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Let $\mathfrak{A}$ be a locally convex separated unitary algebra over the complex field. If T and S are fixed elements of $\mathfrak{A}$ and S is invertible, it is possible to define on $\mathfrak{A}$ the linear operator

$$M(Y) = M(T, S)(Y) = TYS^{-1}$$

for all $Y \in \mathfrak{A}$. The purpose of this paper is to construct a functional calculus with analytic functions for the operator $M(T, S)$, by means of T and S, in order to obtain "multiplicative variants" of some results of M. Rosenblum. In the last section these results are applied to normal operators and matrices.

In what follows U will be a locally convex algebra i.e., an algebra which is a locally convex space and where the multiplication is separately continuous. The topology on $\mathfrak{A}$ is defined by a family of seminorms $\{p_\alpha\}_{\alpha \in J}$ and the space is assumed to be quasi-complete, i.e., every Cauchy net is convergent. It is also supposed that the mappings $Y \to YZ$ (resp. $Z \to YZ$) are uniformly continuous when Z (resp. Y) belongs to a bounded set.

Denoting $B(\mathfrak{A})$ the algebra of all continuous linear operators on $\mathfrak{A}$, the topology on $B(\mathfrak{A})$ is defined by the family of seminorms $\{p_{\alpha,B}\}_{\alpha \in J, B \in \mathfrak{B}}$, where

$$p_{\alpha,B}(L) = \sup_{Y \in B} p(L(Y)) ,$$

for each $L \in B(\mathfrak{A})$, $\mathfrak{B}$ being the family of all bounded sets of $\mathfrak{A}$.

When $T, S \in \mathfrak{A}$ and S is invertible, it is shown that the spectrum $\sigma(M)$ of the operator

$$M(Y) = TYS^{-1}$$

is contained in the set $\sigma(T) \cdot \sigma(S^{-1})$ (Proposition 2.4) hence, taking a complex-valued function f, analytic in a neighbourhood of the set $\sigma(T) \cdot \sigma(S^{-1})$, we can construct the operator $f(M)$ in each point of $\mathfrak{A}$, as well by means of the functional calculus of T, S and they are equal (Theorem 2.9). Since the logarithm of an element of $\mathfrak{A}$ does not always exist, this case is more general than Rosenblum's results (see [3] and Proposition 3.1).

All the statements of this paper can be applied to linear operators on Banach spaces or on locally convex ones, with supplementary properties.

1. **Preliminaries.** It is useful to recall some of the concepts and results contained in [1] and [4].

On a locally convex algebra it is possible to define the spectrum $\sigma(T)$ and the resolvent $\rho(T)$ of an element $T \in \mathfrak{A}$ (these sets are considered in the complex compactified plane $C_\infty = C \cup \{\infty\}$) [4].

We recall that $C_\infty \ni \lambda \in \rho(T)$ if there is a neighbourhood V_λ of λ such that:

1^0 $(\mu I - T)^{-1} \in \mathfrak{A}$ for any $\mu \in V_\lambda \cap C$ (here I is the identity of $\mathfrak{A}$).

2^0 the set $\{(\mu I - T)^{-1}; \mu \in V_\lambda \cap C\}$ is bounded in $\mathfrak{A}$.

In the following we shall write for μI simply μ.

For each $T \in \mathfrak{A}$ let us denote by $\mathfrak{F}(T)$ the set of all analytic functions in a neighbourhood of $\sigma(T) = \complement\rho(T)$. Then, if Γ is a contour (throughout we mean by "contour" a finite system of curves, admissible for the integral calculus) "surrounding" $\sigma(T)$, contained in the domain of definition of $f \in \mathfrak{F}(T)$, we put, by definition

$$
f(T) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\Gamma f(\lambda)(\lambda - T)^{-1}d\lambda & \text{if } \sigma(T) \not\ni \infty \\[2ex] f(\infty) + \dfrac{1}{2\pi i}\displaystyle\int_\Gamma f(\lambda)(\lambda - T)^{-1}d\lambda & \text{if } \sigma(T) \ni \infty \end{cases}
$$

where the integral exists since the space is quasi-complete [1], [4]. For such functions we have the "spectral mapping theorem", namely

$$
\sigma(f(T)) = f(\sigma(T)) [4] .
$$

Let us remark that if $L \in B(\mathfrak{A})$, $f \in \mathfrak{F}(L)$ and Γ is a contour in the domain of f "surrounding" $\sigma(L)$, then we may define the expression $f(L)(Y)$, by using the natural extension of the formula given above for the elements of $\mathfrak{A}$. Since $\mathfrak{A}$ is quasi-complete, the integrals do also exist.

