NORM CONVERGENCE OF MARTINGALES OF RADON-NIKODYM DERIVATIVES GIVEN A $\sigma$-LATTICE

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R. B. Darst and G. A. DeBoth

Suppose that $\{\mathcal{M}_k\}$ is an increasing sequence of sub $\sigma$-lattices of a $\sigma$-algebra $\mathcal{X}$ of subsets of a non-empty set $\Omega$. Let $\mathcal{M}$ be the sub $\sigma$-lattice generated by $\bigcup_k \mathcal{M}_k$. Suppose that $L^\phi$ is an associated Orlicz space of $\mathcal{X}$-measurable functions, where $\phi$ satisfies the $A_2$-condition, and let $h \in L^\phi$. It is verified that the Radon-Nikodym derivative, $f_k$, of $h$ given $\mathcal{M}_k$ is in $L^\phi$ and shown that the sequence $\{f_k\}$ converges to $f$ in $L^\phi$, where $f$ is the Radon-Nikodym derivative of $h$ given $\mathcal{M}$.

1. Introduction. H. D. Brunk defined conditional expectation given a $\sigma$-lattice and established several of its properties in [1]. Subsequently S. Johansen [5] described a Radon-Nikodym derivative given a $\sigma$-lattice and showed that the Radon-Nikodym derivative was the conditional expectation in the appropriate case. Then H. D. Brunk and S. Johansen [2] proved an almost everywhere martingale convergence theorem for the Radon-Nikodym derivatives given an increasing sequence of $\sigma$-lattices. We shall establish norm convergence of these derivatives in $L_1$ and in the Orlicz spaces $L^\phi$, where $\phi$ satisfies the $A_2$-condition. The theory of these Orlicz spaces can be found in [6], so we shall assume and build upon the results therein. Thereby, we can place fewer restrictions on $\phi$ and obtain $L_1$-convergence as a byproduct.

2. Notation. Let $\mathcal{X}$ be a $\sigma$-algebra of subsets of a (non-empty) set $\Omega$, and let $\mu$ be a non-negative (bounded) $\sigma$-additive function defined on $\mathcal{X}$.

For our purposes the following information about $\phi$ will suffice: $\phi$ is an even, convex function defined on the real numbers, $\mathbb{R}$, with $\phi(0) = 0$ and $\phi(x) \neq 0$ for some $x$. Moreover, there exists $K > 0$ with $\phi(2x) \leq K\phi(x)$ for all $x \in \mathbb{R}$. This latter property is called the $A_2$-condition; it implies

$$\phi(x + y) = \phi\left(2\left(\frac{x + y}{2}\right)\right) \leq K\phi\left(\frac{x + y}{2}\right) \leq \left(\frac{K}{2}\right)[\phi(x) + \phi(y)].$$

Then $L^\phi$ denotes the collection of (real valued) $\mathcal{X}$-measurable functions $h$ defined on $\Omega$ with $\int_\Omega \phi(h)d\mu < \infty$. Since $\phi$ is convex and not
identically zero, \( L^\Phi \subset L_1 \); \( L^\phi \) is usually a proper subset of \( L^\Phi \) if \( \lim_{x \to \infty} \Phi(x)/x = \infty \). This latter property and \( \lim_{x \to 0} \Phi(x)/x = 0 \) are required of an Orlicz space; but, these two properties are not necessary for our estimates to be valid. Examples are \( \Phi(x) = |x|^p, 1 \leq p < \infty \).

Let \( h \in L^\Phi \) and \( \lambda(E) = \int_E h d\mu \), where \( E \in \mathcal{M} \). Let \( \mathcal{M} \) be a sub \( \sigma \)-lattice of \( \mathcal{F} \) and let \( f \) be the Radon-Nikodym derivative of \( \lambda \) with respect to \( \mu \). Thus, \( f \) is the \( \mathcal{M} \)-measurable function defined on \( \Omega \) (\( \emptyset \): the empty set, \( \Omega \), and \( \{ f > a \} \) belong to \( \mathcal{M} \), for all \( a \in \mathbb{R} \)) satisfying

\[
\lambda(A \cap [f \leq b]) \leq b \mu(A \cap [f \leq b]), \quad \text{where} \ A \in \mathcal{M} \ \text{and} \ b \in \mathbb{R} ,
\]
and

\[
\lambda([f > a] \cap B^c) \geq a \mu([f > a] \cap B^c),
\]
where \( B^c = \Omega - B, B \in \mathcal{M} \), and \( a \in \mathbb{R} \).

