A GENERAL PHILLIPS THEOREM FOR $C^*$-ALGEBRAS AND SOME APPLICATIONS

DONALD CURTIS TAYLOR
A GENERAL PHILLIPS THEOREM FOR C*-ALGEBRAS
AND SOME APPLICATIONS

DONALD CURTIS TAYLOR

In this paper Phillips's theorem is extended to a C*-algebra setting and, by virtue of this extension, several results on interpolation are generalized and improved.

1. Introduction. Let $N$ be the set of positive integers with the discrete topology and let $m(N)$ denote the bounded complex functions on $N$. We may identify $m(N)$ with $C(\beta N)$, where $\beta N$ denotes the Stone-Cech compactification of $N$. A well known and useful result due to Phillips is the following.

THEOREM. Let $\{f_n\}$ be a sequence in the dual of $C(\beta N)$ that converges weak* to zero. Then

$$\lim_{m \to \infty} \sum_{p \in m} |f_n(\delta_p)| = 0$$

uniformly in $n$, where $\delta_p$ is the characteristic function of the set $\{p\}$.

In §3 we extend this result to a C*-algebra setting and we give several applications of this result. For example, we extend and improve several results on interpolation due to Bade [3] and Akemann [2]. A commutative version of our result was proved by Conway [7].

2. Preliminaries. Let $A$ be a C*-algebra. By a double centralizer on $A$, we mean a pair $(R, S)$ of functions from $A$ to $A$ such that $aR(b) = S(a)b$ for $a, b \in A$, and we denote the set of all double centralizers on $A$ by $M(A)$. If $(R, S) \in M(A)$, then $R$ and $S$ are continuous linear operators on $A$ and $||R|| = ||S||$. So $M(A)$ under the usual operations of addition, multiplication, and involution is a C*-algebra, where $||(R, S)|| = ||R||$. If we define the map $\mu_o: A \to M(A)$ by the formula $\mu_o(a) = (L_a, R_a)$, where $L_a(b) = ab$ and $R_a(b) = ba$ for all $b \in A$, then $\mu_o$ is an isometric *-isomorphism from $A$ into $M(A)$ and $\mu_o(A)$ is a closed two-sided ideal of $M(A)$. Hence throughout this paper we will view $A$ as a closed two-sided ideal of $M(A)$. For a more detailed account of the theory of double centralizers on a C*-algebra, we refer the reader to [4] and [13].

Let $B$ be a C*-algebra and let $A$ be a closed two-sided ideal of $B$. We define the strict topology $\beta_s$ for $B$ to be that locally convex topology generated by the seminorms $(\lambda_a)_{a \in A}$ and $(\rho_a)_{a \in A}$, where $\lambda_a(x) = ||ax||$.
and $\rho_a(x) = ||xa||$, and we let $B_{\beta A}$ denote $B$ under the strict topology generated by $A$. When $A$ and $B$ are understood (specifically, when $B = M(A)$) we let $\beta$ denote the strict topology for $B$ generated by $A$. The topological algebra $M(A)_\beta$ is complete and the unit ball of $A$ is $\beta$ dense in the unit ball of $M(A)$.

We will now state a result due to Busby that is very useful in computing the double centralizer algebra of a C*-algebra.

**Theorem 2.1.** Let $B$ be a C*-algebra, let $A$ be a closed two-sided ideal of $B$, and let $A^0 = \{x \in B|xA = 0\}$. Let the map $\mu: B \to M(A)$ be defined by $\mu(x) = (L_x, R_x)$, where $L_x(a) = xa$ and $R_x(a) = ax$ for each $a$ in $A$. Then the following statements are true:

1. The map $\mu$ is a $*$-homomorphism of $B$ into $M(A)$; consequently, $\mu$ is an isometry if and only if $A^0 = 0$.
2. If $A^0 = 0$ and every $\beta_A$-Cauchy net in the unit ball of $A$ converges in the $\beta_A$ topology to some element of the unit ball of $B$, then $\mu$ is an isometric $*$-isomorphism of $B$ onto $M(A)$.

**Proof.** For a proof, see [4, Proposition 3.7, p. 83].

**Corollary 2.2.** If $B$ is a W*-algebra and $A^0 = 0$, then $\mu$ is an isometric $*$-isomorphism of $B$ onto $M(A)$.

**Proof.** Let $\{a_s\}$ be a $\beta_A$-Cauchy net in the unit ball of $A$. Since the unit ball of $B$ is compact in the weak operator topology, we can assume that $\{a_s\}$ converges in the weak operator topology to some element $x$ in the unit ball of $B$. Since $\{a_s\}$ is $\beta_A$-Cauchy, it is straightforward by [4, Th. 3.9(i), p. 84] to show that $\{a_s\}$ converges to $x$ in the $\beta_A$-topology. The conclusion now follows from Theorem 2.1.