2. **A functional calculus of the operator M.** For two sets A, B in the complex compactified plane with the property that if one contains the point ∞ then the other does not contain zero, we denote by $A \cdot B$ the set $\{\lambda\mu; \lambda \in A, \mu \in B\}$. Also, if $A \not\ni 0$, we denote by A^{-1} the set $\{1/\lambda; \lambda \in A\}$. In the following we need the next geometrical result:

LEMMA 2.1. *Let K and F be two closed sets in C_∞, $0 \notin K \not\ni \infty$. If $\lambda_0 \notin K \cdot F$ and V_0 is a closed neighbourhood of λ_0 disjoint from $K \cdot F$, then there is an open set $G_0 \supset K$ such that $V_0 \cap \overline{G}_0 \cdot F = \varnothing$. Moreover, if Γ_0 is a contour in G_0 which surrounds K and separates it from zero, then $\Gamma_\lambda = \lambda\Gamma_0^{-1}$ is a contour such that the set F is "outside" it,*

for each $\lambda \in V_0,\ 0 \neq \lambda \neq \infty$.

Proof. If we put

$$G_n = \left\{ \mu \in C;\ \mathrm{dist}\,(\mu, K) < \frac{1}{n} \right\},$$

on account of the compactness of F, K and V_0 in C_∞, there is an index n_0 such that $V_0 \cap \bar{G}_{n_0} \cdot F = \varnothing$. We take $G_0 = G_{n_0}$. We can actually suppose that $\bar{G}_0 \not\ni 0$. Now, let Γ_0 be a contour in G_0 surrounding K and separating it from zero. Then for $\lambda \in V_0,\ 0 \neq \lambda \neq \infty$ we have $\Gamma_\lambda = \lambda \Gamma_0^{-1} \subset V_0 \cdot G_0^{-1}$ and $V_0 \cdot G_0^{-1} \cap F = \varnothing$.

LEMMA 2.2. *Let* Γ *be a system of curves in the complex plane, admissible for the integral calculus,* $V_0 \subset C_\infty$ *a closed set and* $F: V_0 \times \Gamma \to \mathfrak{A},\ G: \Gamma \to \mathfrak{A}$ *two continuous functions. Then for any* $\lambda \in V_0 \cap C$ *we can define on* $\mathfrak{A}$ *the linear continuous operator*

$$R_\lambda(Y) = \int_\Gamma F(\lambda, \xi)\, Y G(\xi)\, d\xi\ ,$$

for each $Y \in \mathfrak{A}$.

Proof. By our assumption on the algebra $\mathfrak{A}$, it is easy to see that the product of two continuous functions is also a continuous function (since if U is a neighbourhood of zero in $\mathfrak{A}$ and B is a bounded set then there are two neighbourhoods of zero U_1 and U_2 such that $U_1 B \subset U$ and $B U_2 \subset U$). Since the mapping $\xi \to F(\lambda, \xi) Y G(\xi)$ is continuous on Γ for each $\lambda \in V_0 \cap C$ and the algebra is quasi-complete, then the integral

$$\int_\Gamma F(\lambda, \xi)\, Y G(\xi)\, d\xi$$

exists as an element of $\mathfrak{A}$.

Obviously, it defines a linear operator on $\mathfrak{A}$ denoted by $R_\lambda(Y)$. To see that R is a continuous operator on $\mathfrak{A}$, let us denote by B_1 the set $\{F(\lambda, \xi);\ \lambda \in V_0,\ \xi \in \Gamma\}$ and by B_2 the set $\{G(\xi);\ \xi \in \Gamma\}$ which are bounded in $\mathfrak{A}$. If $U = \{T \in \mathfrak{A};\ p_\alpha(T) < \varepsilon\}$ then, by our hypothesis, there is a neighbourhood U_0 of zero in $\mathfrak{A}$ such that $B_1 U_0 B_2 \subset (2\pi/|\Gamma|)\, U$, where $|\Gamma|$ is the length of Γ. Thus we have

$$p_\alpha(R_\lambda(Y)) \leqq \frac{1}{2\pi} \int_\Gamma |F(\lambda, \xi)\, Y G(\xi)\,|_\alpha\, |d\xi\,| < \varepsilon$$

whenever $Y \in U_0$, hence R_λ is continuous.

