strongly long is convex. Hence if two different orbits of G_α lie in the same orbital of G_α, both are strongly long.*

We now apply the XG_α terminology to *orbitals* of G_α, being assured that an orbital $\text{Conv}(\Delta)$ is contained in that XG_α containing the orbit Δ.

The *α-configuration* of G is defined to be the *po*-set (*o*-set if Ω is totally ordered) of orbitals of G_α, partitioned into $SFxG_\alpha$, $WFxG_\alpha$, $SLnG_\alpha$, and $WLnG_\alpha$, with the point α distinguished; together with the involution $\Delta \rightarrow \Delta'$. α is called the *origin*. (Actually, the α-configuration is completely determined by the *po*-set of orbitals, the subset of fixed points, the origin, and the involution.) We want to show that this configuration is actually independent of α. By an *o*-isomorphism from the α-configuration of (G, Ω) onto the β-configuration of (K, Σ), we mean a *po*-set isomorphism ψ from the *po*-set orbitals of G_α onto that of K_β such that $(XG_\alpha)\psi = XK_\beta$ for each XG_α, $\{\alpha\}\psi = \{\beta\}$, and $(\Delta\psi)'^{\beta} = (\Delta'^{\alpha})\psi$ for all orbitals Δ of G_α. When there is such an *o*-isomorphism, we shall say that the two configurations are "the same configuration".

For any $f \in G$, an *o*-automorphism of (G, Ω) is provided by θ_Ω, defined by $\omega\theta_\Omega = \omega f$, and θ_G, defined by $g\theta_G = f^{-1}gf$. Hence the map $\Delta \rightarrow \Delta f$ is an *o*-isomorphism from the α-configuration onto the β-configuration. Moreover, if $\alpha f_1 = \alpha f_2$, with $f_1, f_2 \in G$, then $f_1 f_2^{-1} \in G_\alpha$, so for each α-full set Δ, $\Delta f_1 f_2^{-1} = \Delta$ and thus $\Delta f_1 = \Delta f_2$. This proves the fundamental

THEOREM 9. *Let G be a coherent subgroup of $A(\Omega)$. Let $\alpha, \beta \in \Omega$ and pick $f \in G$ such that $\alpha f = \beta$. Then $\Delta \rightarrow \Delta f$ furnishes a canonical o-isomorphism (independent of the choice of f) from the α-configuration onto the β-configuration. The canonical o-isomorphism from the α-configuration onto the β-configuration, followed by that from the β-configuration onto the γ-configuration, yields the canonical o-isomorphism from the α-configuration onto the γ-configuration.*

Hence we may speak of the *configuration* of G without reference to a particular point of Ω. Obviously if two o-permutation groups are o-isomorphic, they have the same configuration. Of course we can state a similar definition of configuration in terms of orbits rather than orbitals. Two groups having the same orbit configurations necessarily have the same orbital configurations; but not conversely (Examples 2 and 3). However, the orbit configuration is determined by the orbital configuration together with the number of orbits in each orbital. When we speak of configurations, we shall mean *orbital* configurations unless specified otherwise.

Two distinct points $\beta < \gamma$ of Ω have three possible relationships: $\beta < \gamma$, $\beta > \gamma$, and β incomparable with γ. G is o-2-*transitive* if for any $\beta, \gamma, \sigma, \tau \in \Omega$ such that β and γ are related in the same way as are σ and τ, there exists $g \in G$ such that $\beta g = \sigma$ and $\gamma g = \tau$. If G is o-2-transitive, G must have precisely one positive orbit and precisely one negative orbit (unless Ω is trivially ordered); and precisely one incomparable orbit (unless Ω is totally ordered). Conversely, it is easy to see that if G has such a configuration, G is o-2-transitive. Thus o-2-transitive groups can be characterized in terms of *orbit* configurations; though not in terms of orbital configurations (Example 3), except among the class of l-permutation groups.

We shall be interested also in those groups whose *orbital* configurations are the same as the *orbit* configurations described above for o-2-transitive groups. These groups are characterized by the property that for any $\beta, \gamma, \sigma, \tau \in \Omega$ such that β and γ are related as are σ and τ, there exists $g_1 \in G$ such that $\beta g_1 = \sigma$ and $\gamma g_1 \leqq \tau$; and $g_2 \in G$ such that $\beta g_2 = \sigma$ and $\gamma g_2 \geqq \tau$. Such groups will be called o-2-semitransitive. An o-2-semitransitive l-permutation group is automatically o-2-transitive.

The regular groups can of course be characterized as those whose configurations consist entirely of (strongly) fixed points.

Groups lying between the extremes of o-2-transitivity and regularity can be found among the examples at the end of the paper. See especially Examples 5 and 8. When Ω is totally ordered, the o-anti-isomorphism $\Delta \rightarrow \Delta'$ reduce the problem of determining the o-set of all orbitals to that of determining the o-set of positive orbitals. It can be shown that every o-set occurs as the o-set of positive orbitals for some transitive $(A(\Omega), \Omega)$.

If Δ is an orbit of G_α, the canonically corresponding orbit of G_β will be denoted by Δ_β. In particular, $\Delta_\alpha = \Delta$. Δ_β is to be thought of as "the Δ orbit of G_β". Of course, $(\Delta_\alpha)f = \Delta_{\alpha f}$. Since $\Delta \rightarrow \Delta f$ also yields a a canonical isomorphism from the set of α-full sets onto the set of (αf)-full sets, we may apply the same notation to α-full sets Δ_α, and in particular to orbitals of G_α.

PROPOSITION 10. *If $\alpha g \in SFxG_\alpha$, $g \in G$, then for each orbit(al) Δ of G_α, Δg is another orbit(al) of G_α, and it lies in the same XG_α as Δ.*

Proof. Proposition 7.

4. **O-blocks.** By *o-block* of an *o*-permutation group (G, Ω), we mean a *convex* subset $\square \neq \Delta \subseteqq \Omega$ having the property that for any $g \in G$, $\Delta g = \Delta$ or $\Delta g \cap \Delta = \square$. If the convexity requirement is removed, one has simply a *block* as defined in [18, §6]. Of course, these two concepts coincide when Ω is trivially ordered. The intersection of any collection of *o*-blocks is an *o*-block (provided it is not empty) and the union of any tower of *o*-blocks is an *o*-block. If Δ is an *o*-block, the *o-block system* $\tilde{\Delta}$ is the *po*-set (*o*-set if Ω is totally ordered) of translates Δg $(g \in G)$ of Δ. Since G is transitive, the *o*-block systems of G correspond to the convex G-congruences, where a G-congruence is said to be *convex* if its congruence classes are convex.

We partially order the blocks containing α by inclusion, obtaining a complete lattice, of which the *o*-blocks containing α form a complete sublattice; and similarly for the subgroups of G containing G_α.

THEOREM 11. *Let (G, Ω) be a coherent o-permutation group. In the well known o-correspondence $\Delta \rightarrow \{g \in G \mid \Delta g = \Delta\}$ and $C \rightarrow \alpha C$ between the lattice of blocks containing α and the lattice of subgroups containing G_α, the convex subgroups C correspond precisely to the o-blocks Δ.*

Proof. Clearly if Δ is convex, $\{g \in G \mid \Delta g = \Delta\}$ is convex. Now assume that C is convex. Suppose $\alpha c \leqq \beta \leqq \alpha d$, $c, d \in C$. Pick $f \in G$ such that $\alpha f = \beta$. Use coherence to pick $s \in G$ such that $\alpha s = \alpha d$ and $f \leqq s$. Since $d \in C$ and $sd^{-1} \in G_\alpha \subseteqq C$, $s \in C$. Similarly, pick $t \in C$ such that $t \leqq f$. Since C is convex, $t \leqq f \leqq s$ implies $f \in C$, so that $\beta = \alpha f \in \alpha C$. Therefore αC is convex. This result fails without coherence (Example 7).

