

# Pacific Journal of Mathematics

**A QUASI-KUMMER FUNCTION**

WAZIR HUSAN ABDI

## A QUASI-KUMMER FUNCTION

WAZIR HASAN ABDI

**A particular integral of Kummer's inhomogeneous differential equation is obtained when the right hand member belongs to a general class of multiform functions. A few basic properties of the solution function are established.**

1. **Introduction.** Let  $\sigma$  be a complex constant but not negative real. We denote by  $\mathcal{R}_\sigma$  the Riemann surface of  $z^\sigma$ . Suppose  $f(z)$  to be analytic everywhere on the disc  $K_R: |z| < R$ , and let

$$\mathcal{H}_R^\sigma \equiv \{\phi(z) \mid \phi = z^\sigma f(z); f \text{ analytic on } K_R\}.$$

Then, to each element of the multiplier space  $\mathcal{R}_\sigma$ , there corresponds one and only one element of the product space  $\mathcal{H}_R^\sigma$  which is regular everywhere in the domain  $K_R$  of the analytic component  $f(z)$  slit and screwed in the usual way, if necessary. A few subspaces of  $\mathcal{H}_R^\sigma$  are:

$$\mathcal{H}_{R(\rho)}^\sigma: \{\phi \mid K_R, R \leq \rho < \infty\}$$

$$\mathcal{H}_{\pi(p)}^\sigma: \{\phi \mid \text{analytic component a polynomial of degree } p\}$$

and

$$\mathcal{H}_{\infty(k)}^\sigma: \{\phi \mid |f^{(m)}(0)| \leq Bk^m, (B, k) > 0\}.$$

Now consider the equation

$$(1.1) \quad z \frac{d^2 W}{dz^2} + (b - z) \frac{dW}{dz} - aW = \phi(z).$$

The associated homogeneous problem leads to Kummer's confluent hypergeometric and other well-known transcendental functions. But the properties of the particular integral of the inhomogeneous equation have been studied in detail only recently by Babister [1] who has considered a few particular cases. In this paper we take the general classes  $\mathcal{H}_{R(\rho)}^\sigma$ ,  $\mathcal{H}_{\infty(e)}^\sigma$ ,  $\mathcal{H}_{\pi(p)}^\sigma$  and use Frobenius's method to show that in each case a particular integral of (1.1) exists and belongs to some similar subspace of  $\mathcal{H}_R^{\sigma+1}$ . We also give some basic properties of the solution-function which we have called quasi-Kummer.

As  $f(z)$  is analytic on  $K_R$ ,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, |z| < R.$$

Accordingly, a formal series solution of (1.1) is given by

$$(1.2) \quad {}_1A_1 \left( \begin{matrix} \sigma \\ z \end{matrix} \middle| \begin{matrix} a \\ b \end{matrix}; f(z) \right) = \sum_{n=0}^{\infty} \frac{(\sigma + a + 1)_n P_n(\sigma; a, b; f)}{(\sigma + 1)_{n+1} (\sigma + b)_{n+1}} z^{n+\sigma+1},$$

where

$$(1.3) \quad P_n(\sigma; a, b; f) = \sum_{m=0}^n \frac{(\sigma + 1)_m (\sigma + b)_m}{(\sigma + a + 1)_m} \frac{f^{(m)}(0)}{m!}$$

and  $(\nu)_n$  denotes the Pochhammer product  $\nu(\nu + 1) \cdots (\nu + n - 1)$ .

2. Some subsidiary results. In order to establish our main results, we require some formulae which will be stated in the form of lemmas. For convenience we write:

$$\begin{aligned} \alpha &= |\sigma + a + 1| & \lambda &= \sec(1/2 \arg(\sigma + a + 1)) \\ \beta &= |\sigma + b| & \mu &= \sec(1/2 \arg(\sigma + b)) \\ \gamma &= |\sigma + 1| & \nu &= \sec(1/2 \arg(\sigma + 1)). \end{aligned}$$

Also, it is assumed that  $\alpha, \beta, \gamma$  are all finite and positive.

