

# Pacific Journal of Mathematics

**NORM CONVERGENCE OF MARTINGALES OF  
RADON-NIKODYM DERIVATIVES GIVEN A  $\sigma$ -LATTICE**

RICHARD BRIAN DARST AND GENE ALLEN DEBOTH

## NORM CONVERGENCE OF MARTINGALES OF RADON-NIKODYM DERIVATIVES GIVEN A $\sigma$ -LATTICE

R. B. DARST AND G. A. DEBOTH

Suppose that  $\{\mathcal{M}_k\}$  is an increasing sequence of sub  $\sigma$ -lattices of a  $\sigma$ -algebra  $\mathcal{A}$  of subsets of a non-empty set  $\Omega$ . Let  $\mathcal{M}$  be the sub  $\sigma$ -lattice generated by  $\bigcup_k \mathcal{M}_k$ . Suppose that  $L^\phi$  is an associated Orlicz space of  $\mathcal{A}$ -measurable functions, where  $\phi$  satisfies the  $\Delta_2$ -condition, and let  $h \in L^\phi$ . It is verified that the Radon-Nikodym derivative,  $f_k$ , of  $h$  given  $\mathcal{M}_k$  is in  $L^\phi$  and shown that the sequence  $\{f_k\}$  converges to  $f$  in  $L^\phi$ , where  $f$  is the Radon-Nikodym derivative of  $h$  given  $\mathcal{M}$ .

1. Introduction. H. D. Brunk defined conditional expectation given a  $\sigma$ -lattice and established several of its properties in [1]. Subsequently S. Johansen [5] described a Radon-Nikodym derivative given a  $\sigma$ -lattice and showed that the Radon-Nikodym derivative was the conditional expectation in the appropriate case. Then H. D. Brunk and S. Johansen [2] proved an almost everywhere martingale convergence theorem for the Radon-Nikodym derivatives given an increasing sequence of  $\sigma$ -lattices. We shall establish norm convergence of these derivatives in  $L_1$  and in the Orlicz spaces  $L^\phi$ , where  $\phi$  satisfies the  $\Delta_2$ -condition. The theory of these Orlicz spaces can be found in [6], so we shall assume and build upon the results therein. Thereby, we can place fewer restrictions on  $\phi$  and obtain  $L_1$ -convergence as a byproduct.

2. Notation. Let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of a (non-empty) set  $\Omega$ , and let  $\mu$  be a non-negative (bounded)  $\sigma$ -additive function defined on  $\mathcal{A}$ .

For our purposes the following information about  $\phi$  will suffice:  $\phi$  is an even, convex function defined on the real numbers,  $R$ , with  $\phi(0) = 0$  and  $\phi(x) \neq 0$  for some  $x$ . Moreover, there exists  $K > 0$  with  $\phi(2x) \leq K\phi(x)$  for all  $x \in R$ . This latter property is called the  $\Delta_2$ -condition; it implies

$$(1) \quad \phi\left(2\left(\frac{x+y}{2}\right)\right) \leq K\phi\left(\frac{x+y}{2}\right) \leq \left(\frac{K}{2}\right)[\phi(x) + \phi(y)].$$

Then  $L^\phi$  denotes the collection of (real valued)  $\mathcal{A}$ -measurable functions  $h$  defined on  $\Omega$  with  $\int_\Omega \phi(h)d\mu < \infty$ . Since  $\phi$  is convex and not

identically zero,  $L^\phi \subset L_1$ ;  $L^\phi$  is usually a proper subset of  $L_1$  if  $\lim_{x \rightarrow \infty} \Phi(x)/x = \infty$ . This latter property and  $\lim_{x \rightarrow 0} \Phi(x)/x = 0$  are required of an Orlicz space; but, these two properties are not necessary for our estimates to be valid. Examples are  $\Phi(x) = |x|^p$ ,  $1 \leq p < \infty$ .

Let  $h \in L^\phi$  and  $\lambda(E) = \int_E h d\mu$ , where  $E \in \mathcal{A}$ . Let  $\mathcal{M}$  be a sub  $\sigma$ -lattice of  $\mathcal{A}$  and let  $f$  be the Radon-Nikodym derivative of  $\lambda$  with respect to  $\mu$ . Thus,  $f$  is the  $\mathcal{M}$ -measurable function defined on  $\Omega$  ( $\phi$ : the empty set,  $\Omega$ , and  $[f > a]$  belong to  $\mathcal{M}$ , for all  $a \in R$ ) satisfying

$$(2) \quad \lambda(A \cap [f \leq b]) \leq b\mu(A \cap [f \leq b]), \quad \text{where } A \in \mathcal{M} \text{ and } b \in R,$$

and

$$(3) \quad \lambda([f > a] \cap B^c) \geq a\mu([f > a] \cap B^c),$$

where  $B^c = \Omega - B$ ,  $B \in \mathcal{M}$ , and  $a \in R$ .