We may make a complete lattice of the set of block systems of G by defining $\tilde{\Gamma} \leqq \tilde{\Delta}$ iff $\Gamma \subseteqq \Delta$, where Γ and Δ are the blocks in $\tilde{\Gamma}$ and $\tilde{\Delta}$ which contain α. Obviously the definition is independent of the choice of α. The set of *o*-block systems forms a complete sublattice. It is proved in [8, Theorem 3] that if Ω is totally ordered, the lattice of *o*-block systems is also totally ordered. Thus Theorem 11 gives us

COROLLARY 12. *The convex subgroups of G which contain G_α are totally ordered under inclusion.*

For the special case of *l*-permutation groups, this was proved by Holland [5]. His result mentioned only the convex prime *l*-subgroups containing G_α, but since G_α is prime, every subgroup containing it must automatically be a prime l-subgroup, and thus the two results coincide.

PROPOSITION 13. *A block Δ of G which contains α must be α-full and symmetric with respect to α.*

THEOREM 14. *Let G be a coherent subgroup of $A(\Omega)$, and let $\Delta = \Delta_\alpha$ be a convex α-full set. Then $\Gamma = \{\beta \in \Omega \mid \Delta_\beta = \Delta_\alpha\}$ is a (symmetric) o-block of G.*

Proof. $C = \{g \in G \mid \Delta g = \Delta\}$ is a convex subgroup of G containing G_α. But $\Gamma = \alpha C$, which is an o-block of G by Theorem 11.

It is immediate from the proof of Theorem 14 that even if Δ is not convex, Γ is still a block of G. This can also be deduced from the statement of the theorem. For if we throw away the order on Ω, leaving Ω trivially ordered and G coherent, then Δ becomes convex, so by the theorem, Γ is a block of G. Similar remarks apply to many of the theorems to come.

THEOREM 15. *Let G be a coherent subgroup of $A(\Omega)$. If Δ is an α-full o-block of G, then Δ' is also an (α-full) o-block of G, and $\{\beta \in \Omega \mid \Delta_\beta = \Delta_\alpha\}$ is the translate of Δ' which contains α.*

Proof. Let Γ be the o-block $\{\beta \in \Omega \mid \Delta_\beta = \Delta_\alpha\}$. Pick $f \in G$ such that $\alpha \in \Delta f$. Then Γf, also an o-block, is equal to $\{\eta \in \Omega \mid \Delta_\eta = \Delta f\} = \{\eta \in \Omega \mid \alpha \in \Delta_\eta\}$ (because Δ is a block) $= \{\alpha g \mid \alpha \in \Delta_{\alpha g} = \Delta_\alpha g\} = \Delta'$.

COROLLARY 16. *Let Δ be a weakly long orbit of G_α. Then Δ is an o-block of G. Indeed, if $\alpha g \neq \alpha$, $g \in G$, then $\Delta g \cap \Delta = \square$.*

Proof. Theorems 15 and 14. Thus for an α-full o-block Δ, Δ' need not lie in the same o-block system as Δ.

When Ω is totally ordered, we may complete Ω by Dedekind cuts and consider Ω to be a subset of its Dedekind completion $\bar{\Omega}$ (without end points). Each $f \in A(\Omega)$ can be extended to $f \in A(\bar{\Omega})$ by defining $\bar{\omega} f$ to be $\sup\{\beta f \mid \beta \in \Omega, \beta \leqq \bar{\omega}\}$. $A(\Omega)$ is an l-subgroup of $A(\bar{\Omega})$, but in general is not transitive even on $\bar{\Omega}\backslash\Omega$. A point $\bar{\omega} \in \bar{\Omega}$ is *α-full* if it is fixed by G_α. Equivalently, $\bar{\omega}$ is α-full if it is the sup (inf) of an α-full segment of Ω. If $\bar{\omega} \in \Omega$, then $\bar{\omega}$ is α-full iff $\bar{\omega} \in FxG_\alpha$. For any α-full point $\bar{\omega}_\alpha$, and for any $g \in G, \bar{\omega}_{\alpha g} = \bar{\omega}_\alpha g$ is the (αg)-full point canonically corresponding to $\bar{\omega}_\alpha$.

PROPOSITION 17. *Suppose that Ω is totally ordered and that $\bar{\omega}_\alpha$ is* *an α-full point. Then $\{\beta \in \Omega \mid \bar{\omega}_\beta = \bar{\omega}_\alpha\}$ is an o-block of G.*

Proof. $\{\eta \in \Omega \mid \eta \leqq \bar{\omega}_\alpha\}$ is an α-full segment of Ω. Apply Theorem 14.

LEMMA 18. *Suppose Ω is totally ordered. Let Δ be an α-full set. If $\alpha g \geqq \alpha$, then* $(\inf \Delta)g \geqq \inf \Delta$ *and* $(\sup \Delta)g \geqq \sup \Delta$.

Proof. Pick $1 \leqq k \in G$ such that $\alpha k = \alpha g$. Since Δ is α-full, $\Delta g = \Delta k$.

It is easily checked that

LEMMA 19 ([7, Lemma 3]). *Let $\alpha \in \Delta \subseteqq \Omega$. Suppose that $\Delta g = \Delta$ for each $g \in G$ such that $\alpha g \in \Delta$. Then Δ is a block of G.*

LEMMA 20. *Suppose that $\alpha \in \Delta \subseteqq \Omega$, Ω totally ordered, and that Δ is convex, α-full, and symmetric with respect to α. Let Π be any cofinal subset of Δ. Then Δ is an o-block of G provided only that $\alpha g \in \Pi$, $g \in G$, implies* $\inf \Delta g \not> \inf \Delta$ *and* $\sup \Delta g \not> \sup \Delta$.

Proof. By the first lemma, we see first that $\Delta g = \Delta$ when $\alpha \leqq \alpha g \in \Pi$; and next that $\Delta g = \Delta$ when $\alpha \leqq \alpha g \in \Delta$. In view of the second lemma, the conclusion follows from the symmetry of Δ.

THEOREM 21. *Let G be a coherent subgroup of $A(\Omega)$, Ω totally ordered. Suppose G has a (long) orbital Δ cofinal with Ω, so that Δ' is a (long) orbital coinitial with Ω. Then $\{\beta \in \Omega \mid \Delta' < \beta < \Delta\}$ is an o-block of G.*

Proof. By transitivity, terminal orbitals must be long. Now let Π be the α-full set $\Gamma = \{\beta \in \Omega \mid \Delta' < \beta < \Delta\}$ and let $\bar{\sigma} = \sup \Gamma$. We show first that if $\alpha < \alpha g \in \Gamma$, $g \in G$, then $\bar{\sigma} g \not> \bar{\sigma}$. For suppose $\bar{\sigma} g > \bar{\sigma}$. Pick $h \in G$ such that $\bar{\sigma} h < \alpha$. Since Δ is cofinal with Ω, we can pick $\delta \in \Delta$ such that $\delta h > \bar{\sigma}$. Now pick $k \in G_\alpha$ such that $(\bar{\sigma} g)k > \delta$. Since $k \in G_\alpha$ and Γ is α-full, $(\alpha g)k \in \Gamma$, so that $\alpha g k \leqq \bar{\sigma}$. Since $(\alpha g k)h \leqq \bar{\sigma} h < \alpha$, we can use coherence to pick $h \leqq f \in G$ such that $(\alpha g k)f = \alpha$. But $\bar{\sigma} g k f \geqq \bar{\sigma} g k h > \delta h > \bar{\sigma}$, contradicting the fact that $\bar{\sigma}$ is α-full. Therefore $\bar{\sigma} g \not> \bar{\sigma}$ when $\alpha < \alpha g \in \Gamma$. Similarly, $(\inf \Gamma)f \not< \inf \Gamma$ when $\alpha > \alpha f \in \Gamma$, and thus since Γ is symmetric, $(\inf \Gamma)g \not> \inf \Gamma$ when $\alpha < \alpha g \in \Gamma$. By the last lemma, Γ is an o-block of G.

