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ALGEBRAS**

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ON THE OPERATOR $M(Y) = TYS^{-1}$ IN LOCALLY CONVEX ALGEBRAS

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Let \mathfrak{A} be a locally convex separated unitary algebra over the complex field. If T and S are fixed elements of \mathfrak{A} and S is invertible, it is possible to define on \mathfrak{A} the linear operator

$$M(Y) = M(T, S)(Y) = TYS^{-1}$$

for all $Y \in \mathfrak{A}$. The purpose of this paper is to construct a functional calculus with analytic functions for the operator $M(T, S)$, by means of T and S , in order to obtain "multiplicative variants" of some results of M. Rosenblum. In the last section these results are applied to normal operators and matrices.

In what follows U will be a locally convex algebra i.e., an algebra which is a locally convex space and where the multiplication is separately continuous. The topology on \mathfrak{A} is defined by a family of seminorms $\{p_\alpha\}_{\alpha \in J}$ and the space is assumed to be quasi-complete, i.e., every Cauchy net is convergent. It is also supposed that the mappings $Y \rightarrow YZ$ (resp. $Z \rightarrow YZ$) are uniformly continuous when Z (resp. Y) belongs to a bounded set.

Denoting $B(\mathfrak{A})$ the algebra of all continuous linear operators on \mathfrak{A} , the topology on $B(\mathfrak{A})$ is defined by the family of seminorms $\{p_{\alpha, B}\}_{\alpha \in J, B \in \mathfrak{B}}$, where

$$p_{\alpha, B}(L) = \sup_{Y \in B} p_\alpha(L(Y)) ,$$

for each $L \in B(\mathfrak{A})$, \mathfrak{B} being the family of all bounded sets of \mathfrak{A} .

When $T, S \in \mathfrak{A}$ and S is invertible, it is shown that the spectrum $\sigma(M)$ of the operator

$$M(Y) = TYS^{-1}$$

is contained in the set $\sigma(T) \cdot \sigma(S^{-1})$ (Proposition 2.4) hence, taking a complex-valued function f , analytic in a neighbourhood of the set $\sigma(T) \cdot \sigma(S^{-1})$, we can construct the operator $f(M)$ in each point of \mathfrak{A} , as well by means of the functional calculus of T, S and they are equal (Theorem 2.9). Since the logarithm of an element of \mathfrak{A} does not always exist, this case is more general than Rosenblum's results (see [3] and Proposition 3.1).

All the statements of this paper can be applied to linear operators on Banach spaces or on locally convex ones, with supplementary properties.

1. **Preliminaries.** It is useful to recall some of the concepts and results contained in [1] and [4].

On a locally convex algebra it is possible to define the spectrum $\sigma(T)$ and the resolvent $\rho(T)$ of an element $T \in \mathfrak{A}$ (these sets are considered in the complex compactified plane $C_\infty = C \cup \{\infty\}$) [4].

We recall that $C_\infty \ni \lambda \in \rho(T)$ if there is a neighbourhood V_λ of λ such that:

- 1° $(\mu I - T)^{-1} \in \mathfrak{A}$ for any $\mu \in V_\lambda \cap C$ (here I is the identity of \mathfrak{A}).
- 2° the set $\{(\mu I - T)^{-1}; \mu \in V_\lambda \cap C\}$ is bounded in \mathfrak{A} .

In the following we shall write for μI simply μ .

For each $T \in \mathfrak{A}$ let us denote by $\mathfrak{F}(T)$ the set of all analytic functions in a neighbourhood of $\sigma(T) = \complement \rho(T)$. Then, if Γ is a contour (throughout we mean by “contour” a finite system of curves, admissible for the integral calculus) “surrounding” $\sigma(T)$, contained in the domain of definition of $f \in \mathfrak{F}(T)$, we put, by definition

$$f(T) = \begin{cases} \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - T)^{-1} d\lambda & \text{if } \sigma(T) \not\ni \infty \\ f(\infty) + \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - T)^{-1} d\lambda & \text{if } \sigma(T) \ni \infty \end{cases}$$

where the integral exists since the space is quasi-complete [1], [4]. For such functions we have the “spectral mapping theorem”, namely

$$\sigma(f(T)) = f(\sigma(T)) [4].$$

Let us remark that if $L \in B(\mathfrak{A})$, $f \in \mathfrak{F}(L)$ and Γ is a contour in the domain of f “surrounding” $\sigma(L)$, then we may define the expression $f(L)(Y)$, by using the natural extension of the formula given above for the elements of \mathfrak{A} . Since \mathfrak{A} is quasi-complete, the integrals do also exist.

