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## **ASYMPTOTICS FOR A CLASS OF WEIGHTED EIGENVALUE PROBLEMS**

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**Abstract:** This paper deals with the asymptotic behavior at infinity of the solutions to  $\mathcal{L}(y) = \lambda wy$  on  $[a, \infty)$  where  $\mathcal{L}$  is an  $n$ th order ordinary linear differential operator,  $\lambda$  is a nonzero complex number and  $w$  is a suitably chosen positive valued continuous functions. As an application the deficiency indices of certain symmetric differential operators in Hilbert space are computed.

1. Preliminaries. Throughout the first three sections  $\mathcal{L}$  will denote an operator of the form,

$$(1.1) \quad \mathcal{L}(y) = y^{(n)} + \sum_{k=2}^n p_k y^{(n-k)} \quad \text{on } [a, \infty),$$

where each of  $p_2, \dots, p_n$  is a continuous complex valued function on  $[a, \infty)$ . In view of the transformation indicated on p.309 of [2] it results in no great loss of generality to take the coefficient of  $y^{(n-1)}$  to be zero, and in order to simplify the exposition we shall do this. We shall be concerned with the behavior at infinity of the solutions to

$$(1.2) \quad \mathcal{L}(y) = \lambda wy \quad \text{on } [a, \infty)$$

where  $\lambda$  is a nonzero complex number and  $w$  is an appropriate weight (i.e., positive valued continuous function). For a given  $\mathcal{L}$  we shall consider the weights  $w$  indicated by the following definition.  $\mathcal{L}(a, \infty)$  denotes the Banach space of all complex valued measurable functions which are absolutely Lebesgue integrable on  $[a, \infty)$ .

**DEFINITION.** If  $\mathcal{L}$  is as in 1.1 the statement that  $w$  is an  $\mathcal{L}$ -admissible weight means that

- (1)  $w$  is differentiable, strictly increasing, and unbounded on  $[a, \infty)$ ;
- (2) each of  $[w'/w^{1+1/n}]'$  and  $[(w'/w)^2(1/w^{1/n})]$  is continuous on  $[a, \infty)$  and is in  $\mathcal{L}(a, \infty)$ ; and
- (3)  $p_j/w^{(j-1)/n} \in \mathcal{L}(a, \infty)$  for  $j = 2, 3, \dots, n$ .

For example if  $\mathcal{L}(y)(t) = y''(t) \pm t^\alpha y(t)$  for  $t \geq 1$  and  $w(t) = t^\beta$  then  $w$  will be an  $\mathcal{L}$ -admissible weight if and only if  $\beta > 0$  and  $\beta > 2(\alpha + 1)$ .

We shall demonstrate that when  $w$  is an  $\mathcal{L}$ -admissible weight the solutions of 1.2 have a particularly simple asymptotic behavior and

we shall establish that every operator of the form 1.1 has admissible weights.

Our asymptotic theorem relies on the classic perturbation theorem of Norman Levinson [2, Thorem 8.1 p. 92 or 10]. Recent related works include [3, 7, 8, 9, 11, and 12]. The results in § 4 complement those of reference [13].

**2. Results.** Our main results are stated in the following two theorems.

**THEOREM 1.** *If  $\mathcal{L}$  is as in 1.1 and  $U$  is a continuous function on  $[a, \infty)$  there is an  $\mathcal{L}$ -admissible weight  $w$  with  $w(t) \geq U(t)$  for  $t \geq a$ .*

**THEOREM 2.** *If  $\mathcal{L}$  is as in 1.1,  $w$  is an  $\mathcal{L}$ -admissible weight, and  $\lambda$  is a nonzero complex number then equation 1.2 has  $n$  linearly independent solutions  $y_1, \dots, y_n$  such that for  $k = 0, \dots, n - 1$*

$$y_j^{(k)}(t)w^{\alpha_k}(t)e^{-\mu_j h(t)} \longrightarrow \mu_j^k \text{ as } t \longrightarrow \infty,$$

where

$$h(t) = \int_a^t w^{1/n},$$

$\mu_1, \dots, \mu_n$  are the distinct  $n$ th roots of  $\lambda$ , and  $\alpha_{k-1} = (n - 2k + 1)/2n$  for  $k = 1, \dots, n$ .