If $B$ is a W*-algebra, then it is straightforward to show that $A^0$ is a two-sided ideal of $B$ that is closed in the weak operator topology. Hence $A^0$ has an identity $q$ that commutes with each element of $B$. If follows that the quotient algebra $B/A^0$ is isometrically $*$-isomorphic to the W*-algebra $(1-q)B(1-q)$. Now define the map $\mu': B/A^0 \to M(A)$ by the formula $\mu'(x + A^0) = \mu(x)$ for each $x$ in $B$. Since $\ker \mu = A^0$, we see that $\mu'$ is well defined. Due to the fact that $\{x \in B/A^0 | x(A/A^0) = 0\} = \{0\}$, we get

**Corollary 2.3.** If $B$ is a W*-algebra, then $M(A)$ is a W*-algebra and the map $\mu'$ is an isometric $*$-isomorphism of $B/A^0$ onto $M(A)$; that is, $M(A) \cong M(A/A^0)$.

**Example.** Let $H$ be a Hilbert space, let $B(H)$ be the bounded linear operators on $H$, and let $B_0(H)$ be the compact linear operators
on $H$. It is well known that $B_0(H)$ is a closed two-sided ideal of $B(H)$. Since $B(H)$ is a $W^*$-algebra and $\{x \in B(H) | xB_0(H) = 0\} = \{0\}$, we have that $B(H)$ is the double centralizer algebra of $B_0(H)$.

**Example.** Let $B$ be a finite dimensional $C^*$-algebra, let $S$ be a locally compact paracompact Hausdorff space, and let $\beta(S)$ denote the Stone-Cech compactification of $S$. Let $C(\beta(S), B)$ denote the space of all $B$-valued continuous functions on $\beta(S)$ and let $C_0(S, B) = \{x \in C(\beta(S), B) | x(t) = 0, t \in \beta(S) - S\}$. It is clear that under the usual pointwise operations and sup-norm that $C(\beta(S), B)$ is a $C^*$-algebra and $C_0(S, B)$ is a closed two-sided ideal of $C(\beta(S), B)$. Now it is straightforward to show that a $\beta$-Cauchy net in the unit ball of $C_0(S, B)$ converges to a $B$-valued continuous function on $S$ that is uniformly bounded. Since a bounded $B$-valued continuous function on $S$ can be uniquely extended to $B$-valued continuous functions on $\beta(S)$, Theorem 2.1 gives us that $C(\beta(S), B)$ is the double centralizer algebra of $C_0(S, B)$.

**Proposition 2.4.** Let $B$ be a $C^*$-algebra and $A$ a closed two-sided ideal of $B$. Then $B^*_\beta A$, the dual of $B_\beta A$, can be identified under the natural mapping as a closed subspace of $B^*$.

**Proof.** The proof will follow from a variation of the argument given for [13, Corollary 2.3, p. 635].

**Proposition 2.5.** Let $B$ be a $C^*$-algebra and let $A$ be a closed two-sided ideal of $B$. If $f$ is a bounded linear functional on $B$, then there exists a unique decomposition $f = f^0 + f^1$ such that $f^0 \in B^*_\beta A$ and $f^1 \in A^\perp$. Consequently, $B^* = B^*_\beta A \oplus A^\perp$.

**Proof.** For a proof, see [14, Corollary 2.7].

**Remark.** For each $f \in B^*$ we will always let $f^0$ and $f^1$ denote those unique linear functionals in $B^*_\beta A$ and $A^\perp$ respectively that satisfy $f = f^0 + f^1$.

**Definition.** Let $A$ be a $C^*$-algebra. A subset $K$ of $M(A)^*_\beta$ is said to be **tight** if $K$ is uniformly bounded and if for some, or for each, approximate identity $\{e_i\}$ for $A$ we have

$$\|(1 - e_i)f(1 - e_i)\| \to 0$$

uniformly on $K$. Here $(1 - e_i)f(1 - e_i)(x) = f((1 - e_i)x(1 - e_i))$ for each $x \in M(A)$. 

THEOREM 2.6. Let $A$ be a $C^*$-algebra. Then a subset $K$ of $M(A)^\dagger$ is $\beta$-equicontinuous if and only if $K$ is tight.

Proof. For a proof, see [13, Theorem 2.6, p. 636].

3. A general Phillips theorem for $C^*$-algebras. In this section we will study sequential convergence in the dual of a double centralizer algebra. In particular, we prove a general Phillips theorem for $C^*$-algebras and we give some applications of it.

DEFINITION. An approximate identity $\{e_\lambda | \lambda \in A\}$ for the $C^*$-algebra $A$ is said to be well behaved if and only if the following properties are satisfied:

1. $e_\lambda \geq 0$ for each $\lambda \in A$.
2. If $\lambda_2 > \lambda_1$, then $e_{\lambda_2} e_{\lambda_1} = e_{\lambda_1}$.
3. If $\lambda_1, \lambda_2, \cdots$ is a strictly increasing sequence in $A$ and $\lambda \in A$, then there exists a positive integer $N$ such that for all $n, m > N$ we have $e_\lambda (e_{\lambda_n} - e_{\lambda_m}) = 0$.