Our first result is a preliminary step to an  $L^\phi$  martingale convergence theorem.

**3. The derivative of an  $L^\phi$ -function is an  $L^\phi$ -function.** We shall verify this assertion by establishing a sequence of estimates, the first of which is

$$(4) \quad \int_{[f > a]} \Phi(f) d\mu \leq \int_{[f > a]} \Phi(h) d\mu, \quad \text{for all } a \geq 0.$$

To verify (4), choose  $\delta > 0$  and  $a = a_0 < a_1 < a_2 < \dots$  with  $\Phi(a_k) = \Phi(a_{k-1}) + \delta$ . Let  $A_k = [a_k \leq f < a_{k+1}]$  and notice that (3) implies

$$|\lambda|(\Omega) \geq \lambda([f > a_k]) \geq a_k \mu([f > a_k]).$$

Thus,  $\mu([f > a_k]) \rightarrow 0$  and

$$\int_{[f > a]} \Phi(\cdot) d\mu = \sum_{k=1}^n \int_{A_k} \Phi(\cdot) d\mu + \int_{[f > a_n]} \Phi(\cdot) d\mu = \sum_{k=1}^{\infty} \int_{A_k} \Phi(\cdot) d\mu.$$

Applying (3) again,  $\int_{A_k} h d\mu = \lambda(A_k) \geq a_{k-1} \mu(A_k)$ , so

$$a_{k-1} \leq \frac{1}{\alpha_k} \int_{A_k} h d\mu, \quad \text{where } \alpha_k = \mu(A_k) > 0.$$

Then, applying Jensen's inequality,

$$\Phi(a_{k-1}) \leq \Phi\left(\frac{1}{\alpha_k} \int_{A_k} h d\mu\right) \leq \frac{1}{\alpha_k} \int_{A_k} \Phi(h) d\mu.$$

Next, notice that

$$\int_{A_k} \Phi(f)d\mu \leq \Phi(a_k)\mu(A_k) = (\Phi(a_{k-1}) + \delta)\mu(A_k) \leq \int_{A_k} \Phi(h)d\mu + \delta\mu(A_k) .$$

Thus  $\int_{[f>a]} \Phi(f)d\mu \leq \int_{[f>a]} \Phi(h)d\mu + \delta\mu(\Omega)$ , for all  $\delta > 0$ , which implies (4).

By a similar argument, one obtains

$$(5) \quad \int_{[f\leq a]} \Phi(f)d\mu \leq \int_{[f\leq a]} \Phi(h)d\mu , \quad \text{for all } a \leq 0 .$$

Hence, splitting  $\Omega$  into two pieces,  $[f > 0]$  and  $[f \leq 0]$ , and applying (4) and (5), yields

$$(6) \quad \int_{\Omega} \Phi(f)d\mu \leq \int_{\Omega} \Phi(h)d\mu ;$$

thus verifying Theorem 1.

**THEOREM 1.** *The Radon-Nikodym derivative of an  $L^{\phi}$ -function is an  $L^{\phi}$ -function.*

**4. A Martingale convergence theorem.** Suppose that  $\{\mathcal{M}_k\}_{k=1}^{\infty}$  is an increasing sequence of  $\sigma$ -lattices of subsets of  $\Omega$ , and  $\mathcal{M}$  is the  $\sigma$ -lattice generated by the lattice  $\mathcal{M}_{\infty} = \bigcup_k \mathcal{M}_k$ . Denote by  $\mathcal{A}_k$  the  $\sigma$ -algebra that is generated by  $\mathcal{M}_k$  and by  $\lambda_k$  and  $\mu_k$  the restrictions of  $\lambda$  and  $\mu$  to  $\mathcal{A}_k$ . Let  $h_k$  be an  $\mathcal{A}_k$ -measurable function satisfying  $\lambda(E) = \int_E h_k d\mu$ , where  $E \in \mathcal{A}_k$ , and denote by  $f_k$  the Radon-Nikodym derivative of  $\lambda_k$  with respect to  $\mu_k$  on  $\mathcal{M}_k$ .

