NORM CONVERGENCE OF MARTINGALES OF RADON-NIKODYM DERIVATIVES GIVEN A \( \sigma \)-LATTICE

Richard Brian Darst and Gene Allen DeBoth
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Suppose that $\{\mathcal{M}_k\}$ is an increasing sequence of sub $\sigma$-lattices of a $\sigma$-algebra $\mathcal{A}$ of subsets of a non-empty set $\Omega$. Let $\mathcal{M}$ be the sub $\sigma$-algebra generated by $\bigcup_k \mathcal{M}_k$. Suppose that $L^\Phi$ is an associated Orlicz space of $\mathcal{A}$-measurable functions, where $\Phi$ satisfies the $\Delta_2$-condition, and let $h \in L^\Phi$. It is verified that the Radon-Nikodym derivative, $f_k$, of $h$ given $\mathcal{M}_k$ is in $L^\Phi$ and shown that the sequence $\{f_k\}$ converges to $f$ in $L^\Phi$, where $f$ is the Radon-Nikodym derivative of $h$ given $\mathcal{M}$.

1. Introduction. H. D. Brunk defined conditional expectation given a $\sigma$-lattice and established several of its properties in [1]. Subsequently S. Johansen [5] described a Radon-Nikodym derivative given a $\sigma$-lattice and showed that the Radon-Nikodym derivative was the conditional expectation in the appropriate case. Then H. D. Brunk and S. Johansen [2] proved an almost everywhere martingale convergence theorem for the Radon-Nikodym derivatives given an increasing sequence of $\sigma$-lattices. We shall establish norm convergence of these derivatives in $L_1$ and in the Orlicz spaces $L^\Phi$, where $\Phi$ satisfies the $\Delta_2$-condition. The theory of these Orlicz spaces can be found in [6], so we shall assume and build upon the results therein. Thereby, we can place fewer restrictions on $\Phi$ and obtain $L_1$-convergence as a byproduct.

2. Notation. Let $\mathcal{A}$ be a $\sigma$-algebra of subsets of a (non-empty) set $\Omega$, and let $\mu$ be a non-negative (bounded) $\sigma$-additive function defined on $\mathcal{A}$.

For our purposes the following information about $\Phi$ will suffice: $\Phi$ is an even, convex function defined on the real numbers, $R$, with $\Phi(0) = 0$ and $\Phi(x) \neq 0$ for some $x$. Moreover, there exists $K > 0$ with $\Phi(2x) \leq K\Phi(x)$ for all $x \in R$. This latter property is called the $\Delta_2$-condition; it implies

$$\Phi(x + y) = \Phi\left(2\left(\frac{x + y}{2}\right)\right) \leq K\Phi\left(\frac{x + y}{2}\right) \leq \left(\frac{K}{2}\right) [\Phi(x) + \Phi(y)].$$

Then $L^\Phi$ denotes the collection of (real valued) $\mathcal{A}$-measurable functions $h$ defined on $\Omega$ with $\int_\Omega \Phi(h)d\mu < \infty$. Since $\Phi$ is convex and not
identically zero, \( L^p \subseteq L_i \); \( L^p \) is usually a proper subset of \( L \), if \( \lim_{x \to 0} \Phi(x)/x = \infty \). This latter property and \( \lim_{x \to 0} \Phi(x)/x = 0 \) are required of an Orlicz space; but, these two properties are not necessary for our estimates to be valid. Examples are \( \Phi(x) = |x|^p, 1 \leq p < \infty \).

Let \( h \in L^p \) and \( \lambda(E) = \int_E h \, d\mu \), where \( E \subseteq \mathcal{R} \). Let \( \mathcal{R} \) be a sub \( \sigma \)-lattice of \( \mathcal{R} \) and let \( f \) be the Radon-Nikodym derivative of \( \lambda \) with respect to \( \mu \). Thus, \( f \) is the \( \mathcal{R} \)-measurable function defined on \( \Omega \) (\( \emptyset \); the empty set, \( \varnothing \), and \( \{ f > a \} \) belong to \( \mathcal{R} \), for all \( a \in R \)) satisfying

\[
(2) \quad \lambda(A \cap [f \leq b]) \leq b \mu(A \cap [f \leq b]), \quad \text{where } A \in M \text{ and } b \in R,
\]

and

\[
(3) \quad \lambda([f > a] \cap B^c) \geq a \mu([f > a] \cap B^c),
\]

where \( B^c = \varnothing - B, B \in \mathcal{R} \), and \( a \in R \).