PROPOSITION 2.3. *Let* $T, S \in \mathfrak{A}$ *be such that* $S^{-1} \in \mathfrak{A}$. *If* $\lambda_0 \notin \sigma(T)$. $\sigma(S^{-1})$ *and* V_0 *is a closed neighbourhood of* λ_0 *such that* $V_0 \cap \sigma(T)$. $\sigma(S^{-1}) = \varnothing$, *then there is an open set* $G_0 \supset \sigma(S^{-1}),\ G_0 \not\ni 0$ *such that* V_0.

$G_0^{-1} \subset \rho(T)$.

Moreover, if Γ_0 is a contour in G_0 which surrounds $\sigma(S^{-1})$ and separates it from zero, then we have

$$(\lambda - M(T, S))^{-1}(Y) = \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda - \xi T)^{-1} Y(\xi - S^{-1})^{-1} d\xi$$

for each $Y \in \mathfrak{A}$ and $\lambda \in V_0 \cap C$.

Proof. By the spectral mapping theorem, $\sigma(S^{-1}) = \sigma(S)^{-1}$ does not contain zero and it is a compact set in C, therefore we may apply Lemma 2.1, by putting $F = \sigma(T)$ and $K = \sigma(S^{-1})$. Let G_0 and Γ_0 be as in this lemma. Then $V_0 \cdot \Gamma_0^{-1} \subset V_0 \cdot G_0^{-1}$ does not intersect the set $F = \sigma(T)$, therefore $\lambda \xi^{-1} \in \rho(T)$ for all $\lambda \in V_0 \cap C$ and $\xi \in \Gamma_0$, and $\sigma(T)$ is "outside" $\Gamma_\lambda = \lambda \Gamma_0^{-1}$ $(0 \neq \lambda \neq \infty)$. By Lemma 2.2, the integral

$$\frac{1}{2\pi i} \int_{\Gamma_0} (\lambda - \xi T)^{-1} Y(\xi - S^{-1})^{-1} d\xi = \frac{1}{2\pi i} \int_{\Gamma_0} \left(\frac{\lambda}{\xi} - T\right)^{-1} Y(\xi - S^{-1})^{-1} \frac{d\xi}{\xi}$$

exists, therefore we have for any $\lambda \in V_0$ $(0 \neq \lambda \neq \infty)$

$$(\lambda - M(T, S)) \left(\frac{1}{2\pi i} \int_{\Gamma_0} \left(\frac{\lambda}{\xi} - T\right)^{-1} Y(\xi - S^{-1})^{-1} \frac{d\xi}{\xi} \right)$$

$$= \frac{\lambda}{2\pi i} \int_{\Gamma_0} \left(\frac{\lambda}{\xi} - T\right)^{-1} Y(\xi - S^{-1})^{-1} \frac{d\xi}{\xi}$$

$$- \frac{1}{2\pi i} \int_{\Gamma_0} \left(T - \frac{\lambda}{\xi} + \frac{\lambda}{\xi}\right) \left(\frac{\lambda}{\xi} - T\right)^{-1} Y(\xi - S^{-1})^{-1}(S^{-1} - \xi + \xi) \frac{d\xi}{\xi}$$

$$= -\frac{1}{2\pi i} \int_{\Gamma_0} Y \frac{d\xi}{\xi} + \frac{\lambda}{2\pi i} \int_{\Gamma_0} \left(\frac{\lambda}{\xi} - T\right)^{-1} \frac{d\xi}{\xi}$$

$$+ \frac{1}{2\pi i} \int_{\Gamma_0} Y(\xi - S^{-1})^{-1} d\xi = Y,$$

since

$$\frac{1}{2\pi i} \int_{\Gamma_0} \frac{d\xi}{\xi} = 0,$$

$$\frac{1}{2\pi i} \int_{\Gamma_0} Y(\xi - S^{-1})^{-1} d\xi = Y$$

by the well-known functional calculus for an element of $\mathfrak{A}$ with the spectrum compact in C and

$$\frac{\lambda}{2\pi i} \int_{\Gamma_0} \left(\frac{\lambda}{\xi} - T\right)^{-1} \frac{d\xi}{\xi^2} = -\frac{1}{2\pi i} \int_{\lambda \Gamma_0^{-1}} (\eta - T)^{-1} d\eta = 0,$$

since $\sigma(T)$ is "outside" $\lambda \Gamma_0^{-1}$.