2. **A functional calculus of the operator M.** For two sets A, B in the complex compactified plane with the property that if one contains the point ∞ then the other does not contain zero, we denote by $A \cdot B$ the set $\{\lambda\mu; \lambda \in A, \mu \in B\}$. Also, if $A \not\ni 0$, we denote by A^{-1} the set $\{1/\lambda; \lambda \in A\}$. In the following we need the next geometrical result:

LEMMA 2.1. *Let K and F be two closed sets in C_∞ , $0 \notin K \not\ni \infty$. If $\lambda_0 \notin K \cdot F$ and V_0 is a closed neighbourhood of λ_0 disjoint from $K \cdot F$, then there is an open set $G_0 \supset K$ such that $V_0 \cap \bar{G}_0 \cdot F = \emptyset$. Moreover, if Γ_0 is a contour in G_0 which surrounds K and separates it from zero, then $\Gamma_\lambda = \lambda \Gamma_0^{-1}$ is a contour such that the set F is “outside” it,*

for each $\lambda \in V_0, 0 \neq \lambda \neq \infty$.

Proof. If we put

$$G_n = \left\{ \mu \in C; \text{dist}(\mu, K) < \frac{1}{n} \right\},$$

on account of the compactness of F, K and V_0 in C_∞ , there is an index n_0 such that $V_0 \cap \bar{G}_{n_0} \cdot F = \emptyset$. We take $G_0 = G_{n_0}$. We can actually suppose that $\bar{G}_0 \ni 0$. Now, let Γ_0 be a contour in G_0 surrounding K and separating it from zero. Then for $\lambda \in V_0, 0 \neq \lambda \neq \infty$ we have $\Gamma_\lambda = \lambda \Gamma_0^{-1} \subset V_0 \cdot G_0^{-1}$ and $V_0 \cdot G_0^{-1} \cap F = \emptyset$.

LEMMA 2.2. *Let Γ be a system of curves in the complex plane, admissible for the integral calculus, $V_0 \subset C_\infty$ a closed set and $F: V_0 \times \Gamma \rightarrow \mathfrak{A}, G: \Gamma \rightarrow \mathfrak{A}$ two continuous functions. Then for any $\lambda \in V_0 \cap C$ we can define on \mathfrak{A} the linear continuous operator*

$$R_\lambda(Y) = \int_\Gamma F(\lambda, \xi) YG(\xi) d\xi,$$

for each $Y \in \mathfrak{A}$.

Proof. By our assumption on the algebra \mathfrak{A} , it is easy to see that the product of two continuous functions is also a continuous function (since if U is a neighbourhood of zero in \mathfrak{A} and B is a bounded set then there are two neighbourhoods of zero U_1 and U_2 such that $U_1 B \subset U$ and $B U_2 \subset U$). Since the mapping $\xi \rightarrow F(\lambda, \xi) YG(\xi)$ is continuous on Γ for each $\lambda \in V_0 \cap C$ and the algebra is quasi-complete, then the integral

$$\int_\Gamma F(\lambda, \xi) YG(\xi) d\xi$$

exists as an element of \mathfrak{A} .

Obviously, it defines a linear operator on \mathfrak{A} denoted by $R_\lambda(Y)$. To see that R is a continuous operator on \mathfrak{A} , let us denote by B_1 the set $\{F(\lambda, \xi); \lambda \in V_0, \xi \in \Gamma\}$ and by B_2 the set $\{G(\xi); \xi \in \Gamma\}$ which are bounded in \mathfrak{A} . If $U = \{T \in \mathfrak{A}; p_\alpha(T) < \varepsilon\}$ then, by our hypothesis, there is a neighbourhood U_0 of zero in \mathfrak{A} such that $B_1 U_0 B_2 \subset (2\pi/|\Gamma|)U$, where $|\Gamma|$ is the length of Γ . Thus we have

$$p_\alpha(R_\lambda(Y)) \leq \frac{1}{2\pi} \int_\Gamma |F(\lambda, \xi) YG(\xi)|_\alpha d\xi < \varepsilon$$

whenever $Y \in U_0$, hence R_λ is continuous.