Our first result is a preliminary step to an \( L^p \) martingale convergence theorem.

3. The derivative of an \( L^p \)-function is an \( L^p \)-function. We shall verify this assertion by establishing a sequence of estimates, the first of which is

\[
(4) \quad \int_{[f > a]} \Phi(f) \, d\mu \leq \int_{[f > a]} \Phi(h) \, d\mu, \quad \text{for all } a \geq 0.
\]

To verify (4), choose \( \delta > 0 \) and \( a = a_0 < a_1 < a_2 < \cdots \) with \( \Phi(a_k) = \Phi(a_{k-1}) + \delta \). Let \( A_k = [a_k \leq f > a_{k-1}] \) and notice that (3) implies

\[
|\lambda| (\Omega) = \lambda([f > a_k]) \geq a_k \mu([f > a_k]).
\]

Thus, \( \mu([f > a_k]) \to 0 \) and

\[
\int_{[f > a]} \Phi(\cdot) \, d\mu = \sum_{k=1}^{\infty} \int_{A_k} \Phi(\cdot) \, d\mu + \int_{[f > a_k]} \Phi(\cdot) \, d\mu = \sum_{k=1}^{\infty} \int_{A_k} \Phi(\cdot) \, d\mu.
\]

Applying (3) again,

\[
\int_{A_k} h \, d\mu = \lambda(A_k) \geq a_{k-1} \mu(A_k), \quad \text{so}
\]

\[
a_{k-1} \leq \frac{1}{\alpha_k} \int_{A_k} h \, d\mu, \quad \text{where } \alpha_k = \mu(A_k) > 0.
\]

Then, applying Jensen’s inequality,

\[
\Phi(a_{k-1}) \leq \Phi\left( \frac{1}{\alpha_k} \int_{A_k} h \, d\mu \right) \leq \frac{1}{\alpha_k} \int_{A_k} \Phi(h) \, d\mu.
\]

Next, notice that
\[ \int_{A_k} \Phi(f) d\mu \leq \Phi(a_k) \mu(A_k) = (\Phi(a_{k-1}) + \delta) \mu(A_k) \leq \int_{A_k} \Phi(h) d\mu + \delta \mu(A_k). \]

Thus \[ \int_{[f > a]} \Phi(f) d\mu \leq \int_{[f > a]} \Phi(h) d\mu + \delta \mu(\Omega), \] for all \( \delta > 0 \), which implies (4).

By a similar argument, one obtains

(5) \[ \int_{[f \leq a]} \Phi(f) d\mu \leq \int_{[f \leq a]} \Phi(h) d\mu, \] for all \( a \leq 0 \).

Hence, splitting \( \Omega \) into two pieces, \([f > 0]\) and \([f \leq 0]\), and applying (4) and (5), yields

(6) \[ \int_{[a]} \Phi(f) d\mu \leq \int_{[a]} \Phi(h) d\mu; \]

thus verifying Theorem 1.

**Theorem 1.** The Radon-Nikodym derivative of an \( L^\phi \)-function is an \( L^\phi \)-function.

4. A Martingale convergence theorem. Suppose that \( \{\mathcal{M}_k\}_{k=1}^\infty \) is an increasing sequence of \( \sigma \)-lattices of subsets of \( \Omega \), and \( \mathcal{M} \) is the \( \sigma \)-lattice generated by the lattice \( \mathcal{M}_\infty = \bigcup_k \mathcal{M}_k \). Denote by \( \mathcal{A}_k \) the \( \sigma \)-algebra that is generated by \( \mathcal{M}_k \) and by \( \lambda_k \) and \( \mu_k \) the restrictions of \( \lambda \) and \( \mu \) to \( \mathcal{A}_k \). Let \( h_k \) be an \( \mathcal{A}_k \)-measurable function satisfying \( \lambda(E) = \int_E h_k d\mu \), where \( E \in \mathcal{A}_k \), and denote by \( f_k \) the Radon-Nikodym derivative of \( \lambda_k \) with respect to \( \mu_k \) on \( \mathcal{M}_k \).

**Theorem 2.** The sequence \( \{f_k\} \) converges to \( f \) in \( L^\phi \)-norm:

(7) \[ \lim_{k \to \infty} \int_{\Omega} \Phi(f - f_k) d\mu = 0. \]

**Proof.** To begin, notice that applying (4) and (5) to \( f_k \) yields

(8) \[ \int_{[f_k > a]} \Phi(h_k) d\mu \geq \int_{[f_k > a]} \Phi(f_k) d\mu, \] for all \( a \geq 0 \),

and

(9) \[ \int_{[f_k \leq a]} \Phi(h_k) d\mu \geq \int_{[f_k \leq a]} \Phi(f_k) d\mu, \] for all \( a \leq 0 \).