Analogously

$$\frac{1}{2\pi i}\int_{\Gamma_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}((\lambda - M(T, S))(Y))(\xi - S^{-1})^{-1}\frac{d\xi}{\xi} = Y ,$$

therefore

$$(\lambda - M(T, S))^{-1}(Y) = \frac{1}{2\pi i}\int_{\Gamma_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}Y(\xi - S^{-1})^{-1}\frac{d\xi}{\xi} .$$

If $\lambda = 0$ then $0 \notin \sigma(T)$, hence $T^{-1} \in \mathfrak{A}$ and

$$(-M(T, S))^{-1}(Y) = -T^{-1}YS = \frac{1}{2\pi i}\int_{\Gamma_0} -T^{-1}Y(\xi - S^{-1})^{-1}\frac{d\xi}{\xi}$$

and this finishes the proof.

PROPOSITION 2.4.   *With the same conditions as in the previous proposition, we have*

$$\sigma(M(T, S)) \subset \sigma(T) \cdot \sigma(S^{-1}) .$$

Proof. We can suppose $\sigma(T) \cdot \sigma(S^{-1}) \neq C_\infty$. By the preceding proposition, if $\lambda_0 \notin \sigma(T) \cdot \sigma(S^{-1})$ and V_0 is a closed neighbourhood of λ_0, $V_0 \cap \sigma(T) . \sigma(S^{-1}) = \varnothing$, then for any $\lambda \in V_0 \cap C$ the operator $(\lambda - M(T, S))^{-1}$ exists and, by Lemma 2.2, it is a continuous operator. We have only to prove that the set

$$\{(\lambda - M(T, S))^{-1}; \lambda \in V_0 \cap C\}$$

is a bounded one in $B(\mathfrak{A})$.

For, let $\{p_\alpha\}_{\alpha \in J}$ the family of semi-norms on $\mathfrak{A}$ and $\{p_{\alpha\,B}\}_{\alpha \in J, B \in \mathfrak{B}}$ the family of semi-norms on $B(\mathfrak{A})$ (see the introduction). Define $B_1 = \{(\lambda - \xi T)^{-1}, \lambda \in V_0 \cap C, \xi \in \Gamma_0\}$ and $B_2 = \{(\xi - S^{-1})^{-1}; \xi \in \Gamma_0\}$ which are bounded in $\mathfrak{A}$. Indeed, $V_0 \cdot \Gamma_0^{-1} \subset \rho(T)$ and it is a compact set in C, therefore by reasoning with a finite covering, we obtain the boundedness of the family B_1. A similar argument is valid for B_2. If $B \in \mathfrak{B}$ is arbitrary, then B_1BB_2 is also a bounded set, therefore we have

$$p_{\alpha,B}((\lambda - M(T, S))^{-1}) = \sup_{y \in B} p_\alpha((\lambda - M(T, S))^{-1}(Y))$$

$$\leq \frac{1}{2\pi}\sup_{y \in B}\int_{\Gamma_0} p_\alpha((\lambda - \xi T)^{-1}Y(\xi - S^{-1})^{-1}) |d\xi|$$

$$\leq C(\alpha, B) < \infty ,$$

where we kept the notations of the preceding proposition. Consequently $\lambda_0 \in \rho(M(T, S))$.

COROLLARY 1.   *If $M = M(T, T)$ is an inner automorphism of the algebra $\mathfrak{A}$ then*

$$\sigma(M) \subset \sigma(T) \cdot \sigma(T^{-1}) \ .$$

PROPOSITION 2.5. *Let* T, S *be in* $\mathfrak{A}$ *with* $S^{-1} \in \mathfrak{A}$, *and* $M = M(T, S)$. *If* $\lambda_0 \notin \sigma(M) \cdot \sigma(S)$ *and* V_0 *is a closed neighbourhood of* λ_0 *such that* $V_0 \cap \sigma(M) \cdot \sigma(S) = \phi$, *then there is an open set* $G_0 \supset \sigma(S)$, $G_0 \not\ni 0$ *such that* $V_0 \cdot G_0^{-1} \subset \rho(M)$.

Moreover, if Γ_0 *is a contour in* G_0 *which surrounds* $\sigma(S)$ *and separates it from zero, we have*

$$(\lambda - T)\Big(\frac{1}{2\pi i}\int_{\Gamma_0} (\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1}d\xi\Big) = Y$$

for all $Y \in \mathfrak{A}$ *and* $\lambda \in V_0 \cap C$.