LEMMA 1. For  $0 < R \leq \lambda < \infty$   $\text{Max}_{|z|=R} |f(z)| = M(R)$

$$\begin{aligned} (i) & \quad \left[ \begin{array}{l} \frac{(\gamma)_n \lambda^{n-1}}{|\beta - \alpha + 1|} \left| \frac{(\beta)_{n+1}}{(\alpha)_n} + 1 - \alpha \right| \frac{M(R)}{R^n}, \beta - \alpha + 1 \neq 0 \\ (ii) \quad |P_n(\sigma; a, b; f)| \leq \frac{(\beta)_n \lambda^{n-1}}{|\gamma - \alpha + 1|} \left| \frac{(\gamma)_{n+1}}{(\alpha)_n} + 1 - \alpha \right| \frac{M(R)}{R^n}, \gamma - \alpha + 1 \neq 0 \\ (iii) \quad \frac{(\gamma)_n (n + 1) \lambda^{n-1}}{R^n} \frac{M(R)}{R^n}, \beta = \gamma = \alpha - 1 > 0 \end{array} \right. \end{aligned}$$

Proof. By Cauchy's inequality

$$|P_n(\sigma; a, b; f)| \leq \sum_{m=0}^n \left| \frac{(\sigma + 1)_m (\sigma + b)_m}{(\sigma + a + 1)_m} \right| \frac{M(R)}{R^m},$$

which on applying Erber's estimate [3]:

$$\frac{1}{|(\delta)_n|} \leq \frac{\sec^{n-1}(1/2 \arg \delta)}{(|\delta|)_n}, \quad |\arg \delta| < \pi$$

and after simplification proves the first part of the lemma. The second part follows mutatis mutandis on replacing  $\beta$  by  $\gamma$ . Also in the third case  $|P_n(\sigma; a, b; f)| \leq (\beta)_n \lambda^{n-1} \gamma \sum_{m=0}^n 1/\gamma + m$ . Hence the result.

LEMMA 2. If  $|f^{(m)}(0)| \leq Bk^m, B > 0, k > 0$ , then

$$|P_n(\sigma; a, b; f)| \leq \begin{cases} \frac{B}{\lambda} {}_2F_1 \left[ \begin{matrix} \gamma, \beta \\ \alpha \end{matrix}; k\lambda \right] & k\lambda < 1 \\ \frac{Bl^n}{\lambda} {}_2F_1 \left[ \begin{matrix} \gamma, \beta \\ \alpha \end{matrix}; \frac{k\lambda}{l} \right] & 1 \leq k\lambda < l, \end{cases}$$

*Proof.*

$$|P_n(\sigma; a, b; f)| \leq B \left| \sum_{m=0}^n \frac{(\sigma + 1)_m (\sigma + b)_m}{(\sigma + a + 1)_m} \frac{k^m}{m!} \right|$$

$$\leq \frac{Bl^n}{\lambda} \sum_{m=0}^n \frac{(\gamma)_m (\beta)_m}{(\alpha)_m m!} \left(\frac{k\lambda}{l}\right)^m, \quad 1 \leq k\lambda < l$$

which leads to the second part. The proof of the first part is very straightforward.

**3. Main Theorems.** By Lemma 1 (i), the modulus of the general coefficient in the power-series (1.2) can be majorised by

$$\frac{(\alpha)_n (\gamma)_n}{(\gamma)_{n+1} (\beta)_{n+1}} \left(\frac{\lambda\mu\nu}{R}\right)^n \left| \frac{(\beta)_{n+1}}{(\alpha)_n} + 1 - \alpha \right| \frac{M(R)}{|\lambda|\beta - \alpha + 1|}.$$

Hence the series converges absolutely and uniformly to an analytic function for all  $|z| < R/\lambda\mu\nu$ . Another majorant is provided by Lemma 1 (ii) both leading to

**THEOREM 1.** *If  $\phi \in \mathcal{H}_{R(\lambda)}$ ,  $(\lambda, \mu, \nu) < \infty$ , then  ${}_1A_1\left(\begin{smallmatrix} \sigma \\ z \end{smallmatrix} \middle| \begin{smallmatrix} a \\ b \end{smallmatrix}; f(z)\right) \in \mathcal{H}_{R(\rho)}$ ,  $\rho\lambda\mu\nu \leq R$  and  ${}_1A_1\left(\begin{smallmatrix} \sigma \\ z \end{smallmatrix} \middle| \begin{smallmatrix} a \\ b \end{smallmatrix}; f(z)\right)$  never exceeds*

$$\frac{M(R)|z^{\sigma+1}|}{\beta\gamma\lambda|\beta - \alpha + 1|} \left\{ \beta_2F_1\left[\begin{smallmatrix} \gamma, 1 \\ \gamma + 1 \end{smallmatrix}; \frac{\lambda\mu\nu|z|}{R}\right] + {}_3F_2\left[\begin{smallmatrix} \alpha, \gamma, 1 \\ \beta + 1, \gamma + 1 \end{smallmatrix}; \frac{\lambda\mu\nu|z|}{R}\right] \right\},$$