REMARK. If $S$ is a locally compact paracompact Hausdorff space, then $S$ can be expressed as the union of a collection $\{S_\alpha | \alpha \in I\}$ of pairwise disjoint open and closed $\sigma$-compact subsets of $S$. Since each $C^*$-algebra $C_0(S_\alpha)$ has a countable approximate identity and $C_0(S) \cong (\sum C_0(S_\alpha))_\sigma$, it follows by Proposition 3.1 and Proposition 3.2 that $C_0(S)$ has a well behaved approximate identity. Now let $H$ be a Hilbert space and $\{p_\alpha | \alpha \in I\}$ be a maximal family of orthogonal projections on $H$. It is straightforward to show that $\{p_\alpha | \alpha \in I\}$ is a series approximate identity for $B_0(H)$, the space of all compact operators on $H$, consequently, by Proposition 3.1, $B_0(H)$ has a well behaved approximate identity. Finally, suppose $A$ is a $C^*$-algebra such that $M(A)$ is isometrically isomorphic to $A^{**}$, the bidual of $A$. By some recent results of E. McCharen or by [15, Theorem 5.1, p. 533] $A$ is dual, consequently, $A \cong (\sum B_0(H_\alpha))_\sigma$, where $\{H_\alpha\}$ is a family of Hilbert spaces (see [11]). Hence by Proposition 3.2 $A$ has a well behaved approximate identity.

PROPOSITION 3.1. Let $A$ be a $C^*$-algebra and suppose one of the following conditions holds:

1. $A$ has a countable approximate identity;
2. $A$ has a series approximate identity (see [2, p. 527]).

Then $A$ has a well behaved approximate identity.

Proof. It is straightforward to verify that $A$ has a well behaved approximate identity when (2) holds. Therefore assume $A$ has a
countable approximate identity \( \{c_n\} \). We can also assume \( c_n ^ 0 \), since \( c_n^*c_n \) is an approximate identity for \( A \). Let \( b = \sum_{n=1}^{\infty} c_n/2^n \). Then \( b \) is a strictly positive element of \( A \) in the sense of [1, p. 749]. Hence \( A \) contains a countable increasing abelian approximate identity \( \{d_n\} \) [1, Theorem 1, p. 749]. Let \( A_0 \) denote the maximal commutative subalgebra of \( A \) that contains \( \{d_n\} \). Then we can view \( A_0 \) as \( C_0(\mathbb{M}) \), the complex-valued continuous functions that vanish at \( \infty \) on the maximal ideal space \( \mathbb{M} \) of \( A \). Since \( A_0 \) has a countable approximate identity \( \{d_n\} \), it follows by [5, Theorem 4.1, p. 160] that \( \mathbb{M} \) is \( \sigma \)-compact. It is straightforward to show that \( A_0 \) has a well behaved countable approximate identity \( \{e_n\} \). We now wish to show that \( \{e_n\} \) is an approximate identity for \( A \). Let \( a \in A \) and \( \varepsilon > 0 \). Choose a positive integer \( m \) so that \( ||a - d_m a|| < \varepsilon/2 \) and then choose a positive integer \( N \) so that \( ||(d_m - e_n d_m)|| < \varepsilon/2||a|| \) for integers \( n \geq N \). It follows that \( ||a - e_n a|| \leq ||(1 - e_n)(a - d_m a)|| + ||(d_m - e_n d_m)a|| < \varepsilon \) for \( n \geq N \). Hence \( \{e_n\} \) is a well behaved approximate identity for \( A \) and the proof is complete.

**Proposition 3.2.** Let \( \{A_\delta | \delta \in \Delta\} \) be a family of \( C^* \)-algebras. If each \( A_\delta \) has a well behaved approximate identity, then the sub-direct sum \( \left( \sum_{\delta \in \Delta} A_\delta \right)_0 \) has a well behaved approximate identity (see [12, p. 106] for definition of \( \left( \sum_{\delta \in \Delta} A_\delta \right)_0 \)).

**Proof.** For each \( \delta \in \Delta \) let \( \{e_{\delta, \lambda} | \lambda \in \Lambda_\delta\} \) be a well behaved approximate identity for \( A_\delta \), and let \( \mathcal{F} \) denote the family of all finite subsets of \( \Delta \). Let \( \Sigma \) denote the set of all functions \( \sigma \) whose domain \( D_\sigma \in \mathcal{F} \) and has the property that \( \sigma(\delta) \in A_\delta \) for each \( \delta \in D_\sigma \). We define the binary relation \( \geq \) in \( \Sigma \) by the following formula: \( \sigma_1 \geq \sigma_2 \) if and only if \( D_{\sigma_1} \supseteq D_{\sigma_2} \) and \( \sigma_2(\delta) \geq \sigma_1(\delta) \) for each \( \delta \in D_{\sigma_2} \). It is straightforward to verify that \( \Sigma \) under \( \geq \) is a directed set. Now for each \( \sigma \in \Sigma \) define \( d_\sigma \) in \( \left( \sum_{\delta \in \Delta} A_\delta \right)_0 \) by the following formula: \( d_\sigma(\delta) = e_{\delta, \sigma(\delta)} \) for each \( \delta \in D_\sigma \) and \( d_\sigma(\delta) = 0 \) otherwise. It is straightforward to verify that \( \{d_\sigma | \sigma \in \Sigma\} \) is a well behaved approximate identity for \( \left( \sum_{\delta \in \Delta} A_\delta \right)_0 \).