Proof. We apply Lemma 2.1 with $F = \sigma(M)$ and $K = \sigma(S)$. Therefore, if Γ_0 is a contour as in the quoted lemma, then the integral

$$\frac{1}{2\pi i}\int_{\Gamma_0} (\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1}d\xi$$

exists as an element of $\mathfrak{A}$ for each $\lambda \in V_0 \cap C$.
From the relation

$$\lambda(\lambda - \xi M)^{-1}(Y) - \xi T(\lambda - \xi M)^{-1}S^{-1} = Y$$

we obtain

$$T(\lambda - \xi M)^{-1} = \frac{1}{\xi}(\lambda(\lambda - \xi M)^{-1}(Y) - Y)S \ ,$$

therefore we can write, for $\lambda \in V_0 \cap C$, $\lambda \neq 0$,

$$(\lambda - T)\frac{1}{2\pi i}\int_{\Gamma_0} (\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1}d\xi$$

$$= \frac{\lambda}{2\pi i}\int_{\Gamma_0} (\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1}d\xi$$

$$- \frac{1}{2\pi i}\int_{\Gamma_0} \frac{1}{\xi}(\lambda(\lambda - \xi M)^{-1}(Y) - Y)(-1 + \xi(\xi - S)^{-1})d\xi$$

$$= \frac{1}{2\pi i}\int_{\Gamma_0} \frac{\lambda}{\xi}(\lambda - \xi M)^{-1}(Y)d\xi - \frac{1}{2\pi i}\int_{\Gamma_0} \frac{d\xi}{\xi}Y$$

$$+ \frac{1}{2\pi i}\int_{\Gamma_0} Y(\xi - S)^{-1}d\xi = Y \ ,$$

since

$$\frac{1}{2\pi i}\int_{\Gamma_0} \frac{d\xi}{\xi} = 0 \ ,$$

$$\frac{1}{2\pi i}\int_{\Gamma_0}(\xi - S)^{-1}d\xi = I$$

and

$$\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\lambda}{\xi}(\lambda - \xi M)^{-1}(Y)d\xi = -\frac{1}{2\pi i}\int_{\lambda\Gamma_0^{-1}}(\eta - M)^{-1}(Y)d\eta = 0$$

because $\sigma(M)$ is "outside" $\lambda\Gamma_0^{-1}$.

If $\lambda = 0$, a direct argument proves the validity of the given formula.

PROPOSITION 2.6. *With the same conditions as in the previous proposition, if $\sigma(T) \ni \infty$ then $\sigma(M) \ni \infty$.*

Proof. Let us suppose that $\sigma(M) \not\ni \infty$. Then there is a closed neighbourhood of ∞, let us say V_∞, where we have

$$(\lambda - \xi M)^{-1}(Y) = \sum_{n=0}^{\infty}\frac{\xi^n M^n(Y)}{\lambda^{n+1}}$$

for all $\xi \in \Gamma_0$ where the series is uniformly convergent in $\mathfrak{A}$.

Hence we have

$$\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1}d\xi$$

$$= \frac{1}{2\pi i}\int_{\Gamma_0}\sum_{n=0}^{\infty}\frac{\xi^n M^n(Y)}{\lambda^{n+1}}(\xi - S)^{-1}d\xi$$

$$= \sum_{n=0}^{\infty}\frac{T^n YS^{-n}}{\lambda^{n+1}}\frac{1}{2\pi i}\int_{\Gamma_0}\xi^n(\xi - S)^{-1}d\xi$$

$$= \sum_{n=0}^{\infty}\frac{T^n Y}{\lambda^{n+1}},$$

thus the last series is uniformly convergent in a neighbourhood of ∞ and defines, for $Y = I$, the inverse $\lambda - T$.

Moreover, the set $\{(\lambda - T)^{-1}; \lambda \in V_\infty\}$ is bounded in $\mathfrak{A}$ since the set

$$\left\{\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda - \xi M)^{-1}(I)(\xi - S)^{-1}d\xi; \lambda \in V_\infty\right\}$$

is bounded in $\mathfrak{A}$ (see the proof of Proposition 2.4), thus $\infty \in \rho(T)$.