**3. Proofs.** The proof of Theorem 1 will be facilitated by the following results.

**LEMMA.** *If  $r > 1$  and  $1 < c < d$  there exist positive constants  $M_r$  and  $N_r$ , depending only on  $r$ , and a function  $f$  defined on  $[0, 1]$  such that*

(1)  *$f$  is continuously differentiable, strictly increasing,  $f(0) = c$ ,  $f(1) = d$ , and  $f'(0) = 0 = f'(1)$ ;*

(2)  *$[f'/f^r]'$  exists and is continuous on  $[0, 1]$  and has the value 0 at 0 and at 1; and*

(3)  *$|[f'/f^r]'(x)| \leq M_r c^{1-r}$  and  $[(f'/f)^2 f^{1-r}](x) \leq N_r c^{1-r}$  for all  $x \in [0, 1]$ .*

*Proof.* Given  $r > 1$  and  $1 < c < d$  let  $g: [0, 1] \rightarrow [0, 1]$  be a twice continuously differentiable function such that  $g(0) = 0$ ,  $g(1) = 1$ ,  $g'(x) > 0$  for  $x \in (0, 1)$ , and  $g'(0) = g''(0) = g'(1) = g''(1) = 0$  (e.g. let  $g(x) = h(h(x))$  where  $h(x) = (2x - x^2)^2$ ). Then let  $f: [0, 1] \rightarrow [c, d]$  be given by

$$f = \left\{ c^{1-r} - 6(c^{1-r} - d^{1-r}) \left[ \left( \frac{1}{2} \right) g^2 - \left( \frac{1}{3} \right) g^3 \right] \right\}^{1/(1-r)},$$

clearly  $f(0) = c$  and  $f(1) = d$ . Since each of  $g$  and the function whose value at  $x$  is  $(1/2)x^2 - (1/3)x^3$  is strictly increasing on  $[0, 1]$  and since  $1 - r < 0$  and  $1 < c < d$  we see that  $f$  is strictly increasing on  $[0, 1]$ . Using the above listed properties of  $g$  we see that  $f'$  is continuous on  $[0, 1]$  and that  $f'(0) = 0 = f'(1)$ . Computation shows that

$$[f'/f^r] = (6/(r-1))(c^{1-r} - d^{1-r})(g - g^2)g'$$

and

$$[f'/f^r]' = (6/(r-1))(c^{1-r} - d^{1-r})[(1-2g)(g')^2 + (g-g^2)g''].$$

Hence condition (2) of the lemma is satisfied. Letting  $M_r$  be a bound for  $(6/(r-1))[(1-2g)(g')^2 + (g-g^2)g'']$  on  $[0, 1]$  we see that

$$|[f'/f^r]'(x)| \leq M_r c^{1-r} \quad \text{for } x \in [0, 1].$$

Noting that  $c^{1-r} \geq (f(x))^{1-r} \geq d^{1-r}$  for  $x \in [0, 1]$  and letting  $N_r$  be a bound for  $[(6/(r-1))(g-g^2)g']^2$  on  $[0, 1]$  we see that

$$|[f'/f^r]^2 f^{1-r}(x)| \leq N_r c^{3(1-r)} \leq N_r c^{1-r}$$

for  $x \in [0, 1]$ , and the lemma is proved.