In generalizations of theorems about finite permutation groups, FxG_α often must be expressed as $SFxG_\alpha$ ($= FxG_\alpha$ if G is finite). For example:

THEOREM 22. *Let* (G, Ω) *be a coherent o-permutation group. Then* $SFxG_\alpha$ *is a block of* G.

Proof. $SFxG_\alpha$ is α-full, so $(SFxG_\alpha)g = SFxG_{\alpha g}$. In view of Proposition 7, this says that $\{\beta \in \Omega \mid G_\beta = G_\alpha\}g = \{\gamma \in \Omega \mid G_\gamma = G_{\alpha g}\}$, which is equal to $SFxG_\alpha$ if $G_{\alpha g} = G_\alpha$, and does not meet $SFxG_\alpha$ otherwise.

5. **O-primitive groups.** Following Holland's definition for l-groups [7], we define a coherent subgroup G of $A(\Omega)$, Ω partially ordered, to be *o-primitive* if G has no o-blocks except Ω and the singletons $\{\omega\}$. Theorem 11 establishes Holland's result (obtained in essentially the same way) that G is o-primitive if and only if G_α is a maximal proper convex subgroup of G. O-permutation groups which are primitive are *a fortiori* o-primitive. On the other hand, $A(I)$, I the integers, is o-primitive, but not primitive.

PROPOSITION 23. *Let* (G, Ω) *be a coherent o-permutation group,* Ω *totally ordered. If* G *is o-2-semitransitive, it is o-primitive. If* G *is o-2-transitive, it is primitive.*

An o-group K is *Archimedean* if for any $1 < k, f \in K$, $f < k^n$ for some positive integer n; i.e., if K contains no proper convex subgroups. K is Archimedean iff K is isomorphic as an o-group to an o-subgroup of the additive reals [2, p. 45].

PROPOSITION 24. *Suppose that* (G, Ω) *is regular, with* Ω *totally ordered. Then* (G, Ω) *is o-primitive iff* G *is Archimedean.*

Proof. By Theorem 11, since $G_\alpha = \{1\}$.

This proposition almost characterizes the o-primitive regular groups in terms of their configurations. Unfortunately, it is possible for an Archimedean o-group (the rationals) to be isomorphic as an o-set to a non-Archimedean o-group ($\overleftarrow{Q \times I}$, Q the rationals, I the integers). This is the reason for the word "almost".

Among o-primitive groups on totally ordered sets Ω, there are thus two classes which lie at opposite extremes in terms of the amount of movement possible within G_α: the Archimedean regular groups, which we have almost characterized in terms of their configurations; and the o-2-semitransitive groups, which we have completely characterized in terms of their configurations. The remaining o-primitive groups will be discussed in detail in §7. For now, we apply §4 to o-primitive groups in general.

If $\Delta \subseteq \Omega$ and $\beta, \gamma \in \Omega$, we say that β and γ can be *separated* by

Δ if some translate $\Delta g (g \in G)$ of Δ contains precisely one of β and γ. An orbit $\bar{\omega}G$ *of* G is dense in $\bar{\Omega}$ if it meets every nontrivial segment of $\bar{\Omega}$. Of course, $\bar{\omega}G = \Omega$ if $\bar{\omega} \in \Omega$, and $\bar{\omega}G \cap \Omega = \square$ if $\bar{\omega} \in \bar{\Omega} \backslash \Omega$.

THEOREM 25. *Let (G, Ω) be a coherent o-permutation group. The following are equivalent (except that if Ω is not totally ordered, only the first three make sense):*

(i) *G is o-primitive.*

(ii) *For every segment $\square \neq \Delta \subset \Omega$, any $\beta \neq \gamma \in \Delta$ can be separated by Δ.*

(iii) *For every α-full segment $\square \neq \Delta_\alpha \subset \Omega$, $\Delta_\beta \neq \Delta_\gamma$ for $\beta \neq \gamma$ $(\alpha, \beta, \gamma \in \Omega)$.*

(iv) *For every α-full point $\bar{\omega}_\alpha \in \bar{\Omega}$, $\bar{\omega}_\beta \neq \bar{\omega}_\gamma$ for $\beta \neq \gamma$ $(\alpha, \beta, \gamma \in \Omega)$.*

(v) *For every $\bar{\omega} \in \bar{\Omega}$, $\bar{\omega}G$ is dense in $\bar{\Omega}$.*

Proof. It is clear that each of these conditions imples (i). Now suppose that G is o-primitive. If Δ is a segment, $\square \neq \Delta \subset \Omega$, then a convex G-congruence is given by the relation $\beta \equiv \gamma$ iff β and γ cannot be separated by Δ; and since some pairs $\beta \neq \gamma \in \Omega$ can be separated by Δ, every pair can, so that (ii) holds. For (v), if $\bar{\Gamma}$ were a nontrivial segment of $\bar{\Omega}$ which did not meet $\bar{\omega}G$, then for $\beta \neq \gamma \in \bar{\Gamma} \cap \Omega$ and $\Delta = \{\omega \in \Omega \mid \omega < \bar{\omega}\}$, β and γ could not be separated by Δ. For (iii), we use Theorem 14; and for (iv), Proposition 17. For Ω totally ordered and G an l-subgroup of $A(\Omega)$, the equivalence of (i), (ii), and (v) was shown by Holland [7, Theorem 2]. For Ω trivially ordered, the equivalence of (i) and (ii) was shown by Wielandt [17, Theorem 7.12].

THEOREM 26. *Let (G, Ω) be o-primitive. Then G is balanced and FxG_α is a block of G.*

Proof. Since weakly long orbits are o-blocks, G is balanced, so $FxG_\alpha = SFxG_\alpha$ is a block.

6. **Centralizers.** In Example 8, the map $z: \Omega \rightarrow \Omega$ given by $\beta z = \beta + 1$ lies in the centralizer $Z_{A(\Omega)}G$ of G in $A(\Omega)$. This phenomenon will be of paramount importance in the study of o-primitive groups. Accordingly, we devote this section to the study of centralizers.

When Ω is totally ordered, we shall be interested also in the centralizer of G in $A(\bar{\Omega})$. We define $\bar{F}xG_\alpha = \{\bar{\omega} \in \bar{\Omega} \mid \bar{\omega}G_\alpha = \bar{\omega}\} = \{\bar{\omega} \in \bar{\Omega} \mid G_{\bar{\omega}} \supseteqq G_\alpha\}$ and $\bar{S}\bar{F}xG_\alpha = \{\bar{\omega} \in \bar{\Omega} \mid \bar{\omega}G_\alpha = \bar{\omega}$ and $\alpha G_{\bar{\omega}} = \alpha\} = \{\bar{\omega} \in \bar{\Omega} \mid G_{\bar{\omega}} = G_\alpha\}$. Points in these two sets are α-full. By Proposition 7, $\bar{F}xG_\alpha \cap \Omega = FxG_\alpha$ and $\bar{S}FxG_\alpha \cap \Omega = SFxG_\alpha$. In the two lemmas which follow, if Ω is not totally ordered, one replaces $\bar{\Omega}$ by Ω, $\bar{F}xF_\alpha$ by FxG_α, and $\bar{S}FxG_\alpha$ by $SFxG_\alpha$.