PROPOSITION 2.7. *Let $T, S \in \mathfrak{A}$ be such that $S^{-1} \in \mathfrak{A}$ and $M = M(T, S)$. Suppose M invertible on $\mathfrak{A}$. If $\lambda_0 \notin \sigma(T) \cdot \sigma(M)^{-1}$ and V_0 is a closed neighbourhood of λ_0 such that $V_0 \cap \sigma(T) \cdot \sigma(M)^{-1} = \varnothing$ then there is an open set $G_0 \supset \sigma(M)^{-1}$, $G_0 \not\ni 0$ such that $V_0 \cdot G_0^{-1} \subset \rho(T)$. Moreover, if Γ_0*

is a contour in G_0 which surrounds $\sigma(M)^{-1}$, we have

$$\left(\frac{1}{2\pi i}\int_{r_0}(\lambda - \xi T)^{-1}(\xi M - 1)^{-1}(Y)\frac{d\xi}{\xi}\right)(\lambda - S) = Y$$

for all $Y \in \mathfrak{A}$ and $\lambda \in V_0 \cap C$.

Proof. If $1/\xi \in \rho(M)$ then from the relation

$$\left(M - \frac{1}{\xi}\right)\left(M - \frac{1}{\xi}\right)^{-1}(Y) = Y$$

we obtain

$$\left(M - \frac{1}{\xi}\right)^{-1}(Y)S = \xi\left(T\left(M - \frac{1}{\xi}\right)^{-1}(Y) - YS\right).$$

As in the proof of Proposition 2.5, we may apply Lemma 2.1 with $F = \sigma(T)$ and $K = \sigma(M)^{-1}$ and if Γ_0 surrounds $\sigma(M)^{-1}$, we have for λ in a neighbourhood V_0 of λ_0, $0 \neq \lambda \neq \infty$,

$$\left(\frac{1}{2\pi i}\int_{r_0}(\lambda - \xi T)^{-1}(\xi M - 1)^{-1}(Y)\frac{d\xi}{\xi}\right)(\lambda - S)$$

$$= \left(\frac{1}{2\pi i}\int_{r_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}\frac{d\xi}{\xi}\right)YS + \frac{1}{2\pi i}\int_{r_0}\left(M - \frac{1}{\xi}\right)^{-1}(Y)\frac{d\xi}{\xi^2} = Y,$$

since

$$\frac{1}{2\pi i}\int_{r_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}\frac{d}{\xi^2}) = -\frac{1}{2\pi i}\int_{\lambda r_0^{-1}}(\eta - T)^{-1}d\eta = 0$$

(because $\sigma(T)$ is "outside" $\lambda\Gamma_0^{-1}$) and

$$\frac{1}{2\pi i}\int_{r_0}\left(M - \frac{1}{\xi}\right)^{-1}(Y)\frac{d\xi}{\xi^2} = \frac{1}{2\pi i}\int_{r_0^{-1}}(\eta - M)^{-1}(Y)d\eta = Y$$

as Γ_0^{-1} surrounds $\sigma(M)$.

If $0 \notin \sigma(T)\cdot\sigma(M)^{-1}$ then $T^{-1} \in \mathfrak{A}$ and the formula is immediate.

LEMMA 2.8. *Let F and K be two closed sets, $0 \notin K \not\ni \infty$ and $G \supset K.F$ an open set. Then there is an open set $G_0 \not\ni 0$, $G_0 \supset K$ such that $G \supset \bar{G}_0 \cdot F$. Moreover, if $\Gamma \subset G$ is a contour surrounding $K \cdot F$ then we can take a contour Γ_0 in G_0 surrounding K and separating it from zero such that $\xi \cdot F$ is "inside" Γ for all $\xi \in \Gamma_0$.*

Proof. The set $\complement G$ is compact in C_∞, therefore we can apply Lemma 2.1 for any $\lambda_0 \in \complement G$ and, taking a finite covering of $\complement G$, we obtain the set G_0. If Γ is a contour surrounding $K \cdot F$, we can choose G_0 such that $\bar{G}_0$. F is "inside" Γ, hence there is a contour Γ_0 in G_0

which surrounds K and separates it from zero such that $\Gamma_0 \cdot F$ is "inside" Γ.