*Proof of Theorem 1.* We shall make use of the fact that if  $U$  is a continuous function on  $[a, \infty)$  and  $\gamma$  is a positive number there is a weight  $w$  such that  $U/w^\gamma \in \mathcal{L}(a, \infty)$ . To see this let  $w$  be such that

$$\frac{1 + |U(t)|}{w^\gamma(t)} = \frac{1}{(t-a+1)^2}.$$

Given an  $\epsilon$  as in 1.1 and a continuous function  $U$  on  $[a, \infty)$  choose weights  $v_2, v_3, \dots, v_n$  such that  $p_j/v_j^{(j-1)/n} \in \mathcal{L}(a, \infty)$  for  $j = 2, \dots, n$  and let  $v$  be a weight such that  $v(t) \geq \max\{U(t), v_2(t), \dots, v_n(t)\}$  for all  $t \geq a$ . Next let  $\{c_k\}_{k=1}^\infty$  be a strictly increasing sequence of numbers with  $c_k \geq k^{2n}$  and  $c_k \geq \text{maximum of } v(t) \text{ for } t \in [a+k-1, a+k]$  and let  $f_k$  be a function satisfying the conclusion to the lemma with  $r = 1 + 1/n$ ,  $c = c_k$  and  $d = c_{k+1}$  for each  $k$ . Let  $w$  be defined by

$$w(t) = f_k(t - a - k + 1) \quad \text{for } t \in [a + k - 1, a + k].$$

Clearly then  $w$  satisfies condition (1) in the definition of admissible weight, and since  $w(t) \geq v(t)$ , we see that  $w(t) \geq U(t)$  and  $w$  satisfies condition (3) of the definition. To see that condition (2) is satisfied note that

$$\int_a^\infty |[w'/w^{1+1/n}]'| = \sum_{k=1}^\infty \int_0^1 |[f'_k/f_k^{1+1/n}]'|$$

$$\leq \sum_{k=1}^\infty M_{1+1/n} c_k^{-1/n} \leq M_{1+1/n} \sum_{k=1}^\infty k^{-2} < \infty ,$$

and

$$\int_a^\infty [w'/w]^2 (1/w^{1/n}) = \sum_{k=1}^\infty \int_0^1 [(f'_k/f_k)^2 f_k^{-1/n}]$$

$$\leq \sum_{k=1}^\infty N_{1+1/n} c_k^{-1/n} \leq N_{1+1/n} \sum_{k=1}^\infty k^{-2} < \infty .$$

*Proof of Theorem 2.* We shall establish the theorem by showing that the standard vector-matrix formulation,

$$(3.1) \quad y' = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ (\lambda w - p_n) & -p_{n-1} & -p_{n-2} & \dots & -p_2 & 0 \end{bmatrix} y$$

of equation (1.2) has a fundamental matrix  $Y_0$  such that

$$Q(t) Y_0(t) E(t) \longrightarrow L \quad \text{as } t \longrightarrow \infty ,$$

where

$$Q = \text{diag} [w^{\alpha_1}, \dots, w^{\alpha_n}]$$

with  $\alpha_k = (n - 2k + 1)/2n$  for  $k = 1, \dots, n$ ;

$$E(t) = \text{diag} [e^{-\mu_1 h(t)}, \dots, e^{-\mu_n h(t)}]$$

with  $\mu_1, \dots, \mu_n$  the distinct  $n$ th roots of  $\lambda$  and

$$h(t) = \int_a^t w^{1/n} ;$$

and

$$L = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \mu_1^2 & \mu_2^2 & \dots & \mu_n^2 \\ \dots & \dots & \dots & \dots \\ \mu_1^{n-1} & \mu_2^{n-1} & \dots & \mu_n^{n-1} \end{bmatrix} .$$