The next result extends Phillips' theorem to a \( C^* \)-algebra setting. A commutative version of this result was proved by Conway [7, Theorem 2.2, p. 55].

**Theorem 3.3.** Suppose \( A \) is a \( C^* \)-algebra with a well behaved approximate identity. If \( \{f_\alpha\} \) is a sequence in \( M(A)^* \) that converges weak* to zero, then \( \{f_\alpha^*\} \) is tight and converges weak* to zero.

**Proof.** It is clear that \( \{f_\alpha\} \) is uniformly bounded, so without loss
of generality we can assume \( \{f_n\} \) is uniformly bounded by 1. Since 
\[ \|f_n\| \geq \|f_n\|_{\mathcal{A}} = \|f_n^*\|_{\mathcal{A}} = \|f_n^*\|, \]
we have that \( \{f_n^*\} \) is also uniformly bounded by 1. Let \( \{e_\lambda : \lambda \in \Lambda\} \) be a well behaved approximate identity for \( \mathcal{A} \) and suppose \( \{f_n\} \) is not tight. Then there exists an \( \varepsilon > 0 \) such that 
\[ \sup \| (1 - e_{\lambda_1}) f_n^*(1 - e_{\lambda_2}) \| \geq 4\varepsilon \]
is cofinal in \( \Lambda \) and since a cofinal subnet of a well behaved approximate identity is also one, we may assume

\[ \text{(3.1)} \quad \sup_{n} \| (1 - e_{\lambda}) f_n^*(1 - e_{\lambda}) \| \geq 4\varepsilon \]

for all \( \lambda \in \Lambda \). We may then define inductively sequences \( n_1 < n_2 < \cdots \) and 
\( \lambda_1 < \lambda_2 < \cdots \) such that 
\[ \| (1 - e_{\lambda_1}) f_n^*(1 - e_{\lambda_2}) \| \geq 4\varepsilon \] and 
\[ \| e_{\lambda_1} - f_n^* e_{\lambda_2} - f_n^0 \| < \varepsilon \]
by using the following: (3.1); \( \lim_{n} \| (1 - e_{\lambda}) g(1 - e_{\lambda}) \| = 0, \) \( g \in M(\mathcal{A})^*_\beta; \)
\( \lim_{n} \| e_{\lambda} g e_{\lambda} - g \| = 0, \) \( g \in M(\mathcal{A})^*_\beta. \) It then follows that

\[ \| (1 - e_{\lambda_1}) f_{n_{k+1}}^*(1 - e_{\lambda_2}) \| = \| e_{\lambda_{k+1}} - e_{\lambda_k} \| \| f_{n_k}^*(e_{\lambda_{k+1}} - e_{\lambda_k}) \| \]
\[ \geq 3\varepsilon. \]

We then, for each \( \lambda \), choose \( b_\lambda = b_\lambda^* \) in ball \( \mathcal{A} \) such that 
\[ |f_{n_k}^*(e_{\lambda_{k+1}} - e_{\lambda_k}) b_{\lambda_k}(e_{\lambda_{k+1}} - e_{\lambda_k})| \geq \varepsilon. \]
Define \( a_\lambda = (e_{\lambda_{k+1}} - e_{\lambda_k}) b_{\lambda_k}(e_{\lambda_{k+1}} - e_{\lambda_k}) \) and let 
\( g_\lambda = f_{n_k}^*. \) Then we have:

(i) \( |g_\lambda(a_\lambda)| \geq \varepsilon; \)
(ii) \( a_j a_\lambda = 0 \) for \( j \neq \lambda; \)
(iii) for each \( \lambda \in \Lambda \), there exists a positive integer \( N \) such that 
\( a_\lambda e_{\lambda_j} = 0 \) for \( k \geq N. \)

Now let \( \alpha = \{a_\lambda\}_{\lambda=1}^\infty \) be an element of \( l^\infty \). By virtue of (ii) and 
(iii) the sequence of partial sums \( \{\sum_{k=1}^n a_\lambda a_\lambda\} \) is uniformly bounded by 
\( \|\alpha\|_{\infty} \) and is \( \beta \)-Cauchy. Since \( M(\mathcal{A})_\beta \) is complete [4, Proposition 3.6, 
\( p. 83 \], \( \sum_{k=1}^\infty a_\lambda a_\lambda \) has a \( \beta \)-limit \( \sum_{k=1}^\infty a_\lambda a_\lambda \) that is also bounded by 
\( \|\alpha\|_{\infty}. \) Next, define the bounded linear map \( T: l^\infty \rightarrow M(\mathcal{A}) \) by the formula

\[ T(\alpha) = \sum_{k=1}^\infty a_\lambda a_\lambda \]

for each \( \alpha \in l^\infty. \) Let \( T^* \) denote the adjoint of \( T. \) Since \( T \) is continuous, 
\( T^* \) is a weak* continuous mapping of \( M(\mathcal{A})^* \) into \( (l^\infty)^* \). From our 
hypothesis on \( \{f_n\} \) it follows that \( \{T^*(g_\lambda)\} \) converges to 0 weak*. Hence, by Phillips theorem [8, p. 32],

\[ \lim_{m \rightarrow \infty} \sum_{q=1}^m |T^* g_\lambda(\delta_q)| = \lim_{m \rightarrow \infty} \sum_{q=1}^m |g_\lambda(a_q)| \rightarrow 0 \]

uniformly in \( k \), where \( \delta_k \) is the Kronecker delta function. Therefore there exists a positive integer \( m \) such that 
\( |g_\lambda(a_m)| \leq \sum_{q=m}^\infty |g_\lambda(a_q)| < \varepsilon. \)
This contradicts (i), so \( \{f_n\} \) is tight.

