

# Pacific Journal of Mathematics

**ON EXTREMAL FIGURES ADMISSIBLE RELATIVE TO  
RECTANGULAR LATTICES**

MURRAY SILVER

## ON EXTREMAL FIGURES ADMISSIBLE RELATIVE TO RECTANGULAR LATTICES

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A theorem of Bender states that if a convex figure  $F$  contains no point of the two dimensional lattice  $G$ , where  $G$  is generated by the vectors  $\bar{V}_1$  and  $\bar{V}_2$  having enclosed angle  $\theta$ , then  $A(F) \leq 1/2 P(F) \max (|\bar{V}_1|, |\bar{V}_2| \sin \theta)$  where  $|\bar{V}_1| \leq |\bar{V}_2|$ . In this paper, two questions are answered: (1) Among all convex figures of perimeter  $L$  which are admissible relative to a rectangular lattice  $G$ , which encloses the maximum area? (2) Can the constant  $1/2$  in Bender's theorem be improved? By using the result of (1), the "sharpest possible" inequality of the Bender type is found.

### NOTATION.

$$w_1 = \min (|\bar{V}_1|, |\bar{V}_2| \sin \theta)$$

$$w_2 = \max (|\bar{V}_1|, |\bar{V}_2| \sin \theta)$$

$A(F)$  is the area of  $F$ ,  $P(F)$  is the perimeter of  $F$ . A figure  $F$  is *admissible* relative to the lattice  $G$ , if no points of  $G$  are in the interior of  $F$ .

**THEOREM.** *If  $F$  is an admissible convex figure relative to the lattice  $G$ , then*

- (i) for  $0 < P(F) \leq \pi(w_1^2 + w_2^2)^{1/2}$ ,  $A(F) \leq (P^2(F))/(4\pi)$
- (ii) for  $\pi(w_1^2 + w_2^2)^{1/2} < P(F) < 4w_1 + \pi w_2$ ,

$$A(F) \leq \frac{P^2(F)}{4\pi} - \frac{\left( P^2(F) - \pi \left( w_1 \sin q/2 + w_2 \cos \frac{q}{2} \right) \right)^2}{\pi(4 - \pi \sin q)}$$

where  $q$  is the root of equation (9).

- (iii) for  $4w_1 + \pi w_2 \leq P(F)$ ,  $A(F) \leq 1/2 w_2 P(F) - \pi/4 w_2^2$ .

Further, if  $G$  is rectangular the extremal figures relative to  $G$  are shown for (i), (ii) and (iii) in Figure 1 (i), (ii) and (iii) respectively; in these cases, equality holds.

By Bender's Lemma [1], only rectangular lattices and admissible convex figures symmetric about the lines  $x' = 1/2$ ,  $y' = 1/2$  need be considered ( $x'$  and  $y'$  are coordinates relative to the lattice); in the remainder of this paper only such figures and lattices will be considered.

**DEFINITION.** Let  $G$  be a (rectangular) lattice and denote by  $R$  the set of all admissible rhombi whose vertices lie on the lines  $x' =$