PROPOSITION 2.3. *Let $T, S \in \mathfrak{A}$ be such that $S^{-1} \in \mathfrak{A}$. If $\lambda_0 \notin \sigma(T)$, $\sigma(S^{-1})$ and V_0 is a closed neighbourhood of λ_0 such that $V_0 \cap \sigma(T) = \emptyset$, then there is an open set $G_0 \supset \sigma(S^{-1}), G_0 \ni 0$ such that V_0*

$G_0^{-1} \subset \rho(T)$.

Moreover, if Γ_0 is a contour in G_0 which surrounds $\sigma(S^{-1})$ and separates it from zero, then we have

$$(\lambda - M(T, S))^{-1}(Y) = \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda - \xi T)^{-1} Y(\xi - S^{-1})^{-1} d\xi$$

for each $Y \in \mathfrak{A}$ and $\lambda \in V_0 \cap \mathcal{C}$.

Proof. By the spectral mapping theorem, $\sigma(S^{-1}) = \sigma(S)^{-1}$ does not contain zero and it is a compact set in \mathcal{C} , therefore we may apply Lemma 2.1, by putting $F = \sigma(T)$ and $K = \sigma(S^{-1})$. Let G_0 and Γ_0 be as in this lemma. Then $V_0 \cdot \Gamma_0^{-1} \subset V_0 \cdot G_0^{-1}$ does not intersect the set $F = \sigma(T)$, therefore $\lambda \xi^{-1} \in \rho(T)$ for all $\lambda \in V_0 \cap \mathcal{C}$ and $\xi \in \Gamma_0$, and $\sigma(T)$ is "outside" $\Gamma_\lambda = \lambda \Gamma_0^{-1}$ ($0 \neq \lambda \neq \infty$). By Lemma 2.2, the integral

$$\frac{1}{2\pi i} \int_{r_0} (\lambda - \xi T)^{-1} Y(\xi - S^{-1})^{-1} d\xi = \frac{1}{2\pi i} \int_{r_0} \left(\frac{\lambda}{\xi} - T\right)^{-1} Y(\xi - S^{-1})^{-1} \frac{d\xi}{\xi}$$

exists, therefore we have for any $\lambda \in V_0$ ($0 \neq \lambda \neq \infty$)

$$\begin{aligned} & (\lambda - M(T, S)) \left(\frac{1}{2\pi i} \int_{r_0} \left(\frac{\lambda}{\xi} - T\right)^{-1} Y(\xi - S^{-1})^{-1} \frac{d\xi}{\xi} \right) \\ &= \frac{\lambda}{2\pi i} \int_{r_0} \left(\frac{\lambda}{\xi} - T\right)^{-1} Y(\xi - S^{-1})^{-1} \frac{d\xi}{\xi} \\ &\quad - \frac{1}{2\pi i} \int_{r_0} \left(T - \frac{\lambda}{\xi} + \frac{\lambda}{\xi}\right) \left(\frac{\lambda}{\xi} - T\right)^{-1} Y(\xi - S^{-1})^{-1} (S^{-1} - \xi + \xi) \frac{d\xi}{\xi} \\ &= -\frac{1}{2\pi i} \int_{r_0} Y \frac{d\xi}{\xi} + \frac{\lambda}{2\pi i} \int_{r_0} \left(\frac{\lambda}{\xi} - T\right)^{-1} \frac{d\xi}{\xi} \\ &\quad + \frac{1}{2\pi i} \int_{r_0} Y(\xi - S^{-1})^{-1} d\xi = Y, \end{aligned}$$

since

$$\frac{1}{2\pi i} \int_{r_0} \frac{d\xi}{\xi} = 0,$$

$$\frac{1}{2\pi i} \int_{r_0} Y(\xi - S^{-1})^{-1} d\xi = Y$$

by the well-known functional calculus for an element of \mathfrak{A} with the spectrum compact in \mathcal{C} and

$$\frac{\lambda}{2\pi i} \int_{r_0} \left(\frac{\lambda}{\xi} - T\right)^{-1} \frac{d\xi}{\xi^2} = -\frac{1}{2\pi i} \int_{\lambda \Gamma_0^{-1}} (\eta - T)^{-1} d\eta = 0,$$

since $\sigma(T)$ is "outside" $\lambda \Gamma_0^{-1}$.