Note that \( \{f_n^*\} \) is now equicontinuous on \( M(\mathcal{A})_\beta \) and converges 
pointwise on a dense subset and hence (by a well known result) converges weak*. The proof is now complete.
By virtue of Proposition 3.1 and the previous remark, the following result is an improvement of [13, Theorem II, p. 634].

**Corollary 3.4.** Suppose $A$ has a well behaved approximate identity. If $K$ is a relatively weak* countably compact subset of $M(A)_\beta^*$, then $K$ is tight. Consequently, $M(A)_\beta$ is a strong Mackey space (hence, in particular, is a Mackey space).

**Proof.** The proof that $K$ is tight is similar to the one given for Theorem 3.3. Since $M(A)_\beta$ is a strong Mackey space if and only if each weak* compact subset of $M(A)_\beta^*$ is $\beta$-equicontinuous, it follows from Theorem 2.6 that $M(A)_\beta$ is a strong Mackey space.

**Remark.** In [6, p. 481] Conway showed that if $S$ is the ordinals less than the first uncountable ordinal and $A = C_0(S)$, then $M(A)_\beta$ is not even a Mackey space. Therefore it follows that $C_0(S)$ does not have a well behaved approximate identity.

The next result extends [5, Theorem 5.1, p. 161].

**Corollary 3.5.** If $A$ has a well behaved approximate identity, then $(MA)_\beta^*$ is weakly sequentially complete.

**Proof.** If $\{f_n\}$ is a weak* Cauchy sequence in $M(A)_\beta^*$, then there exists a unique linear functional $f$ in $M(A)^*$ with $f_n \to f$ weak*. It follows that $f_n - f \to 0$ weak*. Thus, by Theorem 3.3, $(f_n - f)^0 \to 0$ weak*. But by virtue of Proposition 2.5 $(f_n - f)^0 = f_n^0 - f^0 = f_n - f^0$. This implies that $f_n \to f^0$ weak*. Hence $f = f^0$ and the proof is complete.

The next result generalizes and improves results due to Bade [3, Theorem 1.1, p. 149] and Akemann [2, Theorem 2.3, p. 527] (see our Corollaries 3.9 and 3.8).

**Theorem 3.6.** Suppose $A$ is a $C^*$-algebra with a well behaved approximate identity $\{e_\lambda | \lambda \in \Lambda\}$. If $X$ is a Banach space and $T: X \to M(A)$ is a bounded linear map with $T(X) + A = M(A)$, then there exists a $\lambda \in \Lambda$ such that $(1 - e_\lambda)M(A)(1 - e_\lambda) = (1 - e_\lambda)T(X)(1 - e_\lambda)$.

**Proof.** For each $\lambda \in \Lambda$ let $E_\lambda$ denote the uniform closure of the linear space $\{e_\lambda a + ae_\lambda - e_\lambda ae_\lambda | a \in M(A)\}$ and let $T_\lambda: X \to M(A)/E_\lambda$ be the bounded linear map defined by $T_\lambda(x) = T(x) + E_\lambda$. We will now show that there exists a $\lambda$ in $\Lambda$ so that $T_\lambda$ maps $X$ onto $M(A)/E_\lambda$. Suppose no such $\lambda$ exists. Let $\lambda_0 \in \Lambda$. By virtue of [10, 487-8] and the fact
that \((M\Lambda/E^\chi)^*\) is isometrically isomorphic to \(E^i\), we can choose \(f_i\) in \(E^i\) so that \(\|f_i\| = 1\) and \(\|T^*(f_i)\| < 1\), where \(T^*\) denotes the adjoint of \(T\). Having defined \(\lambda_1, \lambda_2, \ldots, \lambda_n\) and \(f_1, f_2, \ldots, f_n\) we can choose, by virtue of [13, Corollary 2.2, p. 635], \(\lambda_{n+1} > \lambda_n\) so that

\[
(3.2) \quad \|e_{i_{n+1}}f^*_ne_{i_{n+1}} - f^*_n\| < \frac{1}{n}.
\]

Now as before choose \(f_{n+1}\) in \(E^i_{i_{n+1}}\) so that

\[
(3.3) \quad \|f_{n+1}\| = 1 \quad \text{and} \quad \|T^*(f_{n+1})\| < \frac{1}{n+1}.
\]