THEOREM 2.9. *Let* $T, S \in \mathfrak{A}$ *be such that* $S^{-1} \in \mathfrak{A}$. *If* $\sigma(T) \cdot \sigma(S^{-1}) \neq C_\infty$ *and* f *is an analytic function, defined in an open set containing* $\sigma(T) \cdot \sigma(S^{-1})$, *then there is a contour* Γ_0 *surrounding* $\sigma(S^{-1})$ *and separating it from zero such that* $f(\xi T)$ *is defined for each* $\xi \in \Gamma_0$ *and*

$$f(M(T, S))(Y) = \frac{1}{2\pi i} \int_{\Gamma_0} f(\xi T) Y(\xi - S^{-1})^{-1} d\xi ,$$

for all $Y \in \mathfrak{A}$ *(where the left side is defined as in the Introduction).*

Proof. We apply Lemma 2.5 with $F = \sigma(T)$ and $K = \sigma(S^{-1})$. Let Γ and Γ_0 be as in this lemma. Suppose that $\sigma(M) \ni \infty$. Then, by Proposition 2.4, we must have $\sigma(T) \ni \infty$. Hence, by Proposition 2.3, we can write

$$f(M(T, S)(Y)) = f(\infty) Y + 1/2\pi i \int_\Gamma f(\lambda)(\lambda - M(T, S))^{-1}(Y) d\lambda$$

$$= f(\infty) Y + 1/2\pi i \int_\Gamma f(\lambda) \Big(\frac{1}{2\pi i} \int_{\Gamma_0} (\lambda - \xi T)^{-1} Y(\xi - S^{-1})^{-1} d\xi \Big) d\lambda .$$

By interchanging the order of integration, we obtain

$$\frac{1}{2\pi i} \int_\Gamma f(\lambda) \Big(\frac{1}{2\pi i} \int_{\Gamma_0} (\lambda - \xi T)^{-1} Y(\xi - S^{-1})^{-1} d\xi \Big) d\lambda$$

$$= \frac{1}{2\pi i} \int_{\Gamma_0} \Big(\frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - \xi T)^{-1} d\lambda \Big) Y(\xi - S^{-1})^{-1} d\xi$$

$$= \frac{1}{2\pi i} \int_{\Gamma_0} (f(\xi T) - f(\infty)) Y(\xi - S^{-1})^{-1} d\xi$$

$$= \frac{1}{2\pi i} \int_{\Gamma_0} f(\xi T) Y(\xi - S^{-1})^{-1} d\xi - f(\infty) Y$$

since, by Lemma 2.5, $\sigma(\xi T) = \xi \sigma(T)$ is "inside" Γ for all $\xi \in \Gamma_0$. In this manner we obtain

$$f(M(T, S))(Y) = \frac{1}{2\pi i} \int_{\Gamma_0} f(\xi T) Y(\xi - S^{-1})^{-1} d\xi .$$

If $\sigma(T) \not\ni \infty$ we have $\sigma(M) \not\ni \infty$ and a similar calculus leads to the same formula. No other case is possible because of Proposition 2.6, and this finishes our proof.

3. **Some applications of the functional calculus.** First of all, we shall show that, in a certain sense, the commutator of two elements

[3] can be found again as a function of the operator M.

PROPOSITION 3.1. *Let T, S be in $\mathfrak{A}$, with compact spectra in C and with $S^{-1} \in \mathfrak{A}$. If none of the sets $\sigma(T)$, $\sigma(S)$ and $\sigma(T) \cdot \sigma(S^{-1})$ separates the complex plane, we have the relation*

$$(\log M(T, S))(Y) = (\log T) Y - Y(\log S) ,$$

for all $Y \in \mathfrak{A}$.

Proof. By our assumption, $\log T$, $\log S$ and $\log M(T, S)$ exist and, from Theorem 2.9, it follows that

$$(\log M(T, S))(Y) = \frac{1}{2\pi i} \int_{\Gamma_0} \log (\xi T) Y(\xi - S^{-1})^{-1} d\xi$$

$$= \frac{1}{2\pi i} \int_{\Gamma_0} (\log \xi) Y(\xi - S^{-1})^{-1} d\xi + \frac{1}{2\pi i} \int_{\Gamma_0} (\log T) Y(\xi - S^{-1})^{-1} d\xi$$

$$= Y(\log S^{-1}) + (\log T) T = (\log T) Y - Y(\log S) .$$

Let now $E(\cdot)$ and $F(\cdot)$ two selfadjoint spectral measures on a Hilbert space, defined on Borel sets of the complex plane. Then the mappings

$$\mathfrak{E}(\sigma_1)(Y) = E(\sigma_1) Y$$

$$\mathfrak{F}(\sigma_2)(Y) = YF(\sigma_2)$$

are two commuting spectral measures, Y being an arbitrary linear bounded operator; therefore the mapping

$$(\mathfrak{E} \times \mathfrak{F})(\sigma_1 \times \sigma_2)(Y) = E(\sigma_1) YF(\sigma_2)$$

induces a spectral measure on the space of the operators and it is possible to integrate with respect to it (see [2] for details).