**THEOREM 2.** *The sequence  $\{f_k\}$  converges to  $f$  in  $L^{\phi}$ -norm:*

$$(7) \quad \lim_{k \rightarrow \infty} \int_{\Omega} \Phi(f - f_k)d\mu = 0 .$$

*Proof.* To begin, notice that applying (4) and (5) to  $f_k$  yields

$$(8) \quad \int_{[f_k > a]} \Phi(h_k)d\mu \geq \int_{[f_k > a]} \Phi(f_k)d\mu , \quad \text{for all } a \geq 0 ,$$

and

$$(9) \quad \int_{[f_k \leq a]} \Phi(h_k)d\mu \geq \int_{[f_k \leq a]} \Phi(f_k)d\mu , \quad \text{for all } a \leq 0 .$$

Since  $\lambda_k$  is the restriction of  $\lambda$  to  $\mathcal{A}_k$ , a variation on the theme which established (4) verifies

$$(10) \quad \int_E \Phi(h)d\mu \geq \int_E \Phi(h_k)d\mu , \quad \text{for all } E \in \mathcal{A}_k :$$

To substantiate this latter assertion, suppose  $a \geq 0$ ,  $\delta > 0$ ,  $b > a$ ,  $\Phi(b) = \Phi(a) + \delta$ ,  $E \in \mathcal{A}_k$ ,  $F = E \cap [b \geq h_k > a]$ , and  $\mu(F) > 0$ . Then  $\int_F h_k d\mu = \int_F h d\mu$ , since  $F \in \mathcal{A}_k$ . Moreover,

$$\int_F \Phi(h_k) d\mu \leq \Phi(b) \mu(F) = [\Phi(a) + \delta] \mu(F),$$

and

$$\begin{aligned} \Phi(a) &\leq \Phi\left(\frac{1}{\mu(F)} \int_F h_k d\mu\right) = \Phi\left(\frac{1}{\mu(F)} \int_F h d\mu\right) \\ &\leq \frac{1}{\mu(F)} \int_F \Phi(h) d\mu. \end{aligned}$$

Thus,

$$\int_F \Phi(h_k) d\mu \leq \int_F \Phi(h) d\mu + \delta \mu(F).$$

Hence, appealing to the proof of (4) and to the sentence containing (5), we claim (10). Consequently,

$$(11) \quad \int_{[f_k > a]} \Phi(h) d\mu \geq \int_{[f_k > a]} \Phi(f_k) d\mu, \quad \text{where } a \geq 0 \text{ and } k = 1, 2, \dots,$$

and

$$(12) \quad \int_{[f_k \leq a]} \Phi(h) d\mu \geq \int_{[f_k \leq a]} \Phi(f_k) d\mu, \quad \text{where } a \leq 0 \text{ and } k = 1, 2, \dots.$$

Moreover,  $a\mu([|f_k| > a]) \leq |\lambda|([|f_k| > a]) \leq |\lambda|(\Omega)$ , where  $a \geq 0$ ; thus,

$$(13) \quad \limsup_{n \rightarrow \infty} \int_{[|f_k| > n]} \Phi(f_k) d\mu = 0.$$

So we can truncate the functions and still approximate them uniformly as follows. Whenever  $n$  is a positive integer and  $u$  is a (real valued) function defined on  $\Omega$ , let  $u^n(x) = u(x)$ , where  $|u(x)| \leq n$ , and  $u^n(x) = nu(x)/|u(x)|$  otherwise. Then, using (1) and setting  $M = \max\{(K/2), (K^2/4)\}$ ,

$$\begin{aligned} \int_{\Omega} \Phi(f - f_k) d\mu &= \int_{\Omega} \Phi(\{f - f^n\} + \{f^n - (f_k)^n\} + \{(f_k)^n - f_k\}) d\mu \\ &\leq M(A_n + B_n + C_n), \end{aligned}$$

where

$$A_n = \int_{[|f|>n]} \Phi(f) d\mu ,$$

$$B_n = \int_{\Omega} \Phi(f_n - (f_k)^n) d\mu ,$$

and

$$C_n = \int_{[|f_k|>n]} \Phi(f_k) d\mu .$$

From (4), (5) and (13), we obtain  $A_n \rightarrow 0$  and  $C_n \rightarrow 0$ . Moreover, for each  $\delta > 0$ ,

$$B_n \leq \Phi(2n)\mu([|f^n - (f_k)^n| > \delta]) + \Phi(\delta)\mu(\Omega)$$

$$\leq \Phi(2n)\mu([|f - f_k| > \delta]) + \Phi(\delta)\mu(\Omega) .$$

But, Brunk and Johansen have shown that  $\lim_k \mu([|f - f_k| > \delta]) = 0$ , where  $\delta > 0$ , so Theorem 2 is established.

Because of the approximation properties which are verified in [4], the results of this paper extend immediately to analogous results for the derivatives of additive set functions defined on algebras of subsets of  $\Omega$  given a sub lattice (cf. [3]). Results for vector valued functions with respect to lattices which are related to the results: [7], [8], [9], of J. J. Uhl, Jr. for vector valued functions with respect to algebras should appear subsequently.