Analogously

$$\frac{1}{2\pi i} \int_{r_0} \left(\frac{\lambda}{\xi} - T \right)^{-1} ((\lambda - M(T, S))(Y)) (\xi - S^{-1})^{-1} \frac{d\xi}{\xi} = Y,$$

therefore

$$(\lambda - M(T, S))^{-1}(Y) = \frac{1}{2\pi i} \int_{r_0} \left(\frac{\lambda}{\xi} - T \right)^{-1} Y (\xi - S^{-1})^{-1} \frac{d\xi}{\xi}.$$

If $\lambda = 0$ then $0 \notin \sigma(T)$, hence $T^{-1} \in \mathfrak{A}$ and

$$(-M(T, S))^{-1}(Y) = -T^{-1}YS = \frac{1}{2\pi i} \int_{r_0} -T^{-1}Y(\xi - S^{-1})^{-1} \frac{d\xi}{\xi}$$

and this finishes the proof.

PROPOSITION 2.4. *With the same conditions as in the previous proposition, we have*

$$\sigma(M(T, S)) \subset \sigma(T) \cdot \sigma(S^{-1}).$$

Proof. We can suppose $\sigma(T) \cdot \sigma(S^{-1}) \neq C_\infty$. By the preceding proposition, if $\lambda_0 \notin \sigma(T) \cdot \sigma(S^{-1})$ and V_0 is a closed neighbourhood of λ_0 , $V_0 \cap \sigma(T) \cdot \sigma(S^{-1}) = \emptyset$, then for any $\lambda \in V_0 \cap C$ the operator $(\lambda - M(T, S))^{-1}$ exists and, by Lemma 2.2, it is a continuous operator. We have only to prove that the set

$$\{(\lambda - M(T, S))^{-1}; \lambda \in V_0 \cap C\}$$

is a bounded one in $B(\mathfrak{A})$.

For, let $\{p_\alpha\}_{\alpha \in J}$ the family of semi-norms on \mathfrak{A} and $\{p_{\alpha B}\}_{\alpha \in J, B \in \mathfrak{B}}$ the family of semi-norms on $B(\mathfrak{A})$ (see the introduction). Define $B_1 = \{(\lambda - \xi T)^{-1}, \lambda \in V_0 \cap C, \xi \in \Gamma_0\}$ and $B_2 = \{(\xi - S^{-1})^{-1}; \xi \in \Gamma_0\}$ which are bounded in \mathfrak{A} . Indeed, $V_0 \cdot \Gamma_0^{-1} \subset \rho(T)$ and it is a compact set in C , therefore by reasoning with a finite covering, we obtain the boundedness of the family B_1 . A similar argument is valid for B_2 . If $B \in \mathfrak{B}$ is arbitrary, then $B_1 B B_2$ is also a bounded set, therefore we have

$$\begin{aligned} p_{\alpha, B}((\lambda - M(T, S))^{-1}) &= \sup_{y \in B} p_\alpha((\lambda - M(T, S))^{-1}(Y)) \\ &\leq \frac{1}{2\pi} \sup_{y \in B} \int_{\Gamma_0} p_\alpha((\lambda - \xi T)^{-1} Y (\xi - S^{-1})^{-1}) |d\xi| \\ &\leq C(\alpha, B) < \infty, \end{aligned}$$

where we kept the notations of the preceding proposition. Consequently $\lambda_0 \in \rho(M(T, S))$.

COROLLARY 1. *If $M = M(T, T)$ is an inner automorphism of the algebra \mathfrak{A} then*

$$\sigma(M) \subset \sigma(T) \cdot \sigma(T^{-1}) .$$

PROPOSITION 2.5. *Let T, S be in \mathfrak{A} with $S^{-1} \in \mathfrak{A}$, and $M = M(T, S)$. If $\lambda_0 \notin \sigma(M) \cdot \sigma(S)$ and V_0 is a closed neighbourhood of λ_0 such that $V_0 \cap \sigma(M) \cdot \sigma(S) = \emptyset$, then there is an open set $G_0 \supset \sigma(S)$, $G_0 \ni 0$ such that $V_0 \cdot G_0^{-1} \subset \rho(M)$.*

Moreover, if Γ_0 is a contour in G_0 which surrounds $\sigma(S)$ and separates it from zero, we have

$$(\lambda - T) \left(\frac{1}{2\pi i} \int_{\Gamma_0} (\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1} d\xi \right) = Y$$

for all $Y \in \mathfrak{A}$ and $\lambda \in V_0 \cap \mathbf{C}$.