Since \( \lambda_k \) is the restriction of \( \lambda \) to \( \mathcal{A}_k \), a variation on the theme which established (4) verifies

(10) \[ \int_E \Phi(h) d\mu \geq \int_E \Phi(h_k) d\mu, \] for all \( E \in \mathcal{A}_k \).
To substantiate this latter assertion, suppose $a \geq 0$, $\delta > 0$, $b > a$, $\Phi(b) = \Phi(a) + \delta$, $E \in \mathcal{A}$, $F = E \cap \{b \geq h > a\}$, and $\mu(F) > 0$. Then
\[ \int_F h \, d\mu = \int_F h \, d\mu, \text{ since } F \in \mathcal{A}. \]
Moreover,
\[ \int_F \Phi(h_k) \, d\mu \leq \Phi(b) \mu(F) = [\Phi(a) + \delta] \mu(F), \]
and
\[ \Phi(a) \leq \Phi\left(\frac{1}{\mu(F)} \int_F h \, d\mu\right) = \Phi\left(\frac{1}{\mu(F)} \int_F h \, d\mu\right) \leq \frac{1}{\mu(F)} \int_F \Phi(h) \, d\mu. \]
Thus,
\[ \int_F \Phi(h_k) \, d\mu \leq \int_F \Phi(h) \, d\mu + \delta \mu(F). \]
Hence, appealing to the proof of (4) and to the sentence containing (5), we claim (10). Consequently,
\[ \int_{[f_k > a]} \Phi(h_k) \, d\mu \geq \int_{[f_k > a]} \Phi(f_k) \, d\mu, \]
where $a \geq 0$ and $k = 1, 2, \ldots$, and
\[ \int_{[f_k \leq a]} \Phi(h_k) \, d\mu \geq \int_{[f_k \leq a]} \Phi(f_k) \, d\mu, \]
where $a \leq 0$ and $k = 1, 2, \ldots$.
Moreover, $a \mu([|f_k| > a]) \leq |\lambda| ([|f_k| > a]) \leq |\lambda| (\Omega)$, where $a \geq 0$; thus,
\[ \limsup_{n \to \infty} \int_{[|f_k| > n]} \Phi(f_k) \, d\mu = 0. \]
So we can truncate the functions and still approximate them uniformly as follows. Whenever $n$ is a positive integer and $u$ is a (real valued) function defined on $\Omega$, let $w^*(x) = u(x)$, where $|u(x)| \leq n$, and $w^*(x) = nu(x)|/u(x)|$ otherwise. Then, using (1) and setting $M = \max\{(K/2), (K^2/4)\}$,
\[ \int_D \Phi(f - f_k) \, d\mu = \int_D \Phi((f - f^*) + \{f^* - (f_k)^*\} + \{(f_k)^* - f_k\}) \, d\mu \leq M(A_n + B_n + C_n), \]
where

\[ A_n = \int_{|f| > n} \Phi(f) d\mu, \]
\[ B_n = \int \Phi(f_n - (f_k)^n) d\mu, \]
\[ C_n = \int_{|f| > n} \Phi(f_k^s) d\mu. \]

From (4), (5) and (13), we obtain \( A_n \to 0 \) and \( C_n \to 0 \). Moreover, for each \( \delta > 0 \),

\[ B_n \leq \Phi(2n)\mu(\{|f^n - (f_k)^n| > \delta\}) + \Phi(\delta)\mu(\Omega) \]
\[ \leq \Phi(2n)\mu(\{|f - f_k| > \delta\}) + \Phi(\delta)\mu(\Omega). \]

But, Brunk and Johansen have shown that \( \lim_k \mu(\{|f - f_k| > \delta\}) = 0 \), where \( \delta > 0 \), so Theorem 2 is established.

Because of the approximation properties which are verified in [4], the results of this paper extend immediately to analogous results for the derivatives of additive set functions defined on algebras of subsets of \( \Omega \) given a sub lattice (cf. [3]). Results for vector valued functions with respect to lattices which are related to the results: [7], [8], [9], of J. J. Uhl, Jr. for vector valued functions with respect to algebras should appear subsequently.

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