We will now show that the sequence \(\{f_n\}\) converges weak* to 0. Let \(a \in M\Lambda\) and let \(\varepsilon > 0\). By our hypothesis there exists an \(x \in X\) and a \(c \in A\) such that \(a = T(x) + c\). Now choose \(\lambda \in A\) so that \(\|c - e\lambda c\| < \varepsilon/3\). Next choose a positive integer \(N\) such that for each integer \(n \geq N\) we have \((e_{i_{n+1}} - e_i)e_i = 0\), \(\|x\|/n < \varepsilon/3\), and \(\|e\|/n < \varepsilon/3\). It follows from (3.2), (3.3), and the fact \(f_n \in E^i_{i_n}\) that for each integer \(n \geq N\)

\[
|f_n(a)| \leq |f_n(T(x))| + |f_n^*(e\lambda c)| + |f_n^*(c - e\lambda c)|
\]
\[
\leq \|T^*f_n\| \|x\| + \|c - e\lambda c\| + \|1 - e\lambda_i\|f_n^*(1 - e\lambda_i)e\lambda c)
\]
\[
\leq \varepsilon/3 + \varepsilon/3 + \|f_n^* - e_{i_{n+1}}f_n^*e_{i_{n+1}}\| \|c\|
\]
\[
+ \|e_{i_{n+1}} - e_{i_n}\| f_n^*(e_{i_{n+1}} - e_{i_n}) (e\lambda c)
\]
\[
< \varepsilon.
\]

Hence \(f_n \to 0\) weak*.

Since \(f_n \to 0\) weak*, we have by Theorem 3.3 that \(\{f_n^*\}\) is tight and converges weak* to zero. Moreover, we will show that \(\|f_n^*\| \to 0\). Let \(\varepsilon > 0\). Choose \(\lambda \in A\) so that \(\|(1 - e\lambda) f_n^*(1 - e\lambda)\| < \varepsilon/2\) for each positive integer \(n\). Next choose a positive integer \(N\) so that for each integer \(n \geq N\), \(e\lambda(e_{i_{n+1}} - e_{i_n}) = 0\) and \(3/n < \varepsilon/2\). Since \(f_n \in E^i_{i_n}\), it is straightforward to verify that \(f_n^* = (1 - e_{i_n}) f_n^*(1 - e_{i_n})\). It follows that for \(n \geq N\)

\[
\|f_n^*\| \leq \|(1 - e\lambda) f_n^*(1 - e\lambda)\| + \|e\lambda f_n^* + f_n^* e_{i_n} - e\lambda f_n^* e_{i_n}\|.
\]

Replacing \(f_n^*\) in the second term by \(e_{i_{n+1}} f_n^* e_{i_{n+1}} - g_n, \quad g_n = -f_n^* + e_{i_{n+1}} f_n^* e_{i_{n+1}},\) we get

\[
\|f_n^*\| < \varepsilon/2 + \|e\lambda e_{i_{n+1}} f_n^* e_{i_{n+1}} + e_{i_{n+1}} f_n^* e_{i_{n+1}} - e\lambda e_{i_{n+1}} f_n^* e_{i_{n+1}} e_{i_n} - 3\|f_n^* - e_{i_{n+1}} f_n^* e_{i_{n+1}}\|
\]
\[
< \varepsilon/2 + 0 + \varepsilon/2
\]
\[
\leq \varepsilon.
\]
for \( n \geq N \). Hence \( \| f_n(a) \| \rightarrow 0 \).

Since the map \((x, c) \rightarrow T(x) + c\) is a bounded linear map from \( x \oplus A \) onto \( M(A) \) by hypothesis, the open mapping theorem gives a constant \( k \) such that if \( a \in M(A) \) and \( \| a \| \leq 1 \), then there exists an \( x \in X \) and \( c \in A \) with \( \| x \| + \| c \| \leq k \) and \( T(x) + c = a \). Then we have

\[
|f_n(a)| \leq |f_n(T(x))| + |f_n(c)| \\
\leq \| T^*f_n \| \| x \| + \| f_n^* \| \| c \| \\
\leq k \left( \frac{1}{n} + \| f_n^* \| \right).
\]

This implies that \( \| f_n \| \leq k(1/n + \| f_n^* \|) \). It follows that \( \| f_n \| \rightarrow 0 \), which contradicts the fact that \( \| f_n \| = 1 \). Hence there exists a \( \lambda_0 \) in \( A \) so that \( T_{\lambda_0} \) maps \( X \) onto \( M(A)/E_{\lambda_0} \).

Finally choose \( \lambda > \lambda_0 \). Let \( a \in M(A) \). Since \( T_{\lambda_0} \) maps \( X \) onto \( M(A)/E_{\lambda_0} \), there exists an \( x \in X \) and \( b \in E_{\lambda_0} \) such that \( T(x) = a + b \).

Due to the fact that \((1 - e_{\lambda_0})b(1 - e_{\lambda_0}) = 0\), we have \((1 - e_{\lambda_0})T(x)(1 - e_{\lambda_0}) = (1 - e_{\lambda_0})a(1 - e_{\lambda_0}) \). Hence \((1 - e_{\lambda_0})T(X)(1 - e_{\lambda_0}) = (1 - e_{\lambda_0})M(A)(1 - e_{\lambda_0}) \) and our proof is complete. The idea of this proof comes from \([2, \text{Theorem } 2.3, \text{p. } 527]\).