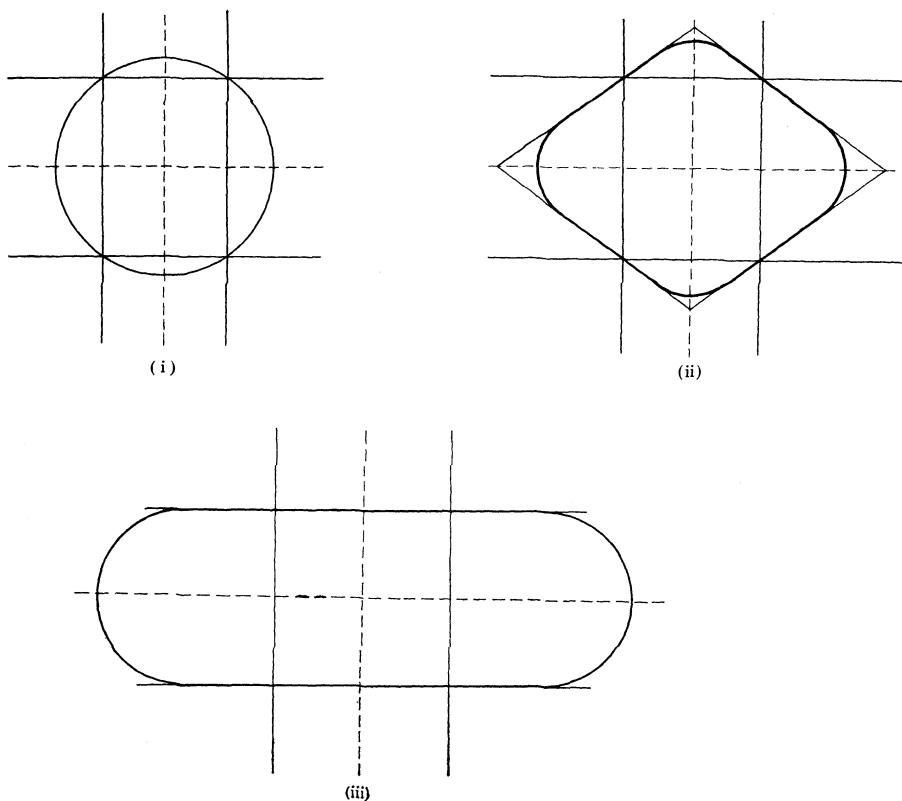


FIGURE 1

$1/2$ ,  $y' = 1/2$  and each of whose sides pass through at least one lattice point of  $G$  (see Figure 2).  $R(\varphi)$  denotes the rhombus in  $R$  with base angle  $\varphi$  (Figure 2) where  $0 \leq \varphi \leq \pi$  ( $\varphi = 0$  and  $\varphi = \pi$  yield the two infinite strips).

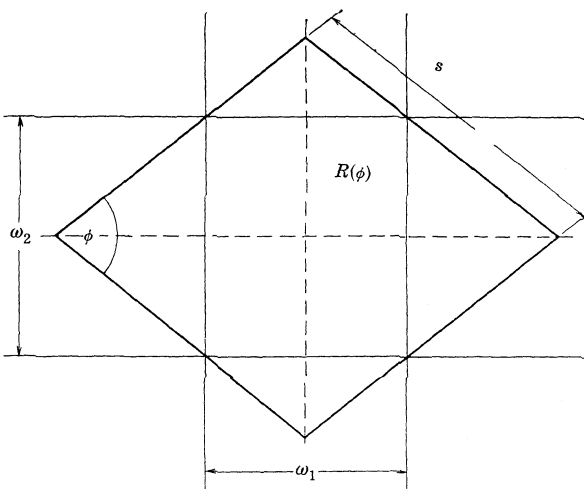


FIGURE 2

LEMMA 1. *Every figure  $F$  is contained in at least one rhombus  $R(\varphi)$  of the set  $R$ .*

*Proof.* Let  $g$  be one of the four lattice points which contain the intersection of the lines  $x' = 1/2$  and  $y' = 1/2$ . Consider the following two cases: (i)  $g$  is a boundary point of  $F$ , (ii)  $g$  is not a boundary point of  $F$ .

(i) Since  $F$  is convex and  $g$  is a boundary point, there exists a line of support  $S$  of  $F$  at the point  $G$ . Construct the three remaining lines symmetric to  $S$  about the lines  $x' = 1/2$  and  $y' = 1/2$ . By the symmetry of  $F$ , all four of these lines are support lines of  $F$  and the rhombus formed contains  $F$  and belongs to  $R$ .

(ii) Since  $g$  is exterior to  $F$ , there exists a line  $S'$  which separates  $g$  and  $F$ . Construct  $S$  through  $g$  parallel to  $S'$ . Clearly  $S$  lies in the exterior of  $F$  and the proof is completed as in (i).