$\beta - \alpha + 1 \neq 0,$

or

$$\frac{M(R)|z^{\sigma+1}|}{\beta\gamma\lambda|\gamma - \alpha + 1|} \left\{ \gamma_2F_1\left[\begin{smallmatrix} \beta, 1 \\ \beta + 1 \end{smallmatrix}; \frac{\lambda\mu\nu|z|}{R}\right] + {}_3F_2\left[\begin{smallmatrix} \alpha, \beta, 1 \\ \beta + 1, \gamma + 1 \end{smallmatrix}; \frac{\lambda\mu\nu|z|}{R}\right] \right\},$$

$\gamma - \alpha + 1 \neq 0$

or

$$\frac{1}{\beta\gamma\lambda} {}_2F_2\left[\begin{smallmatrix} \gamma, 2 \\ \gamma + 1, 1 \end{smallmatrix}; \frac{\lambda\mu\nu|z|}{R}\right], \quad \beta = \gamma = \alpha - 1 > 0.$$

Similarly from Lemma 2, we easily obtain

**THEOREM 2.** *If  $\phi \in \mathcal{H}_{\infty(k)}$ ,  $(\lambda, \mu, \nu) < \infty$ , then  ${}_1A_1\left(\begin{smallmatrix} \sigma \\ z \end{smallmatrix} \middle| \begin{smallmatrix} a \\ b \end{smallmatrix}; f(z)\right) \in \mathcal{H}_{\infty}^{\sigma+1}$  and  ${}_1A_1\left(\begin{smallmatrix} \sigma \\ z \end{smallmatrix} \middle| \begin{smallmatrix} a \\ b \end{smallmatrix}; f(z)\right)$  is dominated by*

$$\frac{B|z^{\sigma+1}|}{\beta\gamma\lambda} {}_2F_1\left[\begin{matrix} \gamma, \beta \\ \alpha \end{matrix}; k\lambda\right] {}_2F_2\left[\begin{matrix} \alpha, 1 \\ \gamma+1, \beta+1 \end{matrix}; \mu\nu|z|\right], k < \frac{1}{\lambda}$$

$$\frac{B|z^{\sigma+1}|}{\beta\gamma\lambda} {}_2F_1\left[\begin{matrix} \gamma, \beta \\ \alpha \end{matrix}; \frac{k\lambda}{l}\right] {}_2F_2\left[\begin{matrix} \alpha, 1 \\ \gamma+1, \beta+1 \end{matrix}; \mu\nu l|z|\right], 1 \leq k\lambda < l.$$

Now, if  $f(z)$  is a polynomial of degree  $p$ , then for all nonnegative integers  $r$ ,  $P_{p+r}(\sigma; a, b; f) = P_p(\sigma; a, b; f)$ . Hence, denoting the set of nonpositive integers by  $Z^0$  we have

**THEOREM 3.** *If  $\phi \in \mathcal{H}_{z(p)}^\sigma$ ,  $(\sigma+1, \sigma+a+1, \sigma+b) \notin Z^0$ , then*

$${}_1A_1\left(\sigma \left| \begin{matrix} a \\ b \end{matrix} \right.; f(z)\right) = \sum_{n=0}^{p-1} \frac{(\sigma+a+1)_n P_n(\sigma; a, b; f)}{(\sigma+1)_{n+1} (\sigma+b)_{n+1}} z^{\sigma+1+n}$$

$$+ \frac{(\sigma+a+1)_p P_p(\sigma; a, b; f)}{(\sigma+1)_{p+1} (\sigma+b)_{p+1}} z^{\sigma+1+p} {}_2F_2\left[\begin{matrix} \sigma+a+1+p, 1 \\ \sigma+b+1+p, \sigma+2+p \end{matrix}; z\right].$$