The next result is a generalization of Phillips theorem that \( c_0 \) is not complemented in \( l^\infty \). It also shows (i) (using Conway’s result that \( C_0(S) \) is complemented in \( C(S) \) implies \( S \) is pseudo-compact) that \( A = C_0(S) \) is never complemented in \( C(S) \) when \( S \) is paracompact and noncompact, (ii) the compacts are uncomplemented in \( B(H) \) unless \( H \) is finite dimensional.

**Corollary 3.7.** Let \( A \) be a \( C^* \)-algebra with well behaved approximate identity. If \( A \) is without an identity, then \( A \) is not complemented in \( M(A) \).

**Proof.** Suppose \( A \) is complemented in \( M(A) \); that is, suppose there exists, a closed subspace \( X \) of \( M(A) \) such that \( X \oplus A = M(A) \). Then by Theorem 3.6 there exists a \( \lambda \in A \) such that \((1 - e_{\lambda})X(1 - e_{\lambda}) = (1 - e_{\lambda})M(A)(1 - e_{\lambda}) \). Since \( e_{\lambda} \) is not an identity for \( A \), there exists an \( a \in A \) such that \((1 - e_{\lambda})a(1 - e_{\lambda}) \neq 0 \). It follows that there exists an \( x \) in \( X \) such that \((1 - e_{\lambda})x(1 - e_{\lambda}) = (1 - e_{\lambda})a(1 - e_{\lambda}) \), or equivalently, \( x = (1 - e_{\lambda})a(1 - e_{\lambda}) + e_{\lambda}x - e_{\lambda}x \). But this implies that \( x = 0 \), since \( x \in A \cap X \). This contradicts the fact that \((1 - e_{\lambda})a(1 - e_{\lambda}) \neq 0 \). Hence \( A \) is not complemented in \( M(A) \) and the proof is complete.

**Corollary 3.8.** Let \( B \) be a \( W^* \)-algebra and let \( A \) be a closed two-sided ideal of \( B \) with a well behaved approximate identity \( \{ e_{\lambda} \mid \lambda \in A \} \).

If \( X \) is a Banach space and \( T : X \rightarrow B \) is a bounded linear map such
that $T(X) + A = B$, then there exists a $\lambda$ in $A$ such that

$$(1 - e_\lambda)T(X)(1 - e_\lambda) = (1 - e_\lambda)B(1 - e_\lambda).$$

Proof. Let $A^0 = \{x \in B \mid xA = 0\}$. Since $A^0$ is a two-sided ideal of $B$ that is closed in the weak operator topology, $A^0$ has an identity $q$ that commutes with each element of $B$. Let $X_0 = \{x \in X \mid qT(x) = 0\}$. Then define the bounded linear map $T_0 : X_0 \to B/A^0$ by the formula $T_0(x) = T(x) + A^0$ for each $x$ in $X_0$. We now wish to show that $T_0(X_0) + A/A^0 = B/A^0$. Let $a \in B$. It is clear that $a + A^0 = a - qa + A^0$. By hypothesis, there exists an $x \in X$ and a $c \in A$ such that $T(x) + c = (1 - q)a$. This means $qT(x) = q(1 - q)a - qc = 0$, so $x \in X_0$. Hence $T_0(X_0) + A/A^0 = B/A^0$. By Corollary 2.3 $M(A) = B/A^0$. Therefore, by Theorem 3.6, there exists $\lambda$ in $A$ such that

$$(3.4) (1 - e_\lambda)B(1 - e_\lambda) = (1 - e_\lambda)T(X_0)(1 - e_\lambda)/A^0.$$

We will now show that $(1 - e_\lambda)B(1 - e_\lambda) = (1 - e_\lambda)T(X)(1 - e_\lambda)$. Let $a \in B$. Then by virtue of (3.4) there exists an $x \in X_0$ and $c \in A^0$ such that $(1 - e_\lambda)a(1 - e_\lambda) = (1 - e_\lambda)T(x)(1 - e_\lambda) + c$. This implies $(1 - e_\lambda)(1 - q)a(1 - e_\lambda) = (1 - e_\lambda)T(x)(1 - e_\lambda)$. Hence

$$(3.5) (1 - e_\lambda)(1 - q)B(1 - e_\lambda) = (1 - e_\lambda)T(X_0)(1 - e_\lambda).$$

Now let $b \in B$. By hypothesis there exists a $y \in X$ such that $qT(y) = qb$. Set $a = b - T(y)$. By (3.5) there exists an $x \in X_0$ such that

$$(1 - e_\lambda)T(x)(1 - e_\lambda) = (1 - e_\lambda)(1 - q)a(1 - e_\lambda).$$

It follows that

$$(1 - e_\lambda)b(1 - e_\lambda) = (1 - e_\lambda)((1 - q)b + qb)(1 - e_\lambda) = (1 - e_\lambda)((1 - q)b + qT(y))(1 - e_\lambda) = (1 - e_\lambda)((1 - q)b - (1 - q)T(y) + T(y))(1 - e_\lambda) = (1 - e_\lambda)((1 - q)(b - T(y))(1 - e_\lambda) + (1 - e_\lambda)T(y)(1 - e_\lambda) = (1 - e_\lambda)T(x)(1 - e_\lambda) + (1 - e_\lambda)T(y)(1 - e_\lambda) = (1 - e_\lambda)T(x + y)(1 - e_\lambda).$$

Hence $(1 - e_\lambda)B(1 - e_\lambda) = (1 - e_\lambda)T(X)(1 - e_\lambda)$ and our proof is complete.