Using this notation we begin by letting  $Y$  be a fundamental matrix for equation (3.1). Since  $h$  is strictly increasing on  $[a, \infty)$  we may let  $g$  be the function inverse to it ( $h(g(s)) = s$  for  $s \geq 0$ ) and let  $Z(s) = Q(g(s)) Y(g(s))$  for  $s \geq 0$ . Noting that  $g'(s) = 1/h'(g(s))$  and

that  $Q(g(s))$  is nonsingular we see that  $Z$  is a fundamental matrix for

$$(3.2) \quad z'(s) = [1/h'(g(s))]$$

$$[Q(g(s))M(g(s))Q^{-1}(g(s)) + Q'(g(s))Q^{-1}(g(s))]z(s)$$

where  $M$  is the coefficient matrix on the right hand side of equation (3.1). Computation shows that equation (3.2) is the same as

$$(3.3) \quad z'(s) = [A + \alpha(s)D + R(s)]z(s), \quad s \geq 0$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \lambda & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} (n \times n),$$

$$\alpha(s) = [w'/w^{1/n}](g(s)),$$

$$D = \text{diag} [\alpha_1, \cdots, \alpha_n],$$

and  $R(s)$  is the  $n \times n$  matrix having

$$[(-p_j/w^{(j-1)/n})(1/w^{1/n})](g(s))$$

as its  $n, n - j + 1$  entry for  $2 \leq j \leq n$  and zero for all other entries. Since  $h(t) \rightarrow \infty$  as  $t \rightarrow \infty$  and

$$\int_0^{h(b)} |[p_j/w^{(j-1)/n}](1/w^{1/n})](g(s))| ds = \int_a^b |(p_j/w^{(j-1)/n}](t)| dt$$

we see from condition (3) of the definition of  $\mathcal{L}$ -admissible weight that  $|R| \in \mathcal{L}(0, \infty)$  since by condition (2) of the definition it is the case that  $[w'/w^{1+1/n}]'$  and  $[w'/w]^2(1/w^{1/n})$  are in  $\mathcal{L}(a, \infty)$  we see from similar "changes of variable" that  $\alpha'$  and  $\alpha^2$  are in  $\mathcal{L}(0, \infty)$ . Since  $\alpha' \in \mathcal{L}(a, \infty)$  and  $\alpha'$  is continuous,  $\alpha$  has a limit at  $\infty$  and since  $\alpha^2 \in \mathcal{L}(0, \infty)$  this limit must be zero. The characteristic roots of  $A$  are  $\mu_1, \cdots, \mu_n$ . Hence for  $j = 1, \cdots, n$  we may let  $\lambda_j$  be the continuous function such that  $\lambda_j(s) \rightarrow \mu_j$  as  $s \rightarrow \infty$  and  $\lambda_j(s)$  is a characteristic root of  $A + \alpha(s)D$  for  $s \geq 0$ .

We now shall show that  $\lambda_j - \mu_j \in \mathcal{L}(0, \infty)$  for each  $j$ . Following the procedure used in [9] we note that

$$\begin{aligned} 0 &= \det [A + \alpha(s)D - \lambda_j(s)I] \\ &= (-1)^{n+1}\lambda + \pi_{i=1}^n (\alpha_i \alpha(s) - \lambda_j(s)) \\ &= (-1)^{n+1}\lambda + (-\lambda_j(s))^n + (-\lambda_j(s))^{n-1} \alpha(s) \sum_{i=1}^n \alpha_i + \alpha^2(s)F(s), \end{aligned}$$

where  $F$  is a bounded function. (Recall that  $\alpha(s) \rightarrow 0$  and  $\lambda_j(s) \rightarrow \mu_j$  as  $s \rightarrow \infty$ .) Noting that  $\sum_{i=1}^n \alpha_i = 0$  we then have

$$0 = (-1)^{n+1}\lambda + (-\lambda_j(s))^n + \alpha^2(s)F(s)$$

and

$$0 = (-1)^{n+1}\lambda + (-\mu_j)^n.$$

From which we conclude that

$$(\lambda_j(s))^n - \mu_j^n = -(-1)^n \alpha^2(s)F(s)$$

and

$$|\lambda_j(s) - \mu_j| \left| \sum_{i=1}^n (\lambda_j(s))^{n-i} \mu_j^{i-1} \right| \leq |\alpha^2(s)F(s)|.$$

Since  $\lambda_j(s) \rightarrow \mu_j \neq 0$  as  $s \rightarrow \infty$ , since  $\alpha^2 \in \mathcal{L}(a, \infty)$  and since  $F$  is bounded we see that  $|\lambda_j(s) - \mu_j|$  is for all large  $s$  dominated by a function in  $\mathcal{L}(0, \infty)$ ; hence  $\lambda_j - \mu_j \in \mathcal{L}(0, \infty)$ .