**4. Contiguity relations.** As the generalized power series (1.2) is uniformly convergent, a number of interesting contiguity relations can be obtained by applying the operator  $d/dz$  or  $\delta (\equiv z d/dz)$  termwise. For example

$$(4.1) \quad \frac{d}{dz} {}_1A_1\left(\sigma \left| \begin{matrix} a \\ b \end{matrix} \right.; f(z)\right) = \sigma {}_1A_1\left(\sigma-1 \left| \begin{matrix} a+1 \\ b+1 \end{matrix} \right.; f(z)\right)$$

$$+ {}_1A_1\left(\sigma \left| \begin{matrix} a+1 \\ b+1 \end{matrix} \right.; \frac{df}{dz}\right).$$

$$(4.2) \quad (b-a-1) {}_1A_1\left(\sigma \left| \begin{matrix} a \\ b \end{matrix} \right.; f(z)\right) = (\sigma+b-1) {}_1A_1\left(\sigma \left| \begin{matrix} a \\ b-1 \end{matrix} \right.; f(z)\right)$$

$$- (\sigma+a+1) {}_1A_1\left(\sigma \left| \begin{matrix} a+1 \\ b \end{matrix} \right.; f(z)\right) + {}_1A_1\left(\sigma+1 \left| \begin{matrix} a \\ b-1 \end{matrix} \right.; \frac{df}{dz}\right)$$

$$- {}_1A_1\left(\sigma+1 \left| \begin{matrix} a+1 \\ b \end{matrix} \right.; \frac{df}{dz}\right).$$

Now, as  ${}_1A_1$  can be written as

$$\frac{z^{\sigma+1} f(0)}{(\sigma+1)(\sigma+b)} + \frac{(\sigma+a+1)z^{\sigma+2}}{(\sigma+1)(\sigma+b)} \sum_{n=0}^{\infty} \frac{(\sigma+a+2)_n z^n}{(\sigma+2)_{n+1} (\sigma+b+1)_{n+1}} \left\{ f(0) \right.$$

$$\left. + \sum_{m=1}^{n-1} \frac{(\sigma+1)_m (\sigma+b)_m}{(\sigma+a+1)_m} \frac{f^{(m)}(0)}{m!} \right\},$$

we have on simplification

$$(4.3) \quad {}_1A_1\left(\sigma \left| \begin{matrix} a \\ b \end{matrix} \right.; f(z)\right) - {}_1A_1\left(\sigma+1 \left| \begin{matrix} a \\ b \end{matrix} \right.; \frac{f(z) - f(0)}{z}\right)$$

$$= \frac{f(0)z^{\sigma+1}}{(\sigma+1)(\sigma+b)} + \frac{(\sigma+a+1)f(0)z^{\sigma+2}}{(\sigma+1)_2(\sigma+b)_2} {}_2F_2 \left[ \begin{matrix} \sigma+a+2, & 1 \\ \sigma+b+2, & \sigma+3 \end{matrix}; z \right].$$

As particular cases, we see that

$${}_1A_1 \left( \begin{matrix} \sigma-1 \\ z \end{matrix} \middle| \begin{matrix} a \\ b \end{matrix}; e^{\sigma z} \right) = \Lambda_{\rho,\sigma}(a, b; z) \text{ and } {}_1A_1 \left( \begin{matrix} \sigma-1 \\ z \end{matrix} \middle| \begin{matrix} a \\ b \end{matrix}; 1 \right) = \theta_\sigma(a, b; z)$$

where  $\Lambda$  and  $\theta$  are Babister's nonhomogeneous confluent functions, so (4.1) and (4.3) reduce to known results [1], (4.236), (4.189). Also from (4.3)

$$\begin{aligned} & {}_1A_1 \left( \begin{matrix} \sigma \\ z \end{matrix} \middle| \begin{matrix} a \\ b \end{matrix}; {}_pF_q \left[ \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right] \right) \\ (4.4) \quad & - \frac{a_1 \dots a_p}{b_1 \dots b_q} {}_1A_1 \left( \begin{matrix} \sigma \\ z \end{matrix} \middle| \begin{matrix} a \\ b \end{matrix}; {}_{p+1}F_{q+1} \left[ \begin{matrix} a_1+1, \dots, a_p+1 & 1 \\ b_1+1, \dots, b_q+1 & 2 \end{matrix}; z \right] \right) \\ & = \frac{z^{\sigma+1}}{(\sigma+1)(\sigma+b)} + \frac{(\sigma+a+1)z^{\sigma+2}}{(\sigma+1)_2(\sigma+b)_2} {}_2F_2 \left[ \begin{matrix} \sigma+a+2, & 1 \\ \sigma+b+2, & \sigma+3 \end{matrix}; z \right] \end{aligned}$$

with the usual restriction on the parameters.