Let $B$ be a $C^*$-algebra, let $\Omega$ be a compact Hausdorff space, and let $C(\Omega, B)$ denote the space of all $B$-valued continuous functions on $\Omega$. Let $Q$ be a closed subset of $\Omega$. A linear subspace $X$ of $C(\Omega, B)$ is said to interpolate $C(Q, B)$ if $X|Q = C(Q, B)$. More briefly, we call $Q$ an interpolation set for $X$. In [3] Bade investigated a class of theorems
which state for appropriate $B$, $\Omega$, $Q$, and $X$ that if $X$ interpolates $C(Q, B)$, then $X$ interpolates $C(V, B)$ for some closed neighborhood $V$ of $Q$. In particular, Bade showed (see [3, Theorem 1.1, Theorem 2.1, pp. 149, 157]) that this happens whenever the following hold: $B$ is the complex numbers; $\Omega = \beta(S)$, where $S$ is a locally compact, $\sigma$-compact or discrete, Hausdorff space; $Q = \beta S - S$; $X$ is a closed linear subspace of $C(\Omega, B)$. We will now give a natural specialization of Theorem 3.6 that extends Bade's results to a noncommutative setting.

**Corollary 3.9.** Let $B$ be a finite dimensional $C^*$-algebra and let $S$ be a locally compact paracompact Hausdorff space. Let $X$ be a closed linear subspace of $C(\beta(S), B)$ such that $X|\beta(S) - S = C(\beta(S) - S, B)$. Then there exists a closed neighborhood $V$ of $\beta(S) - S$ in $\beta(S)$ such that $X|V = C(V, B)$.

**Proof.** It is straightforward to show that $C_0(S, B)$ has a well behaved approximate identity $\{e_\lambda|\lambda \in \Lambda\}$ such that each $e_\lambda$ has compact support. Since the double centralizer algebra of $C_0(S, B)$ is $C(\beta(S), B)$, the conclusion follows from Theorem 3.6.

**References**


Received June 22, 1970. This research was supported in part by the National Science Foundation, under grant No. GP-15736.

UNIVERSITY OF MISSOURI

AND

LOUISIANA STATE UNIVERSITY
The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.
Louis I. Alpert and L. V. Toralballa, *An elementary definition of surface area in $E^{n+1}$ for smooth surfaces* ................................................................. 261
Eamon Boyd Barrett, *A three point condition for surfaces of constant mean curvature* ................................................................. 269
Jan-Erik Björk, *On the spectral radius formula in Banach algebras* ........... 279
Peter Botta, *Matrix inequalities and kernels of linear transformations* ........ 285
Bennett Eisenberg, *Baxter’s theorem and Varberg’s conjecture* ............... 291
Heinrich W. Guggenheimer, *Approximation of curves* .............................. 301
A. Hedayat, *An algebraic property of the totally symmetric loops associated with Kirkman-Steiner triple systems* ............................. 305
Richard Howard Herman and Michael Charles Reed, *Covariant representations of infinite tensor product algebras* ......................... 311
Domingo Antonio Herrero, *Analytic continuation of inner function-operators* ................................................................. 327
Franklin Lowenthal, *Uniform finite generation of the affine group* ............ 341
Stephen H. McCleary, *0-primitive ordered permutation groups* ............... 349
Malcolm Jay Sherman, *Disjoint maximal invariant subspaces* ................... 373
Mitsuru Nakai, *Radon-Nikodým densities and Jacobians* ......................... 375
Mitsuru Nakai, *Royden algebras and quasi-isometries of Riemannian manifolds* ................................................................. 397
Russell Daniel Rupp, Jr., *A new type of variational theory sufficiency theorem* ................................................................. 415
Helga Schirmer, *Fixed point and coincidence sets of biconnected multifunctions on trees* ................................................................. 445
Murray Silver, *On extremal figures admissible relative to rectangular lattices* ................................................................. 451
James DeWitt Stein, *The open mapping theorem for spaces with unique segments* ................................................................. 459
Arne Stray, *Approximation and interpolation* ........................................... 463
Donald Curtis Taylor, *A general Phillips theorem for $C^*$-algebras and some applications* ................................................................. 477
Florian Vasilescu, *On the operator $M(Y) = T Y S^{-1}$ in locally convex algebras* ................................................................. 489
Philip William Walker, *Asymptotics for a class of weighted eigenvalue problems* ................................................................. 501
Kenneth S. Williams, *Exponential sums over GF($2^n$)* ............................ 511