Proof. We apply Lemma 2.1 with $F = \sigma(M)$ and $K = \sigma(S)$. Therefore, if Γ_0 is a contour as in the quoted lemma, then the integral

$$\frac{1}{2\pi i} \int_{\Gamma_0} (\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1} d\xi$$

exists as an element of \mathfrak{A} for each $\lambda \in V_0 \cap \mathbf{C}$.

From the relation

$$\lambda(\lambda - \xi M)^{-1}(Y) - \xi T(\lambda - \xi M)^{-1}S^{-1} = Y$$

we obtain

$$T(\lambda - \xi M)^{-1} = \frac{1}{\xi}(\lambda(\lambda - \xi M)^{-1}(Y) - Y)S ,$$

therefore we can write, for $\lambda \in V_0 \cap \mathbf{C}$, $\lambda \neq 0$,

$$\begin{aligned} & (\lambda - T) \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1} d\xi \\ &= \frac{\lambda}{2\pi i} \int_{\Gamma_0} (\lambda - \xi M)^{-1}(Y)(\xi - S)^{-1} d\xi \\ &\quad - \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\xi} (\lambda(\lambda - \xi M)^{-1}(Y) - Y)(-1 + \xi(\xi - S)^{-1}) d\xi \\ &= \frac{1}{2\pi i} \int_{\Gamma_0} \frac{\lambda}{\xi} (\lambda - \xi M)^{-1}(Y) d\xi - \frac{1}{2\pi i} \int_{\Gamma_0} \frac{d\xi}{\xi} Y \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma_0} Y(\xi - S)^{-1} d\xi = Y , \end{aligned}$$

since

$$\frac{1}{2\pi i} \int_{\Gamma_0} \frac{d\xi}{\xi} = 0 ,$$

$$\frac{1}{2\pi i} \int_{r_0} (\xi - S)^{-1} d\xi = I$$

and

$$\frac{1}{2\pi i} \int_{r_0} \frac{\lambda}{\xi} (\lambda - \xi M)^{-1}(Y) d\xi = -\frac{1}{2\pi i} \int_{\lambda r_0^{-1}} (\eta - M)^{-1}(Y) d\eta = 0$$

because $\sigma(M)$ is “outside” $\lambda\Gamma_0^{-1}$.

If $\lambda = 0$, a direct argument proves the validity of the given formula.

PROPOSITION 2.6. *With the same conditions as in the previous proposition, if $\sigma(T) \ni \infty$ then $\sigma(M) \ni \infty$.*

Proof. Let us suppose that $\sigma(M) \not\ni \infty$. Then there is a closed neighbourhood of ∞ , let us say V_∞ , where we have

$$(\lambda - \xi M)^{-1}(Y) = \sum_{n=0}^{\infty} \frac{\xi^n M^n(Y)}{\lambda^{n+1}}$$

for all $\xi \in \Gamma_0$ where the series is uniformly convergent in \mathfrak{A} .

Hence we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{r_0} (\lambda - \xi M)^{-1}(Y) (\xi - S)^{-1} d\xi \\ &= \frac{1}{2\pi i} \int_{r_0} \sum_{n=0}^{\infty} \frac{\xi^n M^n(Y)}{\lambda^{n+1}} (\xi - S)^{-1} d\xi \\ &= \sum_{n=0}^{\infty} \frac{T^n Y S^{-n}}{\lambda^{n+1}} \frac{1}{2\pi i} \int_{r_0} \xi^n (\xi - S)^{-1} d\xi \\ &= \sum_{n=0}^{\infty} \frac{T^n Y}{\lambda^{n+1}}, \end{aligned}$$

thus the last series is uniformly convergent in a neighbourhood of ∞ and defines, for $Y = I$, the inverse $\lambda - T$.

Moreover, the set $\{(\lambda - T)^{-1}; \lambda \in V_\infty\}$ is bounded in \mathfrak{A} since the set

$$\left\{ \frac{1}{2\pi i} \int_{r_0} (\lambda - \xi M)^{-1}(I) (\xi - S)^{-1} d\xi; \lambda \in V_\infty \right\}$$

is bounded in \mathfrak{A} (see the proof of Proposition 2.4), thus $\infty \in \rho(T)$.