LEMMA 27. *Let $z: \Omega \rightarrow \bar{\Omega}$ be a function which centralizes G, and let $\bar{\omega}_\alpha = \alpha z$. Then $\bar{\omega}_\alpha \in \bar{F}xG_\alpha$, and for all $\beta \in \Omega$, $\beta z = \bar{\omega}_\beta$. If z is one-to-one, $\bar{\omega}_\alpha \in \bar{S}FxG_\alpha$.*

Proof. For any $g \in G$, $\alpha zg = \alpha gz$; so that $\alpha z \in \bar{F}xG_\alpha$, and $\alpha z \in \bar{S}'FxG_\alpha$ if z is one-to-one. Now let $\beta \in \Omega$ and pick $k \in G$ such that $\alpha k = \beta$. Then $\beta z = \alpha kz = \alpha zk = \omega_\alpha k = \omega_{\alpha k} = \omega_\beta$.

COROLLARY 28. *$Z_{S(\Omega)}G = Z_{A(\Omega)}G$, where $S(\Omega)$ is the symmetric group on Ω.*

Proof. If $\bar{\omega}_\alpha \in \bar{F}xG_\alpha$, then for any $\alpha \leqq \beta \in \Omega$, $\bar{\omega}_\alpha \leqq \bar{\omega}_\beta$ by coherence.

LEMMA 29. *Let $\bar{\omega}_\alpha \in \bar{F}xG_\alpha$. Define $z: \bar{\Omega} \rightarrow \bar{\Omega}$ by setting $\beta z = \bar{\omega}_\beta$ for $\beta \in \Omega$, and $\bar{\gamma} z = \sup\{\beta z \mid \beta \leqq \bar{\gamma}\}$ for $\bar{\gamma} \in \bar{\Omega}$. Then z centralizes G. If $\bar{\omega}_\alpha \in \bar{S}FxG_\alpha$, z is one-to-one.*

Proof. For $g \in G$, $\beta \in \Omega$, $\beta gz = \bar{\omega}_{\beta g} = \bar{\omega}_\beta g = \beta zg$. It follows that $\bar{\gamma} gz = \bar{\gamma} zg$ for $\bar{\gamma} \in \bar{\Omega}$. If $\bar{\omega}_\alpha \in \bar{S}FxG_\alpha$, z is one-to-one on Ω and hence on $\bar{\Omega}$.

For finite permutation groups, Kuhn [9] established a correspondence between $Z_{S(\Omega)}G$ and FxG_α. Again FxG_α must be expressed as $SFxG_\alpha$.

THEOREM 30. *Let G be a coherent subgroup of $A(\Omega)$ and let $Z = Z_{A(\Omega)}G = Z_{S(\Omega)}G$. If $z \in Z$ and if $\omega_\alpha = \alpha z \in SFxG_\alpha$, then $\beta z = \omega_\beta$ for all $\beta \in \Omega$. Conversely, if $\omega_\alpha \in SFxG_\alpha$ and if $z: \Omega \rightarrow \Omega$ is defined by setting $\beta z = \omega_\beta$ for $\beta \in \Omega$, then $z \in Z$. Z is a po-group and $z \leftrightarrow \alpha z$ gives an o-isomorphism between the po-set Z and the po-set $SFxG_\alpha$.*

COROLLARY 31. *The po-sets which occur as $SFxG_\alpha$ for coherent o-permutation groups (G, Ω) are precisely those po-sets which are carriers of po-groups. The o-sets which occur in this way with Ω totally ordered are those which are carriers of o-groups.*

Proof. Theorem 30 and Corollary 2.

THEOREM 32. *Let G be a coherent subgroup of $A(\Omega)$, Ω totally ordered. Let $\alpha < \omega_\alpha \in SFxG_\alpha$ and let $z \in Z_{A(\Omega)}G$ be defined by $\beta z = \omega_\beta$, $\beta \in \Omega$. For $\gamma \in \Omega$, $B(\gamma, \omega_\gamma) = \mathrm{Conv}\{\gamma z^i \mid i \in I\}$, I the integers, is the smallest o-block of G containing γ and ω_γ, and the collection of $B(\gamma, \omega_\gamma)$'s forms an o-block system of G. Since $(\delta z)g = (\delta g)z$ for*

$g \in G$, $\delta \in \Omega$, the action of g on $B(\gamma, \omega_\gamma)$ is determined by its action on (γ, ω_γ), and we shall say that z is a period of G.

Proof. If $g \in G$ is such that $\gamma g = \gamma z^i$ for some i, then for any j, $(\gamma z^j)g = \gamma g z^j = \gamma z^{j+i}$. Apply Lemma 20 to show that $B(\gamma, \omega_\gamma)$ is an o-block of G. The rest is clear.

THEOREM 33. *Let (G, Ω) be o-primitive, Ω totally ordered, and let $Z = Z_{A(\bar{\Omega})}G$. Let $z \in Z$ and let $\bar{\omega}_\alpha = \alpha z \in \bar{F}xG_\alpha = \bar{S}FxG_\alpha$. Then for $\beta \in \Omega$, $\beta z = \bar{\omega}_\beta$; and for $\bar{\gamma} \in \bar{\Omega}$, $\bar{\gamma} z = \sup\{\beta z \mid \beta \in \Omega, \beta \leqq \bar{\gamma}\}$. Conversely, if $\bar{\omega}_\alpha \in \bar{F}xG_\alpha$ and if z is defined by $\beta z = \bar{\omega}_\beta$ for $\beta \in \Omega$ and $\bar{\gamma} z = \sup\{\beta z \mid \beta \in \Omega, \beta \leqq \bar{\gamma}\}$ for $\gamma \in \bar{\Omega}$, then $z \in Z$. Z is an o-group and $z \leftrightarrow \alpha z$ gives an o-isomorphism between the o-set Z and the o-set $\bar{F}xG_\alpha$.*

Proof. $\bar{F}xG_\alpha = \bar{S}FxG_\alpha$ because G_α is a maximal proper convex subgroup of G. If $z \in Z$, then Ωz is a dense subset of $\bar{\Omega}$ by Theorem 25, so since z preserves order, $\bar{\gamma} z = \sup\{\beta z \mid \beta \in \Omega, \beta \leqq \bar{\gamma}\}$ for $\bar{\gamma} \in \bar{\Omega}$. Conversely, $\beta z = \bar{\omega}_\beta$ maps Ω one-to-one onto a dense subset of $\bar{\Omega}$, so $\bar{\gamma} z = \sup\{\beta z \mid \beta \in \Omega, \beta \leqq \bar{\gamma}\}$ extends z to an o-permutation of $\bar{\Omega}$.

COROLLARY 34. *If G is o-2-semitransitive, $Z_{A(\bar{\Omega})}G$ is trivial. If G is o-primitive and regular, $Z_{A(\bar{\Omega})}G$ is isomorphic as an o-group to the integers or the reals.*

Proof. Use the theorem. In the regular case, G is the regular representation of a subgroup of the reals, and every proper Dedekind complete subgroup of the reals is discrete. In the next section we shall deal with the remaining o-primitive groups.

PROPOSITION 35. *For any totally ordered Ω and any subset F of $A(\Omega)$, $Z_{A(\Omega)}F$ is a* (not necessarily transitive) *l-subgroup of $A(\Omega)$.*

Proof. Since an l-group is a distributive lattice, if z_1 and z_2 commute with $f \in F$, then $(z_1 \vee z_2)f = z_1 f \vee z_2 f = f z_1 \vee f z_2 = f(z_1 \vee z_2)$.

7. Periodically o-primitive groups. We assume from now on that *Ω is totally ordered.* Earlier we noted that o-2-semitransitive groups and Archimedean regular groups are o-primitive. Now we assume that G is one of the remaining o-primitive groups and prove that it looks strikingly like the group in Example 8.