Thus all the hypotheses of Theorem 8.1 p. 92 of [2] are satisfied and noting that the  $j$ th column of  $L$  is an eigenvector of  $A$  corresponding to  $\mu_j$  we are able to conclude that there exist numbers  $s_1, \dots, s_n$  and a fundamental matrix  $Z_0$  for equation (3.3) such that

$$Z_0(s)G(s) \longrightarrow L \quad \text{as } s \longrightarrow \infty$$

where

$$G(s) = \exp \left\{ \text{diag} \left[ - \int_{s_1}^s \lambda_1, \dots, - \int_{s_n}^s \lambda_n \right] \right\}.$$

Since  $\lambda_j - \mu_j \in \mathcal{L}(a, \infty)$  it follows that there is a nonsingular diagonal constant matrix  $H$  such that

$$Z_0(s)H \text{diag} [e^{-\mu_1 s}, \dots, e^{-\mu_n s}] \longrightarrow L \quad \text{as } s \longrightarrow \infty.$$

(See the procedure followed at the end of the proof of Theorem 2.3 in [12].) Since each of  $Z_0 H$  and  $Z$  is a fundamental matrix for equation (3.3) there is a constant nonsingular matrix  $C$  such that  $Z_0 H = ZC$ . Letting  $Y_0$  be  $YC$  and recalling that  $Z(s) = Q(g(s))Y(g(s))$  we have

$$Q(g(s))Y_0(g(s)) \text{diag} [e^{-\mu_1 s}, \dots, e^{-\mu_n s}] \longrightarrow L \quad \text{as } s \longrightarrow \infty.$$

Hence  $Q(t)Y_0(t)E(t) \rightarrow L$  as  $t \rightarrow \infty$  and the theorem is proved.

4. Application. If  $w$  is a weight on  $[a, \infty)$  we denote by  $\mathcal{L}^2(w; a, \infty)$  the Hilbert space of all complex valued measurable  $y$  such that

$$\int_a^\infty |y|^2 w < \infty$$

with the obvious inner product. If  $\mathcal{L}$  is an  $n$ th formally self-adjoint (in the sense defined in [2]; see in particular 13 and 14 p. 204) operator,  $w$  is a weight,

$$\mathcal{D} = \{y \mid y \in \mathcal{L}^2(w; a, \infty), y^{(n-1)} \text{ is absolutely continuous} \\ \text{and } (1/w) \mathcal{L}(y) \in \mathcal{L}^2(w; a, \infty)\},$$

$$\mathcal{D}' = \{y \mid y \in \mathcal{D} \text{ and has compact support interior to } [a, \infty)\}.$$

and  $L$  and  $L'_0$  are the restriction of  $(1/w) \mathcal{L}$  to  $\mathcal{D}$  and  $\mathcal{D}'_0$  respectively then  $L'_0$  is a densely defined symmetric operator in  $\mathcal{L}^2(w; a, \infty)$ , hence admits a closure  $L_0$  in this space, and  $L_0^* = L$  where  $*$  denotes adjoint operator in  $\mathcal{L}^2(w; a, \infty)$ . Verification of these assertions closely parallels that for the case  $w \equiv 1$  found in [1], [4], and [11].

The deficiency indices of  $L_0$  are  $(n_1, n_2)$  where  $n_j$  is the dimension of the subspace of solutions to

$$\mathcal{L}(y) = (-1)^{j+1} i w y$$

which lie in  $\mathcal{L}^2(w; a, \infty)$ . (Actually for  $\mathcal{L}$  formally self-adjoint any  $\lambda$  in the upper half plane may be used for  $i$  and any  $\lambda$  in the lower half plane for  $-i$ . See [4 Theorem 19, p. 1232, 5, and 6].)