PROPOSITION 2.7. *Let $T, S \in \mathfrak{A}$ be such that $S^{-1} \in \mathfrak{A}$ and $M = M(T, S)$. Suppose M invertible on \mathfrak{A} . If $\lambda_0 \notin \sigma(T) \cdot \sigma(M)^{-1}$ and V_0 is a closed neighbourhood of λ_0 such that $V_0 \cap \sigma(T) \cdot \sigma(M)^{-1} = \emptyset$ then there is an open set $G_0 \supset \sigma(M)^{-1}$, $G_0 \not\ni 0$ such that $V_0 \cdot G_0^{-1} \subset \rho(T)$. Moreover, if Γ_0*

is a contour in G_0 which surrounds $\sigma(M)^{-1}$, we have

$$\left(\frac{1}{2\pi i}\int_{r_0}(\lambda - \xi T)^{-1}(\xi M - 1)^{-1}(Y)\frac{d\xi}{\xi}\right)(\lambda - S) = Y$$

for all $Y \in \mathfrak{A}$ and $\lambda \in V_0 \cap C$.

Proof. If $1/\xi \in \rho(M)$ then from the relation

$$\left(M - \frac{1}{\xi}\right)\left(M - \frac{1}{\xi}\right)^{-1}(Y) = Y$$

we obtain

$$\left(M - \frac{1}{\xi}\right)^{-1}(Y)S = \xi\left(T\left(M - \frac{1}{\xi}\right)^{-1}(Y) - YS\right).$$

As in the proof of Proposition 2.5, we may apply Lemma 2.1 with $F = \sigma(T)$ and $K = \sigma(M)^{-1}$ and if Γ_0 surrounds $\sigma(M)^{-1}$, we have for λ in a neighbourhood V_0 of λ_0 , $0 \neq \lambda \neq \infty$,

$$\begin{aligned} &\left(\frac{1}{2\pi i}\int_{r_0}(\lambda - \xi T)^{-1}(\xi M - 1)^{-1}(Y)\frac{d\xi}{\xi}\right)(\lambda - S) \\ &= \left(\frac{1}{2\pi i}\int_{r_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}\frac{d\xi}{\xi}\right)YS + \frac{1}{2\pi i}\int_{r_0}\left(M - \frac{1}{\xi}\right)^{-1}(Y)\frac{d\xi}{\xi^2} = Y, \end{aligned}$$

since

$$\frac{1}{2\pi i}\int_{r_0}\left(\frac{\lambda}{\xi} - T\right)^{-1}\frac{d}{\xi^2} = -\frac{1}{2\pi i}\int_{r_0^{-1}}(\eta - T)^{-1}d\eta = 0$$

(because $\sigma(T)$ is “outside” $\lambda\Gamma_0^{-1}$) and

$$\frac{1}{2\pi i}\int_{r_0}\left(M - \frac{1}{\xi}\right)^{-1}(Y)\frac{d\xi}{\xi^2} = \frac{1}{2\pi i}\int_{r_0^{-1}}(\eta - M)^{-1}(Y)d\eta = Y$$

as Γ_0^{-1} surrounds $\sigma(M)$.

If $0 \notin \sigma(T) \cdot \sigma(M)^{-1}$ then $T^{-1} \in \mathfrak{A}$ and the formula is immediate.

LEMMA 2.8. *Let F and K be two closed sets, $0 \notin K \ni \infty$ and $G \supset K \cdot F$ an open set. Then there is an open set $G_0 \ni 0$, $G_0 \supset K$ such that $G \supset \bar{G}_0 \cdot F$. Moreover, if $\Gamma \subset G$ is a contour surrounding $K \cdot F$ then we can take a contour Γ_0 in G_0 surrounding K and separating it from zero such that $\xi \cdot F$ is “inside” Γ for all $\xi \in \Gamma_0$.*

Proof. The set \mathfrak{G} is compact in C_∞ , therefore we can apply Lemma 2.1 for any $\lambda_0 \in \mathfrak{G}$ and, taking a finite covering of \mathfrak{G} , we obtain the set G_0 . If Γ is a contour surrounding $K \cdot F$, we can choose G_0 such that $\bar{G}_0 \cdot F$ is “inside” Γ , hence there is a contour Γ_0 in G_0

which surrounds K and separates it from zero such that $\Gamma_0 \cdot F$ is “inside” Γ .

THEOREM 2.9. *Let $T, S \in \mathfrak{A}$ be such that $S^{-1} \in \mathfrak{A}$. If $\sigma(T) \cdot \sigma(S^{-1}) \neq C_\infty$ and f is an analytic function, defined in an open set containing $\sigma(T) \cdot \sigma(S^{-1})$, then there is a contour Γ_0 surrounding $\sigma(S^{-1})$ and separating it from zero such that $f(\xi T)$ is defined for each $\xi \in \Gamma_0$ and*

$$f(M(T, S))(Y) = \frac{1}{2\pi i} \int_{\Gamma_0} f(\xi T) Y(\xi - S^{-1})^{-1} d\xi,$$

for all $Y \in \mathfrak{A}$ (where the left side is defined as in the Introduction).