*Proof of the Theorem.* The inequalities are proven by finding the admissible convex figure of perimeter  $L$  which encloses the maximum area (extremal figure). The problem has been reduced to rectangular lattices and symmetric figures which are contained in rhombi of  $R$ . Denote by  $Y(L, \varphi)$  the extremal figure of perimeter  $L$  contained in  $R(\varphi)$ . The existence, uniqueness, form, etc., of the extremal figure are discussed in references [2] and [4], pp. 124-5. For fixed  $L$ , define  $q$  by  $A(Y(L, q)) = \sup_{\varphi} A(Y(L, \varphi))$ . The maximum area is thus attained by the extremal figure contained in the rhombus  $R(q)$ . Since any figure  $F$  is contained in  $R(\varphi)$  for some  $\varphi$  (Lemma 1),  $A(F) \leq A(Y(L, \varphi)) \leq A(Y(L, q))$ . The inequalities (ii) and (iii) are nothing other than  $A(Y(L, q))$  expressed in terms of  $L$  and the lattice constants; (i) means simply that  $Y(L, q)$  is a circle. In (ii) and (iii),  $Y(L, q)$  contains lattice points on its boundary; otherwise, it is easy to construct a figure of perimeter  $L$  having larger area. Hence, for a rectangular lattice, the inequalities of the theorem are the "sharpest possible".

In the remainder of the proof,  $A(Y(L, \varphi))$  and  $A(Y(L, q))$  are determined.

$Y(L, \varphi)$

From Figure 2, it follows for  $0 < \varphi < \pi$

$$(1) \quad S = \frac{1}{2}w_1 \sec \frac{\varphi}{2} + \frac{1}{2}w_2 \csc \frac{\varphi}{2}$$

or

$$(2) \quad S \sin \varphi = w_1 \sin \frac{\varphi}{2} + w_2 \cos \frac{\varphi}{2}$$

$Y(L, \varphi)$  is the parallel figure of radius  $r$  taken about a concentric subrhombus of  $R(\varphi)$  (see reference 3, p. 124-5). Denoting by  $v$  the length of a side of the subrhombus,

$$(3) \quad A(Y(L, \varphi)) = v^2 \sin \varphi + 4rv + \pi r^2 \quad (\text{see Figure 3}).$$

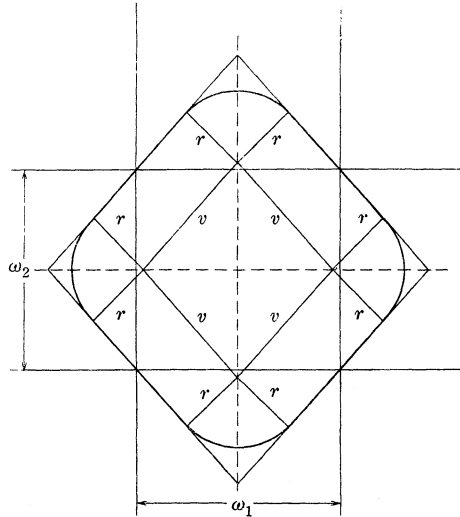


FIGURE 3

The perimeter of  $Y(L, \varphi)$  is given by

$$(4) \quad P(Y(L, \varphi)) = L = 2\pi r + 4v.$$

Use (4) to eliminate  $r$  from (3). After simplification it follows that

$$(5) \quad \frac{L^2}{4\pi} - A(Y(L, \varphi)) = v^2 \left( \frac{4}{\pi} - \sin \varphi \right).$$

The right side of (5) is, of course, the classical isoperimetric deficit. From Figure 3,

$$(6) \quad S \cos \frac{\varphi}{2} = v \cos \frac{\varphi}{2} + r \csc \frac{\varphi}{2}.$$

Using equation (4), eliminate  $r$  from equation (6); use the resulting equation to eliminate  $v$  from equation (5), and finally, use equation (2) to eliminate  $S$ :

$$(7) \quad A(Y(L, \varphi)) = \frac{L^2}{4\pi} - \frac{\left( \frac{L}{\pi} - \left( w_1 \sin \frac{\varphi}{2} + w_2 \cos \frac{\varphi}{2} \right) \right)^2}{\frac{4}{\pi} - \sin \varphi}.$$

Equation (7) is valid for  $0 < \varphi < \pi$ . If  $\varphi = 0$  or  $\pi$ , the extremal figure consists of two parallel lines connected by semicircles [3]. Calculation shows that the area agrees in both cases with (7). Hence, (7) is valid for  $0 \leq \varphi \leq \pi$ .