By use of Theorem 2 we may conclude the following.

**THEOREM 3.** *Let  $\mathcal{L}$  be as in 1.1 and let  $w$  be an  $\mathcal{L}$ -admissible weight.*

(1) *If  $n$  is even and  $\text{Im } \lambda \neq 0$  the dimension of the subspace of solution to equation 1.2 which lie in  $\mathcal{L}^2(w; a, \infty)$  is  $n/2$ .*

(2) *If  $n = 4k + 1 = 2m + 1$  and  $\text{Re } \lambda > 0$  or if  $n = 4k + 3 = 2m + 1$ , and  $\text{Re } \lambda < 0$  the dimension of the subspace is  $m$ .*

(3) *If  $n = 4k + 1 = 2m + 1$ , and  $\text{Re } \lambda < 0$  or if  $n = 4k + 3 = 2m + 1$ , and  $\text{Re } \lambda > 0$  the dimension is  $m + 1$ .*

*Proof.* We begin by noting that for  $c$  real,  $w$  an  $\mathcal{L}$ -admissible weight for some  $\mathcal{L}$ , and  $E \subset [a, \infty)$  with  $E$  of infinite Lebesgue measure (for the first application below we will take  $E = [a, \infty)$ ),

$$(4.1) \quad \int_E \exp \left\{ \int_a^t w^{1/n} [c + (1/n)(w'/w^{1+1/n})] \right\} dt,$$

is finite if  $c < 0$  and infinite if  $c > 0$ . To see this recall that in the proof of Theorem 2 we showed that  $\alpha(s) = [w'/w^{1+1/n}](g(s)) \rightarrow 0$  as  $s \rightarrow \infty$ . Hence  $[w'/w^{1+1/n}](t) = \alpha(h(t)) \rightarrow 0$  as  $t \rightarrow \infty$ . Since  $w(t) \rightarrow \infty$  as  $t \rightarrow \infty$  we then see that  $w^{1/n}(t)[c + (1/n)(w'/w^{1+1/n})(t)] > c$  for  $c > 0$  and  $< c$  for  $c < 0$  for all large  $t$  and the above assertion is immediate.

We next observe from Theorem 2 that if  $w$  is an  $\mathcal{L}$ -admissible



weight then equation 1.2 has  $n$  linearly independent solutions  $U_1, \dots, U_n$  (with  $U_j = (w(a))^{(n-1)/2nj} y_j$ ) such that

$$(4.2) \quad |U_j(t)|^2 w(t) = (1 + o(1)) \exp \left\{ \int_a^t w^{1/n} [2 \operatorname{Re} \mu_j + (1/n)(w'/w^{1+1/n})] \right\}.$$

If  $n = 2m$  and  $\operatorname{Im} \lambda \neq 0$  we may arrange the  $n$ th roots of  $\lambda$  so that

$$\operatorname{Re} \mu_1 < \operatorname{Re} \mu_2 < \dots < \operatorname{Re} \mu_m < 0 < \operatorname{Re} \mu_{m-1} < \dots < \operatorname{Re} \mu_n.$$

Thus each of  $U_1, \dots, U_m$  will lie in  $\mathcal{L}^2(w; a, \infty)$ ; and if  $c_{m+1}, c_{m+2}, \dots, c_n$  are not all zero and  $j$  is the largest integer with  $m + 1 \leq j \leq n$  such that  $c_j \neq 0$  then

$$\sum_{k=1}^m c_{m+k} U_{m+k} = c_j U_j (1 + o(1)) \notin \mathcal{L}^2(w; a, \infty).$$

Hence the first assertion of the theorem is established.