Proof. We apply Lemma 2.5 with $F = \sigma(T)$ and $K = \sigma(S^{-1})$. Let Γ and Γ_0 be as in this lemma. Suppose that $\sigma(M) \ni \infty$. Then, by Proposition 2.4, we must have $\sigma(T) \ni \infty$. Hence, by Proposition 2.3, we can write

$$\begin{aligned} f(M(T, S)(Y)) &= f(\infty)Y + 1/2\pi i \int_{\Gamma} f(\lambda)(\lambda - M(T, S))^{-1}(Y) d\lambda \\ &= f(\infty)Y + 1/2\pi i \int_{\Gamma} f(\lambda) \left(\frac{1}{2\pi i} \int_{\Gamma_0} (\lambda - \xi T)^{-1} Y(\xi - S^{-1})^{-1} d\xi \right) d\lambda. \end{aligned}$$

By interchanging the order of integration, we obtain

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\Gamma} f(\lambda) \left(\frac{1}{2\pi i} \int_{\Gamma_0} (\lambda - \xi T)^{-1} Y(\xi - S^{-1})^{-1} d\xi \right) d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_0} \left(\frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda - \xi T)^{-1} d\lambda \right) Y(\xi - S^{-1})^{-1} d\xi \\ &= \frac{1}{2\pi i} \int_{\Gamma_0} (f(\xi T) - f(\infty)) Y(\xi - S^{-1})^{-1} d\xi \\ &= \frac{1}{2\pi i} \int_{\Gamma_0} f(\xi T) Y(\xi - S^{-1})^{-1} d\xi - f(\infty)Y \end{aligned}$$

since, by Lemma 2.5, $\sigma(\xi T) = \xi\sigma(T)$ is “inside” Γ for all $\xi \in \Gamma_0$.

In this manner we obtain

$$f(M(T, S))(Y) = \frac{1}{2\pi i} \int_{\Gamma_0} f(\xi T) Y(\xi - S^{-1})^{-1} d\xi.$$

If $\sigma(T) \not\ni \infty$ we have $\sigma(M) \not\ni \infty$ and a similar calculus leads to the same formula. No other case is possible because of Proposition 2.6, and this finishes our proof.

3. Some applications of the functional calculus. First of all, we shall show that, in a certain sense, the commutator of two elements

[3] can be found again as a function of the operator M .

PROPOSITION 3.1. *Let T, S be in \mathfrak{A} , with compact spectra in \mathbb{C} and with $S^{-1} \in \mathfrak{A}$. If none of the sets $\sigma(T)$, $\sigma(S)$ and $\sigma(T) \cdot \sigma(S^{-1})$ separates the complex plane, we have the relation*

$$(\log M(T, S))(Y) = (\log T) Y - Y(\log S) ,$$

for all $Y \in \mathfrak{A}$.

Proof. By our assumption, $\log T$, $\log S$ and $\log M(T, S)$ exist and, from Theorem 2.9, it follows that

$$\begin{aligned} (\log M(T, S))(Y) &= \frac{1}{2\pi i} \int_{r_0} \log (\xi T) Y(\xi - S^{-1})^{-1} d\xi \\ &= \frac{1}{2\pi i} \int_{r_0} (\log \xi) Y(\xi - S^{-1})^{-1} d\xi + \frac{1}{2\pi i} \int_{r_0} (\log T) Y(\xi - S^{-1})^{-1} d\xi \\ &= Y(\log S^{-1}) + (\log T) T = (\log T) Y - Y(\log S) . \end{aligned}$$

Let now $E(\cdot)$ and $F(\cdot)$ two selfadjoint spectral measures on a Hilbert space, defined on Borel sets of the complex plane. Then the mappings

$$\begin{aligned} \mathfrak{E}(\sigma_1)(Y) &= E(\sigma_1) Y \\ \mathfrak{F}(\sigma_2)(Y) &= Y F(\sigma_2) \end{aligned}$$

are two commuting spectral measures, Y being an arbitrary linear bounded operator; therefore the mapping

$$(\mathfrak{E} \times \mathfrak{F})(\sigma_1 \times \sigma_2)(Y) = E(\sigma_1) Y F(\sigma_2)$$

induces a spectral measure on the space of the operators and it is possible to integrate with respect to it (see [2] for details).