$Y(L, \varphi)$

From equation (7),  $A(Y(L, \varphi))$  is a single valued continuous function of  $L$  and  $\varphi$  which possesses neither a singularity nor a cusp. To find  $Y(L, \varphi)$ , the isoperimetric deficit

$$(8) \quad D = \frac{\left(\frac{L}{\pi} - \left(w_1 \sin \frac{\varphi}{2} + w_2 \cos \frac{\varphi}{2}\right)\right)^2}{\frac{4}{\pi} - \sin \varphi}$$

must be minimized.

If  $L \leq \pi(w_1^2 + w_2^2)^{1/2}$ , the solution is trivial; viz. the circle. In the remainder of the proof, it is assumed that  $L > \pi(w_1^2 + w_2^2)^{1/2}$ . Setting  $dD/d\varphi = 0$ , the condition for an extremum becomes:

$$(9) \quad L \cos \varphi = (4w_1 + \pi w_2) \cos \frac{\varphi}{2} - (4w_2 + \pi w_1) \sin \frac{\varphi}{2}.$$

The value (s) of  $\varphi$  which yield an extremum of  $D$  must be either 0,  $\pi$  or a root of equation (9).

The case  $w_1 = w_2$  will be treated separately; if not otherwise stated, it is assumed that  $w_2 > w_1$ .

LEMMA 2. *The absolute minimum of  $D(L, \varphi)$  lies in the interval  $0 \leq \varphi \leq \pi/2$ .*

*Proof.* Consider an arbitrary rhombus  $R(\varphi) \in R$ . From the midpoint of  $R(\varphi)$  mark off the distance  $1/2 w_2$  along the line  $x' = 1/2$ ; at this point construct the perpendicular  $d$ . From similar triangles,

$$\frac{d}{S \sin \frac{\varphi}{2}} = \frac{S \cos \frac{\varphi}{2} - \frac{1}{2} w_2}{S \cos \frac{\varphi}{2}}.$$

Using equation (1), eliminate  $S$  and solve for  $d$ :

$$(10) \quad d = \frac{1}{2} w_2 - \frac{1}{2} (w_2 - w_1) \tan \frac{\varphi}{2}.$$

If  $R(\varphi)$  is rotated through  $90^\circ$  about its midpoint, it will not contain a lattice point (the boundary included) if  $d < 1/2 w_1$ . Applying this

condition to equation (10), it follows that (11)  $\tan \varphi/2 > 1$ . Hence,  $R(\varphi)$  does not contain a lattice point when rotated about its midpoint through  $90^\circ$  if  $\varphi > \pi/2$ . Suppose  $Y(L, \varphi)$  is the extremal figure of the rhombus  $R(\varphi)$  where  $\varphi > \pi/2$ . Rotate  $R(\varphi)$  through  $90^\circ$  about its midpoint. By the preceding argument,  $R(\varphi)$  and thus  $Y(L, \varphi)$  contains no lattice point (boundary included). Thus,  $Y(L, \varphi)$  cannot be  $Y(L, q)$ .

Thus,  $q$  is either 0 or a root of equation (8) ( $w_1 \neq w_2$ ).