#### REFERENCES

1. H. D. Brunk, *Conditional expectation given a  $\sigma$ -lattice and applications*, Annals. Math. Statist., **36** (1965), 1339-1350.
2. H. D. Brunk and S. Johansen, *A generalized Radon-Nikodym derivative*, Pacific J. Math., **34** (1970), 585-617.
3. R. B. Darst, *The Lebesgue decomposition, Radon-Nikodym derivative, conditional expectation and martingale convergence for lattices of sets*, Pacific J. Math., **35** (1970), 581-600.
4. R. B. Darst and G. A. DeBoth, *Two approximation properties and a Radon-Nikodym derivative for lattices of sets*, Indiana Univ. Math. J., **21** (1971), 355-362.
5. S. Johansen, *The descriptive approach to the derivative of a set function with respect to a  $\sigma$ -lattice*, Pacific J. Math., **21** (1967), 49-58.
6. M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex functions and Orlicz spaces* (Translation), Groningen, 1961.
7. J. J. Uhl, Jr., *Orlicz spaces of finitely additive set functions*, Studia Math., T. **XXIX** (1967), 19-58.
8. ———, *Applications of Radon-Nikodym theorems to martingale convergence*, Trans. Amer. Math. Soc., **145** (1969), 271-285.

9. J. J. Uhl, Jr., *Martingales of vector valued set functions*, Pacific J. Math., **30** (1969), 533-548.

Received April 7, 1971, R. B. Darst was supported in part by the National Science Foundation under grant no. GP 9470 and G. A. DeBoth was supported by a National Science Foundation Science Faculty Fellowship.

PURDUE UNIVERSITY  
COLORADO STATE UNIVERSITY  
AND  
ST. NORBERT COLLEGE

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. SAMELSON

Stanford University  
Stanford, California 94305

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

C. R. HOBBY

University of Washington  
Seattle, Washington 98105

RICHARD ARENS

University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index. to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.



# Pacific Journal of Mathematics

Vol. 40, No. 3

November, 1972

Wazir Husan Abdi, <i>A quasi-Kummer function</i> . . . . .	521
Vasily Cateforis, <i>Minimal injective cogenerators for the class of modules of zero singular submodule</i> . . . . .	527
W. Wistar (William) Comfort and Anthony Wood Hager, <i>Cardinality of <math>k</math>-complete Boolean algebras</i> . . . . .	541
Richard Brian Darst and Gene Allen DeBoth, <i>Norm convergence of martingales of Radon-Nikodym derivatives given a <math>\sigma</math>-lattice</i> . . . . .	547
M. Edelstein and Anthony Charles Thompson, <i>Some results on nearest points and support properties of convex sets in <math>c_0</math></i> . . . . .	553
Richard Goodrick, <i>Two bridge knots are alternating knots</i> . . . . .	561
Jean-Pierre Gossez and Enrique José Lami Dozo, <i>Some geometric properties related to the fixed point theory for nonexpansive mappings</i> . . . . .	565
Dang Xuan Hong, <i>Covering relations among lattice varieties</i> . . . . .	575
Carl Groos Jockusch, Jr. and Robert Irving Soare, <i>Degrees of members of <math>\Pi_1^0</math> classes</i> . . . . .	605
Leroy Milton Kelly and R. Rottenberg, <i>Simple points in pseudoline arrangements</i> . . . . .	617
Joe Eckley Kirk, Jr., <i>The uniformizing function for a class of Riemann surfaces</i> . . . . .	623
Glenn Richard Luecke, <i>Operators satisfying condition <math>(G_1)</math> locally</i> . . . . .	629
T. S. Motzkin, <i>On <math>L(S)</math>-tuples and <math>l</math>-pairs of matrices</i> . . . . .	639
Charles Estep Murley, <i>The classification of certain classes of torsion free Abelian groups</i> . . . . .	647
Louis D. Nel, <i>Lattices of lower semi-continuous functions and associated topological spaces</i> . . . . .	667
David Emroy Penney, II, <i>Establishing isomorphism between tame prime knots in <math>E^3</math></i> . . . . .	675
Daniel Rider, <i>Functions which operate on <math>\mathbb{F}L_p(T)</math>, <math>1 &lt; p &lt; 2</math></i> . . . . .	681
Thomas Stephen Shores, <i>Injective modules over duo rings</i> . . . . .	695
Stephen Simons, <i>A convergence theorem with boundary</i> . . . . .	703
Stephen Simons, <i>Maximinimax, minimax, and antiminimax theorems and a result of R. C. James</i> . . . . .	709
Stephen Simons, <i>On Ptak's combinatorial lemma</i> . . . . .	719
Stuart A. Steinberg, <i>Finitely-valued <math>f</math>-modules</i> . . . . .	723
Pui-kei Wong, <i>Integral inequalities of Wirtinger-type and fourth-order elliptic differential inequalities</i> . . . . .	739
Yen-Yi Wu, <i>Completions of Boolean algebras with partially additive operators</i> . . . . .	753
Phillip Lee Zenor, <i>On spaces with regular <math>G_\delta</math>-diagonals</i> . . . . .	759