PROPOSITION 3.2. *Suppose that \mathfrak{A} is the algebra of all linear operators on a Hilbert space and $T, S \in \mathfrak{A}(S^{-1} \in \mathfrak{A})$ two normal operators. If E, F are the spectral measures of T and S respectively, then for any function f , analytic in a neighbourhood of $\sigma(T) \cdot \sigma(S^{-1})$, we have*

$$f(M(T, S))(Y) = \iint f(\lambda \mu^{-1}) dE_\lambda Y dE_\mu ,$$

for all $Y \in \mathfrak{A}$.

Proof. Using the same notations as in Theorem 2.9, we have

$$f(M(T, S))(Y) = \frac{1}{2\pi i} \int_{r_0} f(\xi T) Y(\xi - S^{-1})^{-1} d\xi$$

$$\begin{aligned}
 &= \frac{1}{2\pi i} \int_{r_0} f(\xi T) Y \left(\int (\xi - \mu^{-1})^{-1} dF_\mu \right) d\xi \\
 &= \int \left(\frac{1}{2\pi i} \int_{r_0} f(\xi T) (\xi - \mu^{-1})^{-1} d\xi \right) Y dF_\mu \\
 &= \int \left(\frac{1}{2\pi i} \int_{r_0} (Sf(\xi\lambda) dE_\lambda) (\xi - \mu^{-1})^{-1} d\xi \right) Y dF_\mu \\
 &= \iint \left(\frac{1}{2\pi i} \int_{r_0} f(\xi\lambda) (\xi - \mu^{-1})^{-1} d\xi \right) dE_\lambda Y dF_\mu \\
 &= \iint f(\lambda\mu^{-1}) dE_\lambda Y dF_\mu .
 \end{aligned}$$

PROPOSITION 3.3. *Assume that the Hilbert space in the previous proposition is finite dimensional and that*

$$T = \sum_{j=1}^p \lambda_j E_j, \quad S = \sum_{k=1}^q \mu_k F_k$$

(where $\{E_j\}$ and $\{F_k\}$ are now finite orthogonal resolutions of the identity). If f is an analytic function in an open set containing the set $\{\lambda_1, \dots, \lambda_p\} \cdot \{\mu_1^{-1}, \dots, \mu_q^{-1}\}$ then we have

$$f(M(T, S))(Y) = \sum_{j=1}^p \sum_{k=1}^q f(\lambda_j \mu_k^{-1}) E_j Y F_k ,$$

for all $Y \in \mathfrak{A}$.

The proof follows easily from the preceding proposition.

PROPOSITION 3.4. *Suppose that \mathfrak{A} , T and S are as in Proposition 3.3. Then a necessary and sufficient condition that the equation $TY = ZS$ have a solution $Y \in \mathfrak{A}$ is that $\lambda_r = 0$ implies $E_r Z = 0$.*

Proof. Let Y be a solution of the equation $TY = ZS$, hence

$$TYS^{-1} = \sum_{j=1}^p \sum_{k=1}^q \lambda_j \mu_k^{-1} E_j Y F_k = Z .$$

From this relation we obtain easily

$$E_r TYS^{-1} F_s = \lambda_r \mu_s^{-1} E_r Y F_s = E_r Z F_s ,$$

thus $\lambda_r E_s Y F_s = \mu_s E_r Z F_s$.

If $\lambda_r = 0$, since $\mu_s \neq 0$ for all s , we have $E_r Z F_s = 0$, hence $\sum_s E_r Z F_s = E_r Z = 0$.

Conversely, if $\lambda_r = 0$ implies $E_r Z = 0$, let us consider the matrix

$$Y = \sum_{\lambda_j \neq 0} \sum_{k=1}^q \frac{\mu_k}{\lambda_j} E_j Z F_k .$$

We have

$$\begin{aligned} TYS^{-1} &= \sum_{l=1}^q \lambda_l E_l \sum_{\lambda_j \neq 0} \sum_{k=1}^q \frac{\mu_l}{\lambda_j} E_j Z F_k \sum_{t=1}^q \mu_t^{-1} F_t \\ &= \sum_{l=1}^p E_l Z \sum_{t=1}^q F_t = Z, \end{aligned}$$

consequently Y is a solution of the equation.

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