LEMMA 36. *G_α has a first positive long orbital Δ_1. α is the only point between Δ_1' and Δ_1.*

Proof. Since G is not regular, G_α has a long orbital Δ. Since G is balanced, Δ may be assumed negative and thus not cofinal with Ω, so that $\bar{\mu} = \sup \Delta \in \bar{\Omega}$. Pick $g \in G$ such that $\alpha \in \Delta g$ and let $\Delta_1 = \text{Conv}((\bar{\mu}g)G_\alpha)$. Pick an arbitrary $\beta \in \Omega$ such that $\alpha < \beta < \bar{\mu}g$. Since $\bar{\mu}G$ is dense in $\bar{\Omega}$ by Theorem 25, we may pick $h \in G$ such that $\alpha < \bar{\mu}h \leqq \beta$ and $h \leqq g$. $\alpha \in \Delta h$ and thus $\alpha h^{-1} \in \Delta$. Since also $\alpha g^{-1} \in \Delta$, we may pick $k \in G_\alpha$ such that $(\alpha g^{-1})k \geqq \alpha h^{-1}$. Now $\alpha(g^{-1}kh) \geqq \alpha$, but $(\bar{\mu}g)g^{-1}kh \leqq \bar{\mu}kh = \bar{\mu}h$ (since $\bar{\mu}$ is α-full) $\leqq \beta$. Finally, we pick $1 \geqq m \in G$ such that $(\alpha g^{-1}kh)m = \alpha$. Letting $n = g^{-1}khm$, we have $an = \alpha$ and $(\bar{\mu}g)n \leqq \beta$. Since β was arbitrary, there are no points between α and Δ_1, and Δ_1 is thus the first positive orbital. In view of the definition of Δ_1, this implies that Δ_1 is long.

Let us define $\bar{\omega} = \bar{\omega}_\alpha \in \bar{F}xG_\alpha$ to be $\sup \Delta_1$. (Δ_1 is bounded above in Ω because G is not o-2-semitransitive.) Let $z \in Z_{A(\bar{\Omega})}G$ be the o-permutation of $\bar{\Omega}$ associated with $\bar{\omega}_\alpha$ by Theorem 33. For each integer k, we define $\bar{\omega}_k$ to be αz^k. In particular, $\bar{\omega}_0 = \alpha$ and $\bar{\omega}_1 = \bar{\omega}$. We define Δ_k to be $(\bar{\omega}_{k-1}, \bar{\omega}_k) \subseteqq \Omega$, so that $\bar{\Delta}_k = \bar{\Delta}_1 z^{k-1}$. ($\bar{\Delta}_k$ does not include $\bar{\omega}_{k-1}$ or $\bar{\omega}_k$). The new definition of $\bar{\Delta}_1$ agrees with the old. Since G has period z and since the orbitals of G_α are convex, the fact that Δ_1 is an orbital of G_α implies that each Δ_k is an orbital of G_α. Thus for $k > 0$, Δ_k is the k^{th} positive long orbital; and Δ_{-k} is the $k + 1^{\text{st}}$ long orbital to the left of α. Since G is balanced, Δ_k is paired with Δ_{-k+1}. Between Δ_k and Δ_{k+1} lies precisely one point of $\bar{\Omega}$, namely $\bar{\omega}_k$. If $\bar{\omega}_k \in \Omega$, then $\bar{\omega}_k \in FxG_\alpha (= SFxG_\alpha)$.

LEMMA 37. *For any integers n and k and any $g \in G$, $\alpha g \in \Delta_n$ implies $\bar{\omega}_k g \in \bar{\Delta}_{k+n}$.*

Proof. $\bar{\omega}_k g = \alpha z^k g = \alpha g z^k \in \bar{\Delta}_n z^k = \bar{\Delta}_{k+n}$.

COROLLARY 38. $\text{Conv}\{\Delta_k \mid k \textit{ an integer}\} = \Omega$.

Proof. By Lemma 20, this set is an o-block of the o-primitive group G.

LEMMA 39. *Suppose that some $\bar{\omega}_i \in \Omega (i \neq 0)$. Let n be the least positive integer such that $\bar{\omega}_n \in \Omega$. Then $\bar{\omega}_k \in \Omega$ iff k is a multiple of n.*

Proof. $\bar{\omega}_n$ is the least positive point in the symmetric set $SFxG_\alpha$. Proposition 10 guarantees first that if k is a multiple of n, $\bar{\omega}_k \in \Omega$; and then the converse.

Recapitulating, the (strongly) long orbitals Δ_k of G_α form a set

o-isomorphic to the integers; and denoting sup Δ_k by $\bar{\omega}_k$, so that $\bar{\omega}_0 = \alpha$, either the (strongly) fixed points of G_α are precisely those $\bar{\omega}_k$'s such that k is a multiple of some fixed positive integer n, in which case we say that *G has Config* (n), or α is the only fixed point of G_α, in which case we say that *G has Config* (∞).

——(——)|(——)(——) (——)(——)|(——)——

α

Config (2)

MAIN THEOREM 40. *Suppose that G is a coherent subgroup of* $A(\Omega)$, Ω *totally ordered, and that G is o-primitive, but not o-2-semitransitive or regular. Then for some* $n = 1, 2, \cdots, \infty$, *G has Config* (n). $Z_{A(\bar{\Omega})}G$ *is cyclic, having as a generator the o-permutation z of* $\bar{\Omega}$ *defined by* $\beta z = (\bar{\omega}_1)_\beta$ *for* $\beta \in \Omega$ *and* $\bar{\gamma} z = \sup\{\beta z \mid \beta \in \Omega, \beta \leqq \bar{\gamma}\}$ *for* $\bar{\gamma} \in \bar{\Omega}$. *We shall say that z is the period of G and that G is periodically o-primitive.* Δ_{k+1} *is "one period up" from* Δ_k *in the sense that* $\bar{\Delta}_k z = \bar{\Delta}_{k+1}$. *If G has Config(n) for some finite n,* $Z_{A(\Omega)}G$ *is cyclic, having as a generator the o-permutation* $\hat{z}$ *of* Ω *defined by* $\beta\hat{z} = (\bar{\omega}_n)_\beta$, $\beta \in \Omega$; *and if G has Config* (∞), $Z_{A(\Omega)}G$ *is trivial.*

A few comments on this theorem are in order. z generates $Z_{A(\bar{\Omega})}G$ by Theorem 33. The fact that $(\bar{\delta}z)g = (\bar{\delta}g)z$ for $g \in G$, $\bar{\delta} \in \bar{\Omega}$, means that the action of G on Ω is determined by its action on any interval $(\bar{\gamma}, \bar{\gamma}z)$, and in particular on any Δ_k. z is analogous to the function $z: \beta \rightarrow \beta + 1$ of Example 8. If G has Config(n) for some finite n and if $\hat{z}$ is the period associated with $\bar{\omega}_n$, then $\hat{z}$ is nicer than z in that it is in $A(\Omega)$ rather than merely in $A(\bar{\Omega})$, but it suffers the disadvantage of being a larger and ultimately less useful period. In the next section, we shall construct examples of o-primitive groups having all of these configurations. Unfortunately, o-imprimitive groups can also have all of these configurations except Config (1). What o-blocks might there be containing α?