LEMMA 3. For  $0 \leq \varphi \leq \pi/2$  equation (9) has

- (i) exactly one root if  $L < 4w_1 + \pi w_2$
- (ii) exactly one root (viz.,  $\varphi = 0$ ) if  $L = 4w_1 + \pi w_2$
- (iii) no roots if  $L > 4w_1 + \pi w_2$

*Proof.* Form the auxillary functions  $y_1 = L \cos \varphi$  and

$$\begin{aligned} y_2 &= (4w_1 + \pi w_2) \cos \frac{\varphi}{2} - (4w_2 + \pi w_1) \sin \frac{\varphi}{2} \\ &= ((4w_1 + \pi w_2)^2 + (4w_2 + \pi w_1)^2)^{1/2} \sin \left( \beta - \frac{\varphi}{2} \right) \end{aligned}$$

where

$$\tan \beta = \frac{4w_1 + \pi w_2}{\pi w_1 + 4w_2}.$$

Clearly,  $38^\circ < \beta < 45^\circ$ . The roots of equation (9) are the points of intersection of  $y_1$  and  $y_2$ . Divide the problem into three parts

- (i)  $y_1(0) < y_2(0)$ ; i.e.,  $L < 4w_1 + \pi w_2$
- (ii)  $y_1(0) = y_2(0)$ ; i.e.,  $L = 4w_1 + \pi w_2$
- (iii)  $y_1(0) > y_2(0)$ ; i.e.,  $L > 4w_1 + \pi w_2$

$y_1$  and  $y_2$  are cosine and sine curves; the lemma follows from the elementary properties of these curves.

From Lemma 3, it follows for (iii) and (ii) that  $q = 0$ . In case (i),  $D'(0)$  is negative and  $q$  must be the (single) root of equation (9). Thus, the extremal figures have been found and inserting the value of  $q$  into equation (7) gives the theorem (for  $w_1 \neq w_2$ ).

*The Solution for  $w_1 = w_2 = w$ .*

This is the most important single case; viz., the square lattice. Geometrically it is obvious that equation (7) and therefore (8) are symmetric about  $\pi/2$ ; viz.,  $R(\varphi)$  is identical with  $R(\pi - \varphi)$  except for a rotation of  $\pi/2$  about the midpoint. Hence  $Y(L, \varphi)$  is identical with  $Y(L, \pi - \varphi)$ , except for a rotation of  $\pi/2$  about its midpoint.  $\varphi$  can therefore be restricted to the interval  $0 \leq \varphi \leq \pi/2$ . In this case, equation (9) becomes:

$$(12) \quad \left( \cos \frac{\varphi}{2} - \sin \frac{\varphi}{2} \right) \left( \cos \frac{\varphi}{2} + \sin \frac{\varphi}{2} - \frac{(4 + \pi)w}{L} \right) = 0$$

Equation (12) has two roots

$$(13) \quad \varphi = \frac{\pi}{2},$$

$$(14) \quad \sin \varphi = \left( \frac{w}{L} \right)^2 (4 + \pi)^2 - 1.$$

(i)  $0 < L \leq \sqrt{2}\pi w$ . The circle is admissible and  $A(B) \leq (P^2(B))/4\pi$ .  
 (iii)  $L > (4 + \pi)w$ . Equation (14) does not yield an admissible root; since  $D(\varphi)$  is strictly increasing,  $q = 0$ . The extremal figure is of the form shown in Figure 1 (iii) and  $A(B) \leq 1/2wP(B) - 1/4\pi w^2$

(ii)  $\sqrt{2}\pi w < L \leq (4 + \pi)w$ . Case (ii) decomposes into two cases:

(iia)  $\sqrt{2}\pi w < L \leq \sqrt{2}(4 + \pi)w$

(iib)  $\sqrt{2}(4 + \pi)w < L \leq (4 + \pi)w$

*Case (iia)* If  $L \leq 1/\sqrt{2}(4 + \pi)w$ , equation (14) offers no solution; since  $D(\varphi)$  is strictly decreasing,  $q = \pi/2$ . Thus, for all  $L$  in (iia), the extremal figure is contained in  $R(\pi/2)$ . The desired inequality becomes  $A(B) \leq \frac{1}{(4 - \pi)} (-1/4L^2 + \sqrt{8}wL - 2\pi w^2)$ . Note that there is no analogy if  $w_1 \neq w_2$ .

*Case (iib)*  $q$  occurs in  $(0, \pi/2)$  and is given by equation (14). The extremal figure has the form shown in Figure 1 (ii) and  $A(B) \leq (L^2)/(4 + \pi)4 + w^2$ .

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