5. Illustration. The above results are of particular advantage when the analytic component of  $\phi$  involves functions of hypergeometric type because these (for that matter, almost all) special functions belong to one of the classes considered.

For example: (See Table on next page).

The first four are Babister's nonhomogeneous confluent functions, the next three are obtained via a result due to Carlitz [2]:

$$\begin{aligned} & {}_5F_4 \left[ \begin{matrix} a, 1+a/2, b, c, d; \\ a/2, 1+a-b, 1+a-c, 1+a-d \end{matrix}; z \right] \\ & = \frac{(1+a)_n(1+b)_n(1+c)_n(1+d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n n!} \end{aligned}$$

provided that  $a = b + c + d$ . In the last two cases  $P_n(\sigma; a, b; f) = ((\sigma+1)_n)/n!$  or  $n+1$  respectively yielding the results with the usual restriction on the parameters.

Some other properties of  ${}_1A_1 \left( \begin{matrix} \sigma \\ z \end{matrix} \middle| \begin{matrix} a \\ b \end{matrix}; f(z) \right)$  will be discussed in another communication.

$\sigma$	$f(z)$	${}_1A_1 \left( \begin{matrix} \sigma \\ z \end{matrix} \middle  \begin{matrix} a \\ b \end{matrix} ; f(z) \right)$
0	$\frac{{}_2F_1 \Gamma(b)}{\Gamma(a)\Gamma(b-a)} e^{z/1}$	$\Omega(a, b; z)$
$1-b$	$\frac{{}_2F_1 \Gamma(2-b)e^{z/2}}{\Gamma(a-b+1)\Gamma(1-a)}$	$\bar{\Omega}(a, b; z)$
$\sigma-1$	1	$\theta_\sigma(a, b; z)$
$\sigma-1$	$e^{\sigma z}$	$A_{\rho, \sigma}(a, b; z)$
$\sigma$	${}_3F_3 \left[ \begin{matrix} 2\sigma+a+b, \sigma+1+\frac{1}{2}(a+b), a-\frac{1}{2} \\ 2\sigma+b+2, \sigma+\frac{1}{2}(a+b), \sigma+a+b \end{matrix} ; z \right]$	$\frac{z^{\sigma+1}}{(\sigma+1)(\sigma+b)} {}_2F_2 \left[ \begin{matrix} 2\sigma+a+b+1, a \\ 2\sigma+b+2, \sigma+a+b \end{matrix} ; z \right]$
$\sigma$	${}_3F_3 \left[ \begin{matrix} 2\sigma+a+1, \sigma+\frac{1}{2}(a+3), a-b \\ 2\sigma+b+2, \sigma+\frac{1}{2}(a+1), \sigma+a-b+2 \end{matrix} ; z \right]$	$\frac{z^{\sigma+1}}{(\sigma+1)(\sigma+b)} {}_2F_2 \left[ \begin{matrix} 2\sigma+a+2, 1+a-b \\ 2\sigma+b+2, \sigma-a-b+2 \end{matrix} ; z \right]$
$\sigma$	${}_3F_3 \left[ \begin{matrix} 2\sigma+2a+2, \sigma+a+2, 2a-b+1 \\ 2\sigma+b+2, 2a+\sigma+2, 2a-b+\sigma+2 \end{matrix} ; z \right]$	$\frac{z^{\sigma+1}}{(\sigma+1)(\sigma+b)} {}_3F_3 \left[ \begin{matrix} \sigma+a+1, 2\sigma+2a+3, 2a-b+2 \\ 2\sigma+b+2, 2a+\sigma+2, \sigma+2a-b+2 \end{matrix} ; z \right]$
$\sigma$	${}_1F_1 \left[ \begin{matrix} \sigma+a+1 \\ \sigma+b \end{matrix} ; z \right]$	$\frac{z^{\sigma+1}}{(\sigma+1)(\sigma+b)} {}_1F_1 \left[ \begin{matrix} \sigma+a+1 \\ \sigma+b+1 \end{matrix} ; z \right]$
$\sigma$	${}_2F_2 \left[ \begin{matrix} \sigma+a+1, 1 \\ \sigma+b, \sigma+1 \end{matrix} ; z \right]$	$\frac{(\sigma+a+1)z^{\sigma+2}}{(\sigma+1)_2(\sigma+b)_2} {}_2F_2 \left[ \begin{matrix} \sigma+a+2, 2 \\ \sigma+b+2, 3 \end{matrix} ; z \right]$

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