PROPOSITION 41. *If an o-imprimitive group G has Config(n), n finite, then for some integer p,* $1 \leqq p \leqq n/2$, *the nontrivial o-blocks of G containing* α *are precisely the sets Conv* $(\Delta'_k \cup \Delta_k)$, $k = 1, \cdots, p$. *If G has Config* (∞), *this result holds for some* $p \geqq 1$; *or else every Conv* $(\Delta'_k \cup \Delta_k)$ *is an o-block.*

Proof. Every nontrivial o-block containing α is symmetric and thus must be of the form Conv $(\Delta'_k \cup \Delta_k)$ for some $k \geqq 1$. If Conv $(\Delta'_p \cup \Delta_p)$ is an o-block, successive applications of Theorem 21 show that Conv $(\Delta'_k \cup \Delta_k)$ is an o-block for $k = p-1, p-2, \cdots, 1$. By Proposition 10, if n is finite, Conv $(\Delta'_p \cup \Delta_p)$ cannot be an o-block unless

$p \leqq n/2$. All of the possibilities not excluded in the proposition do in fact occur for o-imprimitive l-permutation groups (G, Ω).

COROLLARY 42. *If G has Config* (1), *G is o-primitive.*

COROLLARY 43. *Suppose G has Config(n) for some $n = 1, 2, \cdots, \infty$. Then G is o-imprimitive iff* Conv $(\Delta_1' \cup \Delta_1)$ *is an o-block of G.*

This corollary says that whether G is periodically o-primitive is determined by its configuration and knowledge of whether Conv $(\Delta_1' \cup \Delta_1)$ is an o-block.

We now investigate the consequences of periodicity. By the *support* of $g \in \Omega$ we mean $\{\beta \in \Omega \mid \beta g \neq \beta\}$.

COROLLARY 44. (Holland, [7]). *If G is o-primitive, but not o-2-semitransitive, then any $1 \neq g \in G$ has support bounded neither above nor below.*

COROLLARY 45. (Lloyd, [10]). *If $A(\Omega)$ is o-primitive, then it is either o-2-transitive or the regular representation of an Archimedean o-group.*

Proof. Clearly $A(\Omega)$ is not periodic; and the orbits of $A(\Omega)_\alpha$ are automatically convex.

An l-group is *l-simple* if it has no proper l-ideals.

COROLLARY 46. *An o-primitive l-subgroup G of $A(\Omega)$ is l-simple unless it is o-2-transitive and contains elements of unbounded support.*

Proof. Suppose G is periodically o-primitive. If $1 < g \in G$, then every $\bar{\beta} \in \bar{\Omega}$ is contained in the support of some conjugate of g by Theorem 25. Using periodicity, we apply the argument given at the end of [6] to show that G is l-simple. If G is regular, it is an Archimedean o-group, so it is l-simple. If G is o-2-transitive and contains only elements of bounded support, then G is l-simple by the proof of Theorem 6 of [5]. Note that if Ω is the reals, $A(\Omega)$ is o-2-transitive, but the elements of bounded support form a proper l-ideal.

An *o-ideal* of a *po*-group is a normal convex subgroup which is directed. The proof of Corollary 46 also yields

COROLLARY 47. *Suppose that G is an o-primitive subgroup of $A(\Omega)$, Ω totally ordered. Then G lacks proper o-ideals unless it is o-2-semitransitive and contains elements of unbounded support.*

PROPOSITION 48. *Suppose G (not necessarily o-primitive) has Config(n), n finite. Then any two orbits Δ_j and Δ_k whose subscripts are equal modulo n are o-isomorphic.*

Proof. Proposition 10.

PROPOSITION 49. *Suppose G is periodically o-primitive. Then all long orbitals of G_α have the same cardinality.*

Proof. Let Δ_k be any long orbital of G_α. All proper segments of Δ_k which are coinitial with Δ_k have the same cardinality $\aleph_I$; and all which are cofinal have the same cardinality $\aleph_F$. Furthermore, these cardinalities are independent of k. The proposition follows.

COROLLARY 50. *Suppose that G is periodically o-primitive and that some long orbital of G_α is countable. Then all long orbitals of G_α are o-isomorphic to the rationals and so is Ω.*

We can also deduce analogs of several theorems about nonordered permutation groups. For example, if G is a primitive permutation group, $FxG_\alpha = \{\alpha\}$ unless G is regular and $|\Omega|$ is prime [17, Theorem 7.14]. By Theorem 40, this is almost true if G is an o-primitive o-permutation group. Wielandt [17, Theorem 10.13] shows that if a permutation group G is primitive (and if $|\Omega| > \aleph_0$), then for every orbit $\Delta \neq \{\alpha\}$ of G_α, $|\Delta| + |\Delta'| = |\Omega|$. The proof fails for o-primitive groups, but almost all of the conclusion is given by

COROLLARY 51. *Let G be an o-primitive group. Then for every long orbital Δ of G_α, $|\Delta| + |\Delta'| = |\Omega|$. Except when G is o-2-semitransitive, we can strengthen this to $|\Delta| = |\Omega|$.*

Proof. If G is periodically o-primitive, use Proposition 48 and the fact that G has Config (n). If G is o-2-semitransitive or regular, the conclusion is trivial. It is possible for an o-2-transitive group to have positive and negative orbits of different cardinalities (Example 4).

Wielandt [17, Theorem 10.15] also shows that under somewhat stronger hypotheses, $|\Delta'| = |\Delta|$. This conclusion is given by

COROLLARY 52. *Let G be o-primitive, but not o-2-semitransitive. Then for every orbital Δ of G_α, $|\Delta'| = |\Delta|$.*

8. Full periodically o-primitive groups. For any periodically o-primitive group G, $G \subseteqq Z_{A(\bar{\Omega})}z \cap A(\Omega)$. We shall say that G is *full* if equality obtains. By Proposition 35, a full periodically o-primitive group G is automatically an l-subgroup of $A(\Omega)$ and hence the orbits of G_α are convex.

PROPOSITION 53. *Every periodically o-primitive (G, Ω) is contained in a full group (W, Ω) having the same period z.*

Proof. Take $W = Z_{A(\bar{\Omega})}z \cap A(\Omega)$.

In order to construct groups having Config(n), we characterize those o-sets which occur as Δ_1's for periodically o-primitive groups G for which the orbits of G_α are convex. Let $I_n = \{1, \cdots, n\}$ if n is finite; and let I_n be the integers if $n = \infty$. Let $\Sigma_i = \Delta_i z^{-(i-1)} \subseteqq \bar{\Delta}_1$, $i \in I_n$. The Σ_i's are pairwise disjoint because $\Omega z^k \cap \Omega = \Box$ for $k = 1, \cdots, n-1$ (all k if $n = \infty$). Thus

(a) *$\bar{\Delta}_1$ has a collection $\{\Sigma_i \mid i \in I_n\}$ of dense pairwise disjoint subsets, with $\Sigma_1 = \Delta_1$.*

Since for any $h \in G_\alpha$, $i \in I_n$, $\Sigma_i h = \Delta_i z^{-(i-1)} h = \Delta_i h z^{-(i-1)} = \Delta_i z^{-(i-1)} = \Sigma_i$, we have

(b) *$\{f \in A(\Delta_1) \mid \Sigma_i f = \Sigma_i$ for all $i \in I_n\}$ is transitive on Δ_1.*

For $\bar{\eta} \in \bar{\Delta}_1$, let $L(\bar{\eta}) = \{\bar{\delta} \in \bar{\Delta}_1 \mid \bar{\delta} < \bar{\eta}\}$ and $R(\bar{\eta}) = \{\bar{\delta} \in \bar{\Delta}_1 \mid \bar{\delta} > \bar{\eta}\}$. Suppose $\alpha g \in \Delta_k$, $g \in G$, $k \in I_n$. Let $\bar{\mu} = \alpha g z^{-(k-1)} \in \Sigma_k$. Let $\bar{\nu} = \bar{\omega}_k g^{-1}$ $(= \bar{\omega}_n z^{k-n} g^{-1} = \bar{\omega}_n g^{-1} z^{k-n} \in \Sigma_{n-(k-1)}$ if n finite). Since $g z^{-(k-1)}$ maps $L(\bar{\nu})$ onto $R(\bar{\mu})$ and $g z^{-k}$ maps $R(\bar{\nu})$ onto $L(\bar{\mu})$, we obtain