Our first result is a preliminary step to an \( L^\phi \) martingale convergence theorem.

3. The derivative of an \( L^\Phi \)-function is an \( L^\phi \)-function. We shall verify this assertion by establishing a sequence of estimates, the first of which is

\[
\int_{[f > a]} \Phi(f) d\mu \leq \int_{[f > a]} \Phi(h) d\mu , \quad \text{for all} \ a \geq 0 .
\]

To verify (4), choose \( \delta > 0 \) and \( a = a_0 < a_1 < a_2 < \cdots \) with \( \Phi(a_k) = \Phi(a_{k-1}) + \delta \). Let \( A_k = [a_k \leq f > a_{k-1}] \) and notice that (3) implies

\[
| \lambda | (O) \geq \lambda ([f > a_k]) \geq a_k \mu ([f > a_k]) .
\]

Thus, \( \mu ([f > a_k]) \to 0 \) and

\[
\int_{[f > a]} \Phi(\cdot) d\mu = \sum_{k=1}^\infty \int_{A_k} \Phi(\cdot) d\mu + \int_{[f > a]} \Phi(\cdot) d\mu = \sum_{k=1}^\infty \int_{A_k} \Phi(\cdot) d\mu .
\]

Applying (3) again, \( \int_{A_k} h d\mu = \lambda(A_k) \geq a_k \mu(A_k) \), so

\[
a_{k-1} \leq \frac{1}{\alpha_k} \int_{A_k} h d\mu , \quad \text{where} \ \alpha_k = \mu(A_k) > 0 .
\]

Then, applying Jensen’s inequality,

\[
\Phi(a_{k-1}) \leq \Phi \left( \frac{1}{\alpha_k} \int_{A_k} h d\mu \right) \leq \frac{1}{\alpha_k} \int_{A_k} \Phi(h) d\mu .
\]

Next, notice that
\[
\int_{A_k} \Phi(f) d\mu \leq \Phi(a_k) \mu(A_k) = (\Phi(a_{k-1}) + \delta) \mu(A_k) \leq \int_{A_k} \Phi(h) d\mu + \delta \mu(A_k).
\]
Thus \(\int_{[f > a]} \Phi(f) d\mu \leq \int_{[f > a]} \Phi(h) d\mu + \delta \mu(\Omega)\), for all \(\delta > 0\), which implies (4).

By a similar argument, one obtains
\[
\int_{[f \leq a]} \Phi(f) d\mu \leq \int_{[f \leq a]} \Phi(h) d\mu, \quad \text{for all } a \geq 0.
\]
Hence, splitting \(\Omega\) into two pieces, \([f > 0]\) and \([f \leq 0]\), and applying (4) and (5), yields
\[
\int_{\Omega} \Phi(f) d\mu \leq \int_{\Omega} \Phi(h) d\mu;
\]
thus verifying Theorem 1.

**Theorem 1.** The Radon-Nikodym derivative of an \(L^p\)-function is an \(L^p\)-function.

4. A Martingale convergence theorem. Suppose that \(\{\mathcal{M}_k\}_{k=1}^\infty\) is an increasing sequence of \(\sigma\)-lattices of subsets of \(\Omega\), and \(\mathcal{M}\) is the \(\sigma\)-lattice generated by the lattice \(\mathcal{M}_m = \bigcup_{k=1}^m \mathcal{M}_k\). Denote by \(\mathcal{A}_h\) the \(\sigma\)-algebra that is generated by \(\mathcal{M}_k\) and by \(\lambda_k\) and \(\mu_k\) the restrictions of \(\lambda\) and \(\mu\) to \(\mathcal{A}_h\). Let \(h_k\) be an \(\mathcal{A}_h\)-measurable function satisfying\(\lambda(E) = \int_E h_k d\mu\), where \(E \in \mathcal{A}_h\), and denote by \(f_k\) the Radon-Nikodym derivative of \(\lambda_k\) with respect to \(\mu_k\) on \(\mathcal{M}_k\).