In case  $\operatorname{Im} \lambda \neq 0$  the last two assertions follow analogously upon noting that in Case 2 if  $\operatorname{Im} \lambda \neq 0$  the  $n$ -th roots may be arranged so that

$$\operatorname{Re} \mu_1 < \dots < \operatorname{Re} \mu_m < 0 < \operatorname{Re} \mu_{m+1} < \dots < \operatorname{Re} \mu_n,$$

and that in Case 3 they may be arranged so that

$$\operatorname{Re} \mu_1 < \dots < \operatorname{Re} \mu_{m+1} < 0 < \operatorname{Re} \mu_{m+2} < \dots < \operatorname{Re} \mu_m.$$

If  $\lambda$  is real and positive and  $n = 4k + 1$  the roots may be arranged so that

$$\begin{aligned} \operatorname{Re} \mu_1 &= \operatorname{Re} \mu_2 < \operatorname{Re} \mu_3 \\ &= \operatorname{Re} \mu_4 < \dots < \operatorname{Re} \mu_{2k-1} \\ &= \operatorname{Re} \mu_{2k} < 0 < \operatorname{Re} \mu_{2k+1} \\ &= \operatorname{Re} \mu_{2k+2} < \dots < \operatorname{Re} \mu_{n-2} \\ &= \operatorname{Re} \mu_{n-1} < \operatorname{Re} \mu_n, \end{aligned}$$

and so that if  $\mu_j = \mu_{j+1}$  then  $\operatorname{Im} \mu_{j+1} > 0$ . Then each of  $U_1, \dots, U_{2k}$  is in  $\mathcal{L}(a, \infty)$ , and each of  $U_{2k+1}, \dots, U_n$  is not in  $\mathcal{L}^2(a, \infty)$ . It remains to be shown that no nontrivial linear combination of  $U_{2k+1}, \dots, U_n$  lies in  $\mathcal{L}^2(a, \infty)$  and to do this it is sufficient to show if  $2k + 1 \leq j < n$  with  $j$  odd then no nontrivial linear combination of  $U_j$  and  $U_{j+1}$  lies in  $\mathcal{L}^2(a, \infty)$ .

Suppose that  $c_1 U_j + c_2 U_{j+1} \in \mathcal{L}^2(w; a, \infty)$  with  $c_1$  and  $c_2$  not both zero and  $j$  odd with  $2k + 1 \leq j < n$ . Since  $U_j \notin \mathcal{L}^2(w; a, \infty)$ , it follows that  $c_1 \neq 0$  and  $U_j + c U_{j+1} \in \mathcal{L}^2(w; a, \infty)$  where  $c = c_2/c_1$ . From Theorem 2 and the definition of  $U_1, \dots, U_n$  we have that

$U_j(t) + cU_{j+1}(t) = U_j(t) [1 + c(U_{j+1}(t)/U_j(t))]$  is

$$(4.3) \quad (1 + o(1))U_j(t) \left\{ 1 + (c + o(1)) \exp \left[ \int_a^t 2i (\operatorname{Im} \mu_{j+1}) w^{1/n} \right] \right\}.$$

For all large  $t$ . Hence  $|c| = 1$  for if  $|c| \neq 1$  the term in  $\{ \}$  would be bounded away from zero for all large  $t$  and this would contradict the fact that  $U_j \notin \mathcal{L}^2(w; a, \infty)$ . Letting  $E = \{ t \mid \text{modulus of term in 4.3 in } \{ \} \text{ is } \geq \sqrt{2} \}$  we see since  $w$  is increasing that  $E$  is of infinite measure. (Think of the exponential term in 4.3 or giving the position of a particle on the unit circle at time  $t$  moving counterclockwise at an ever increasing rate.) Hence from 4.3 we see that for some constant  $K$ ,

$$\int_E |U_j|^2 w \leq K \int_a^\infty |U_j + cU_{j+1}|^2 w < \infty.$$

But from 4.1 and 4.2 we see that

$$\int_E |U_j|^2 w = \infty$$

must be the case. This contradiction shows then that  $c_1U_j + c_2U_{j+1} \in \mathcal{L}^2(w; a, \infty)$ .

The proofs of the remaining assertions when  $\lambda$  is real are naalogous.

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