(c) *For any $\bar{\mu}$ in any Σ_k, $k \in I_n$, there exists $\bar{\nu}$ ($\bar{\nu} \in \Sigma_{n-(k-1)}$ if n finite, and $\bar{\nu} \in \bar{\Delta}_1 \backslash \cup \{\Sigma_i\}$ if $n = \infty$) such that there exists an o-isomorphism $s(\bar{\mu}, \bar{\nu})$ of $L(\bar{\nu})$ onto $R(\bar{\mu})$ with $(L(\bar{\nu}) \cap \Sigma_j) s(\bar{\mu}, \bar{\nu}) = R(\bar{\mu}) \cap \Sigma_p$, where $p = j + k - 1$ (mod n if n finite), and there exists an o-isomorphism $t(\bar{\mu}, \bar{\nu})$ of $R(\bar{\nu})$ onto $L(\bar{\mu})$ with $(R(\bar{\nu}) \cap \Sigma_j)$ $t(\bar{\mu}, \bar{\nu}) = L(\bar{\mu}) \cap \Sigma_q$, where $q = j + k$ (mod n if n finite).*

Sets Δ_1 satisfying these conditions will be discussed in the corollaries of the following theorem. When $n = 1$, these conditions state simply that $A(\Delta_1)$ is transitive and that for $\delta \in \Delta_1$, $\{\beta \in \Delta_1 \mid \beta < \delta\}$ is o-isomorphic to $\{\beta \in \Delta_1 \mid \beta > \delta\}$; or equivalently, that Δ_1 is an open interval of some chain Ω for which $A(\Omega)$ is o-2-transitive.

THEOREM 54. *The o-sets which occur as first positive orbits in periodically o-primitive groups G which have Config(n) and for which the orbits of G_α are convex are precisely those o-sets Δ_1 satisfying conditions* (a), (b), *and* (c).

Proof. We construct, for any o-set Δ_1 satisfying these conditions, a full periodically o-primitive group (G, Ω) having Δ_1 as the first positive orbit of G_α. As the construction for $n = \infty$ is similar to and simpler than the construction for finite n, we shall assume that n is finite and leave the case $n = \infty$ to the reader.

Let $\Delta_1(=\Sigma_1), \cdots, \Delta_n$ be pairwise disjoint copies of $\Sigma_1, \cdots, \Sigma_n$, and let Δ be the ordinal sum $\Delta_1 + \cdots + \Delta_n$ with a point α adjoined at the bottom. Let Ω be $\overleftarrow{\Delta \times I}$, I the integers. For each $i \in I$, let $\Delta_i = \{(\sigma, a) \mid \sigma \in \Delta_b\}$, where $i = an + b$ $(1 \leqq b \leqq n)$. This identifies Δ with $\{(\lambda, 0) \mid \lambda \in \Delta\}$. Let $\bar{\omega}_i = \sup \bar{\Delta}_i$. $\bar{\omega}_i \in \Omega$ iff i is a multiple of n. Define $\hat{z} \in A(\Omega)$ by $(\lambda, a)\hat{z} = (\lambda, a + 1)$. Now pick an o-isomorphism w_i from Σ_i onto Δ_i, $i = 1, \cdots, n$, with w_1 the identity map on Δ_1. Since Σ_i is a dense subset of $\bar{\Delta}_1$, we can extend w_i to an o-isomorphism of $\bar{\Delta}_1$ onto $\bar{\Sigma}_i$. We define $z \in A(\bar{\Omega})$ as follows: For $\bar{\beta} \in \bar{\Delta}_i$, $i = 1, \cdots, n-1$, $\bar{\beta}z = \bar{\beta}w_i^{-1}w_{i+1}$, and for $\beta \in \bar{\Delta}_n$, $\bar{\beta}z = \bar{\omega}_n{}^{-1}\hat{z}$. $\bar{\omega}_i z = \bar{\omega}_{i+1}$, $i = 0, \cdots, n-1$. This defines z on $\bar{\Delta} = [\alpha, \bar{\omega}_n)$, and we extend it to $\bar{\Omega}$ so that it has $\hat{z}$ as a period, i.e., we define $(\beta\hat{z}^j)z = (\beta z)\hat{z}^j$ for all $\beta \in [\alpha, \bar{\omega}_n)$, $j \in I$.

We define G to be $Z_{A(\bar{\Omega})} \cap A(\Omega)$, an l-subgroup of $A(\Omega)$. First we show that G is transitive on Ω. It suffices to show that for each $\alpha \neq \lambda \in \Delta$, there exists $g \in G$ such that $\alpha g = \lambda$. $\lambda \in \Delta_k$ for some $k \in I_n$, so that $\bar{\mu} = \lambda w_k^{-1} \in \Sigma_k$. Pick $\bar{\nu} \in \Sigma_{n-(k-1)}$, $s(\bar{\mu}, \bar{\nu})$, and $t(\bar{\mu}, \bar{\nu})$ as in (c). Now we define $g \in G$ as follows: $\alpha g = \lambda$ and $(\bar{\nu}w_{n-(k-1)})g = \bar{\omega}_n$. For $\beta \in (L(\bar{\nu}) \cap \Sigma_j)w_j$, $\beta g = \beta w_j^{-1}s(\bar{\mu}, \bar{\nu})w_{j+(k-1)} \in \Delta_{j+(k-1)}$, where if $j + (k-1) > n$, $w_{j+(k-1)} = w_{j+(k-1)-n}\hat{z}$. For $\beta \in (R(\bar{\nu}) \cap \Sigma_j)w_j$, $\beta g = \beta w_j^{-1}t(\bar{\mu}, \bar{\nu})w_{j+k} \in \Delta_{j+k}$. This defines g on $\Delta = [\alpha, \bar{\omega}_n)$, and we extend it to Ω by defining $(\beta\hat{z}^j)g = (\beta g)\hat{z}^j$ for all $\beta \in [\alpha, \bar{\omega}_n)$, $j \in I$. Since $w_i^{-1}w_{i+1} = z$ and $z^n = \hat{z}$, we have $g \in G$, establishing the transitivity of G.

Each $\bar{\omega}_j$ is fixed by G_α because for $h \in G_\alpha$, $\bar{\omega}_j h = \alpha z^j h = \alpha h z^j = \alpha z^j = \bar{\omega}_j$. By (b), the first positive orbit of G_α is Δ_1, and since G has period z, the j^{th} positive long orbit of G_α is Δ_j, so that G has Config(n). By periodicity, no Conv $(\Delta_j' \cup \Delta_j)$ is an o-block of G, so G is o-primitive, and by construction, it is full.

Corollary 55. *For each $n = 1, 2, \cdots, \infty$, there is a full periodically o-primitive group on the rationals (which are the only countable candidate) having Config(n).*

Proof. Let Δ_1 be the rationals, which satisfy conditions (a), (b), and (c). (Take the Σ_i's to be distinct cosets of the rationals in the reals). By Corollary 50, Ω is o-isomorphic to the rationals.

Corollary 56. *Suppose that Ω is Dedekind complete and that G is a coherent subgroup of $A(\Omega)$. (Do not assume that G is o-primitive). Then*

(1) *G is the regular representation of the integers or the reals,*
or (2) *G is o-2-semitransitive and $|\Omega| = 2^{\aleph_0}$,*
or (3) *G is periodically o-primitive with Config(1) and $|\Omega| = 2^{\aleph_0}$.*
$A(\Omega)$ is o-2-transitive for uncountably many nonisomorphic Dedekind

complete Ω's; *and uncountably many nonisomorphic Dedekind complete* Ω's *support full periodically o-primitive groups having Config*(1).