**Theorem 2.** The sequence \(\{f_k\}\) converges to \(f\) in \(L^p\)-norm:
\[
\lim_{k \to \infty} \int_{\Omega} \Phi(f - f_k) d\mu = 0.
\]

**Proof.** To begin, notice that applying (4) and (5) to \(f_k\) yields
\[
\int_{[f_k > a]} \Phi(h_k) d\mu \geq \int_{[f_k > a]} \Phi(f_k) d\mu, \quad \text{for all } a \geq 0,
\]
and
\[
\int_{[f_k \leq a]} \Phi(h_k) d\mu \geq \int_{[f_k \leq a]} \Phi(f_k) d\mu, \quad \text{for all } a \leq 0.
\]
Since \(\lambda_k\) is the restriction of \(\lambda\) to \(\mathcal{A}_h\), a variation on the theme which established (4) verifies
\[
\int_E \Phi(h) d\mu \geq \int_E \Phi(h_k) d\mu, \quad \text{for all } E \in \mathcal{A}_h:
\]
To substantiate this latter assertion, suppose \( \alpha \geq 0, \delta > 0, b > a, \Phi(b) = \Phi(a) + \delta, E \in \mathcal{A}, F = E \cap [b \geq h_k > a], \) and \( \mu(F) > 0. \) Then
\[
\int_F h_k d\mu = \int_F h d\mu, \text{ since } F \in \mathcal{A}.
\]
Moreover,
\[
\int_F \Phi(h) d\mu \leq \Phi(b) \mu(F) = [\Phi(a) + \delta] \mu(F),
\]
and
\[
\Phi(a) \leq \Phi\left( \frac{1}{\mu(F)} \int_F h d\mu \right) = \Phi\left( \frac{1}{\mu(F)} \int_F h d\mu \right) \leq \frac{1}{\mu(F)} \int_F \Phi(h) d\mu.
\]
Thus,
\[
\int_F \Phi(h) d\mu \leq \int_F \Phi(h) d\mu + \delta \mu(F).
\]
Hence, appealing to the proof of (4) and to the sentence containing (5), we claim (10). Consequently,
\[
\int_{[f_k > a]} \Phi(h) d\mu \leq \int_{[f_k > a]} \Phi(f_k) d\mu,
\]
where \( a \geq 0 \) and \( k = 1, 2, \ldots, \)
and
\[
\int_{[f_k \leq a]} \Phi(h) d\mu \geq \int_{[f_k \leq a]} \Phi(f_k) d\mu,
\]
where \( a \leq 0 \) and \( k = 1, 2, \ldots, \)
Moreover, \( a\mu([|f_k| > a]) \leq |\lambda|([|f_k| > a]) \leq |\lambda|(|\Omega|), \) where \( a \geq 0; \) thus,
\[
\limsup_{n \to \infty} \int_{[|f_k| > a]} \Phi(f_k) d\mu = 0.
\]
So we can truncate the functions and still approximate them uniformly as follows. Whenever \( n \) is a positive integer and \( u \) is a (real valued) function defined on \( \Omega, \) let \( u^n(x) = u(x), \) where \( |u(x)| \leq n, \) and \( u^n(x) = nu(x)/|u(x)| \) otherwise. Then, using (1) and setting \( M = \max\{ (K/2), (K^2/4) \}, \)
\[
\int_\Omega \Phi(f - f_k) d\mu = \int_\Omega \Phi(f - f^n) + (f^n - (f_k)^n) + ((f_k)^n - f_k) d\mu \leq M(A_n + B_n + C_n),
\]
where

\[ A_n = \int_{\{|f| > n\}} \Phi(f) \, d\mu \, , \]
\[ B_n = \int_{\Omega} \Phi(f_n - (f_k)_n) \, d\mu \, , \]

and

\[ C_n = \int_{\{|f_k| > n\}} \Phi(f_k) \, d\mu \, . \]

From (4), (5) and (13), we obtain \( A_n \to 0 \) and \( C_n \to 0 \). Moreover, for each \( \delta > 0 \),

\[ B_n \leq \Phi(2n)\mu(\{|f^n - (f_k)^n| > \delta\}) + \Phi(\delta)\mu(\Omega) \leq \Phi(2n)\mu(\{|f - f_k| > \delta\}) + \Phi(\delta)\mu(\Omega) \, . \]

But, Brunk and Johansen have shown that \( \lim_{n} \mu(\{|f - f_k| > \delta\}) = 0 \), where \( \delta > 0 \), so Theorem 2 is established.

Because of the approximation properties which are verified in [4], the results of this paper extend immediately to analogous results for the derivatives of additive set functions defined on algebras of subsets of \( \Omega \) given a sub lattice (cf. [3]). Results for vector valued functions with respect to lattices which are related to the results: [7], [8], [9], of J. J. Uhl, Jr. for vector valued functions with respect to algebras should appear subsequently.

REFERENCES


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