PROPOSITION 3.2. *Suppose that $\mathfrak{A}$ is the algebra of all linear operators on a Hilbert space and T, $S \in \mathfrak{A}(S^{-1} \in \mathfrak{A})$ two normal operators. If E, F are the spectral measures of T and S respectively, then for any function f, analytic in a neighbourhood of $\sigma(T) \cdot \sigma(S^{-1})$, we have*

$$f(M(T, S))(Y) = \iint f(\lambda \mu^{-1}) dE_\lambda Y dE_\mu ,$$

for all $Y \in \mathfrak{A}$.

Proof. Using the same notations as in Theorem 2.9, we have

$$f(M(T, S))(Y) = \frac{1}{2\pi i} \int_{\Gamma_0} f(\xi T) Y(\xi - S^{-1})^{-1} d\xi$$

$$= \frac{1}{2\pi i} \int_{\Gamma_0} f(\xi T) Y \left(\int (\xi - \mu^{-1})^{-1} dF_\mu \right) d\xi$$

$$= \int \left(\frac{1}{2\pi i} \int_{\Gamma_0} f(\xi T)(\xi - \mu^{-1})^{-1} d\xi \right) Y dF_\mu$$

$$= \int \left(\frac{1}{2\pi i} \int_{\Gamma_0} (Sf(\xi \lambda) dE_\lambda)(\xi - \mu^{-1})^{-1} d\xi \right) Y dF_\mu$$

$$= \int\int \left(\frac{1}{2\pi i} \int_{\Gamma_0} f(\xi \lambda)(\xi - \mu^{-1})^{-1} d\xi \right) dE_\lambda Y dF_\mu$$

$$= \int\int f(\lambda \mu^{-1}) dE_\lambda Y dF_\mu \ .$$

PROPOSITION 3.3. *Assume that the Hilbert space in the previous proposition is finite dimensional and that*

$$T = \sum_{j=1}^{p} \lambda_j E_j, \qquad S = \sum_{k=1}^{2} \mu_k F_k$$

(where $\{E_j\}$ and $\{F_k\}$ are now finite orthogonal resolutions of the identity). If f is an analytic function in an open set containing the set $\{\lambda_1, \cdots, \lambda_p\} \cdot \{\mu_1^{-1}, \cdots, \mu_2^{-1}\}$ then we have

$$f(M(T, S))(Y) = \sum_{j=1}^{T} \sum_{k=1}^{T} f(\lambda_j \mu_k^{-1}) E_j Y F_k \ ,$$

for all $Y \in \mathfrak{A}$.

The proof follows easily from the preceding proposition.

PROPOSITION 3.4. *Suppose that $\mathfrak{A}$, T and S are as in Proposition 3.3. Then a necessary and sufficient condition that the equation $TY = ZS$ have a solution $Y \in \mathfrak{A}$ is that $\lambda_r = 0$ implies $E_r Z = 0$.*

Proof. Let Y be a solution of the equation $TY = ZS$, hence

$$TYS^{-1} = \sum_{j=1}^{p} \sum_{k=1}^{2} \lambda_j \mu_k^{-1} E_j Y F_k = Z \ .$$

From this relation we obtain easily

$$E_r TYS^{-1} F_s = \lambda_r \mu_s^{-1} E_r Y F_s = E_r Z F_s \ ,$$

thus $\lambda_r E_s Y F_s = \mu_s E_r Z F_s$.

If $\lambda_r = 0$, since $\mu_s \neq 0$ for all s, we have $E_r Z F_s = 0$, hence $\sum_s E_r Z F_s = E_r Z = 0$.

Conversely, if $\lambda_r = 0$ implies $E_r Z = 0$, let us consider the matrix

$$Y = \sum_{\lambda_j \neq 0} \sum_{k=1}^{L} \frac{\mu_k}{\lambda_j} E_j Z F_k \ .$$

We have

$$TYS^{-1} = \sum_{l=1}^{q} \lambda_l E_l \sum_{\lambda_j \neq 0} \sum_{k=1}^{q} \frac{\mu_l}{\lambda_j} E_j Z F_k \sum_{t=1}^{q} \mu_t^{-1} F_t$$

$$= \sum_{l=1}^{p} E_l Z \sum_{t=1}^{q} F_t = Z ,$$

consequently Y is a solution of the equation.

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