Proof. Since Ω is Dedekind complete and nontrivial o-blocks of G have no sups in Ω, G must in fact be o-primitive. If g is regular, it is Archimedean, so since Ω is Dedekind complete, G must be isomorphic as an o-permutation group to the regular representation of the integers or the reals. If G has Config(n) for some n, then $n = 1$ because Ω is Dedekind complete.

For the statements about cardinality, we appeal to some interesting results of Babcock [1]. Babcock's Theorem 22 states that a Dedekind complete chain, not the integers, which is homogeneous (and thus in its order topology satisfies the first countability axiom by [16, Theorem 1]) has cardinality $2^{\aleph_0}$. This finishes (2) and (3). When Ω is Dedekind complete, the Config(1) conditions on Δ_1 state precisely that Δ_1^* (Δ_1 with end points) is Dedekind complete and that any two nontrivial closed subintervals of Δ_1^* are o-isomorphic. Babcock constructs uncountably many o-sets satisfying these conditions [1, p. 2]. Moreover, it can be verified that in this special case, Δ_1 is o-isomorphic to Ω, so we get uncountably many nonisomorphic Dedekind complete Ω's supporting full periodically o-primitive groups having Config(1). Of course, for each of these Ω's, $A(\Omega)$ is o-2-transitive.

9. Locally o-primitive groups. Following Holland [7], we say when Ω is totally ordered that G is *locally o-primitive* if in the totally ordered set (Theorem 12) of o-block systems of G, there is a minimal nontrivial system $\tilde{\Delta}$. Certainly o-primitive groups are locally o-primitive. The o-blocks in $\tilde{\Delta}$ are called the *primitive segments* of G. If Γ is a primitive segment, let $G|\Gamma$ denote the restriction of G to Γ, i.e., $\{g|\Gamma : g \in G \text{ and } \Gamma g = \Gamma\}$. Then $(G|\Gamma, \Gamma)$ is o-primitive. As noted in the introduction, every l-group can be embedded in a subdirect product of o-permutation groups (G_i, Ω_i), with each Ω_i totally ordered and G_i a transitive l-subgroup of $A(\Omega_i)$. It can be further arranged that each G_i is locally o-primitive [7].

If for some (and hence each) primitive segment Γ, $G|\Gamma$ is o-2-semitransitive (regular, periodically o-primitive), we shall say that G is *locally o-2-semitransitive* (*regular, periodically o-primitive*). For example, the o-imprimitive groups of Proposition 41 are locally o-2-semitransitive; and if Ω is discrete, G is locally regular with primitive segments o-isomorphic to the integers.

THEOREM 57. *Every locally o-primitive group is locally o-2-semitransitive, locally regular, or locally periodically o-primitive.*

We almost characterize locally o-primitive groups by their configurations with

THEOREM 58. *If G_α has a first positive orbital, then G is locally o-primitive. Conversely, if G is locally o-primitive, then G_α has a first positive orbital (unless G is locally regular and Ω is not discrete).*

Proof. Suppose that G_α has a first positive orbital Δ. By Proposition 13, every o-block $\neq\{\alpha\}$ of G which contains α must contain Δ. Let Γ be the intersection of all such o-blocks. Since $\{\alpha\} \neq \Gamma$, Γ must be a primitive segment of G. Therefore G is locally o-primitive. The converse follows from the previous theorem.

10. Examples.

EXAMPLE 1. Let Ω be the reals and let G be the set of o-permutations of Ω having everywhere a strictly positive derivative. G is an o-2-transitive coherent subgroup of $A(\Omega)$, but it is not an l-subgroup.

EXAMPLE 2. Let Ω be the reals and let G be the linear group $\{ax + b \mid a, b \text{ real}, a > 0\}$. $ax + b$ is positive iff $a = 1$ and $b \geqq 0$. Again G is coherent and o-2-transitive, but not an l-permutation group.

EXAMPLE 3. In Example 2, let H be the coherent subgroup of elements $ax + b$ of G for which a is rational. H is not o-2-transitive, but is o-2-semitransitive. Although H is o-primitive, it is not primitive because the rationals form a block of H.

EXAMPLE 4. Let ω_1 be the first uncountable ordinal; let Σ be the rationals with the usual order; and let Ω be the lexicographic product $\overleftarrow{\Sigma \times \omega_1}$, ordered from the right, i.e., $(\sigma_1, \gamma_1) \leqq (\sigma_2, \gamma_2)$ iff $\gamma_1 < \gamma_2$, or $\gamma_1 = \gamma_2$ and $\sigma_1 \leqq \sigma_2$. $A(\Omega)$ is o-2-transitive. The negative orbit of $A(\Omega)_\alpha$ is countable; the positive orbit is not.

EXAMPLE 5. Let I be the integers with the usual order. $A(I)$ is isomorphic as an o-group to the integers. Let (G, Ω) be the ordered wreath product of $(A(I), I)$ with itself, i.e., $\Omega = \overleftarrow{I \times I}$ and each $g \in G$ is given by $(m, n)g = (m + k_n, n + k)$, where k depends only on g, but k_n depends on n as well as g. In fact, $G = A(\Omega)$, and the configuration of G can be obtained by starting with I, replacing one integer by a set of strongly fixed points o-isomorphic to I, replacing each other integer by a strongly long orbit, and establishing the obvious pairings.

EXAMPLE 6. Let $A(\Omega)$ be as in Example 5. Let G be the coherent subgroup of elements of $A(\Omega)$ which satisfy

(1) $$k_n = k_p \quad \text{if} \quad n \equiv p \pmod 2$$

and

(2) $$k_n \equiv k_p \pmod 2 \quad \text{even if} \quad n \not\equiv p \pmod 2 .$$

None of the long orbits of G_α is convex; indeed, each long orbital of G_α contains precisely two long orbits. The configuration of G consists of alternating strongly long orbitals and o-blocks (each o-isomorphic to the integers) of strongly fixed points.

EXAMPLE 7. In Example 6, replace (2) by (2′) $k_n = -k_p$ if $n \not\equiv p \pmod 2$. Then G is not coherent; indeed no point can be moved to its successor by a positive $g \in G$. (G, Ω) is regular, but not o-isomorphic to the right regular representation of G. $\Delta \to \Delta'$ is not an o-anti-automorphism of the totally ordered set of orbit(al)s of G_α. $\{(i, 0) \mid i \text{ even}\}$ is a block Δ of G which is not convex; but $\{g \in G \mid (0, 0)g \in \Delta\}$ is trivially ordered and hence is a convex subgroup of G.

EXAMPLE 8 (Holland, [6]). The only previously known example of an o-primitive group which is neither o-2-semitransitive nor regular was as follows: Let Ω be the reals and let $G = \{f \in A(\Omega) \mid f$ has period 1, i.e., $(\beta + 1)f = \beta f + 1$ for all $\beta \in \Omega\}$. The map $z: \Omega \to \Omega$, given by $\beta z = \beta + 1$, lies in the centralizer $Z_{A(\Omega)}G$ of G in $A(\Omega)$, and indeed $G = \{f \in A(\Omega) | zf = fz\}$. G is a full periodically o-primitive group having Config(1). (See §7). It is shown in [6] that G is l-simple.

EXAMPLE 9. Let G be the full periodically o-primitive group of Example 8. Let $G^{(m)}$ consist of those elements of G which have m^{th} derivatives and whose first derivatives are positive everywhere. Then $G \supset G^{(1)} \supset G^{(2)} \supset \cdots$. Each $G^{(m)}$ is periodically o-primitive with period 1. The $G^{(m)}$'s are not l-subgroups of $A(\Omega)$ and of course are not full.

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