

Pacific Journal of Mathematics

TWO BRIDGE KNOTS ARE ALTERNATING KNOTS

RICHARD GOODRICK

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H. Schubert introduced a numerical knot invariant called the bridge number of a knot. In particular, he classified the two-bridge knots and proved that they were prime knots. Later, Murasugi showed that if K is an alternating knot then the matrix of K is alternating. The latter is true of two-bridge knots. The purpose of the following is to give a somewhat unusual geometric presentation of two-bridge knots from which it will be seen that they are alternating knots.

By a knot we will mean a polygonal simple closed curve in E^3 . Let C denote the unit circle in the xy plane and f a homeomorphism from C to a knot K . We will assume that K is in a regular position with respect to a projection into the $y = 0$ plane [1] and that those points of K which do not have unique images will be the crossing points of K . Let $f^{-1}(a_1), f^{-1}(a_2), \dots, f^{-1}(a_{2n})$ be the points of C ordered clockwise where a_i are the crossing points of K . If K has a presentation with an associated f such that a_i is an overcrossing point if and only if i is odd, then K is said to be an alternating knot. By a two-bridge knot we mean a nontrivial knot in E^3 which can be represented by two linear segments through a convex cell and two arcs on the boundary of the cell.

THEOREM 1. *If K is a two-bridge knot, then K is an alternating knot.*

Proof. We will start with K in a two-bridge representation (Fig. 1a) and apply several space homeomorphisms to E^3 , so that the resulting representation of K is described by an arc 'monotonely' approaching the center of the cube and four linear segments (Fig. 1b). The proof

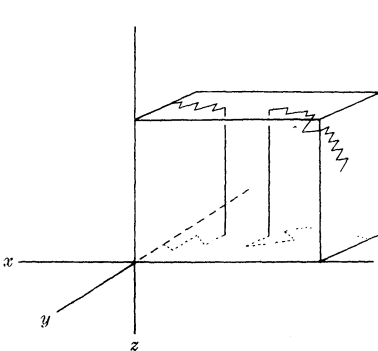


Figure 1a.

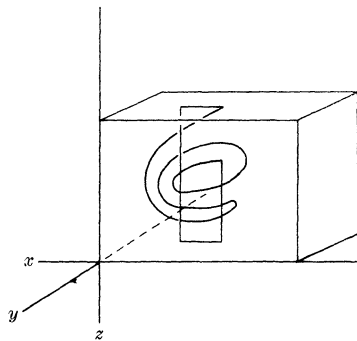


Figure 1b

will be completed by proving a lemma that shows that this representation is an alternating representation.

First assume that the knot K is represented by two arcs $A_i = \{(x, y, z) \mid x = i/3, y = 1/2, 0 \leq z \leq 1\}$, $i = 1, 2$, through the cube $I = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$ and two connecting arcs on the boundary of I , i.e. B_1 and B_2 . Furthermore, we can assume that $B_1 \cup B_2$ does not intersect the planes $y = 0$ and $y = 1$ (Fig. 2).

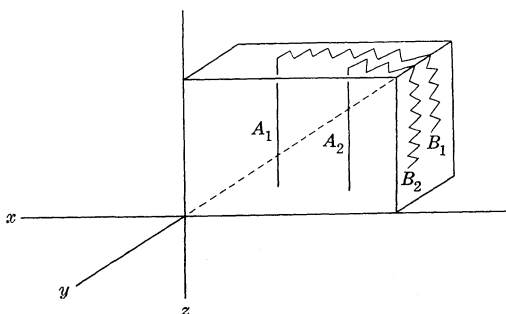


Figure 2.

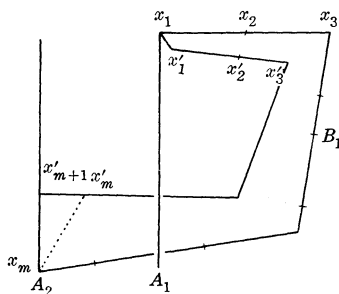


Figure 3.

The first homeomorphism h_1 will move the arc B_1 to an arc starting at the boundary and monotonely approaching the center of I so that it will not cross itself (in the y direction). h_1 will be constructed by the following five steps:

(1) Move B_1 on the boundary of I , leaving the A_i fixed, so that no segment of B_1 lies on the simple closed curve defined by $(\text{boundary of } I) \cap (\text{the plane } y = 1/2)$.

(2) Define L to be the cone from the center of I to B_1 and define O_t to be the annulus $\{(x, y, z) \mid \max(x - 1/2, z - 1/2) = 1/2 - t, 0 \leq y \leq 1\}$, $0 \leq t \leq 1/2$.

(3) From (1) we have $L \cap (A_1 \cup A_2)$ equal to a finite set of points. Hence define ϵ so that the interior of $\bigcup_0^\epsilon O_t \cap L$ contains no point of $A_1 \cup A_2$.

(4) Let x_1, \dots, x_m be the vertices of B_1 ordered from A_1 to A_2 . If $1 \leq k \leq m$, let x'_k be the point common to $O_{k\epsilon/m+1}$ and the linear segment joining x_k to the center of I and let $x'_{m+1} = O_\epsilon \cap A_2$.

(5) $L \cap \bigcup_0^\epsilon O_t$ is a disk whose intersection with K is B_1 . Hence the vertices $x'_1, x'_2, \dots, x'_m, x'_m, \dots, x_1$ determine a simple closed curve which bounds a disk in $\bigcup_0^\epsilon O_t$ whose intersection with K is B_1 . Move B_1 to $x_1, x'_1, \dots, x'_m, x_m$ without moving $A_1 \cup A_2 \cup B_2$. Then move $x'_{m+1}x_mx'_m$ to the segment $x'_{m+1}x'_m$ without moving the rest of K (Fig. 3).

The points of $h_1(B_1)$ approach the center of I in the sense that if x'_i, x'_j are vertices of $h_1(B_1)$ such that $i < j$ and $x'_i \in O_{t_i}, x'_j \in O_{t_j}$, then $t_i < t_j$. Hence if $h_1(K)$ is projected in the y direction, $h_1(B_1)$ will not cross itself.

As $h_1(K) \cap (\text{boundary of } I) = B_2 \cup |x_1|$, we can find a homeomorphism h_2 such that h_2 is fixed on $A_1 \cup \{A_2 - |x'_{m+1}, x_m|\} \cup h_1(B_1)$ and h_2 takes B_2 to an arc on the simple closed curve formed by $(\text{boundary of } I) \cap (\text{plane } y = 1/2)$.

Next, we will define a homeomorphism h_3 which will move $h_1(B_1)$ so that the crossings of $h_3(h_1(B_1))$ will alternate with respect to a projection in the $y = 0$ plane and $h_3(h_1(B_1))$ will still approach the center of I monotonely. Let b_1, b_2, \dots, b_r , be the crossing points of $h_1(B_1)$ ordered from A_1 and let $E_1 = A_1 \cap \{(x, y, z) | z \geq 1/2\}$, $E_2 = A_1 \cap \{(x, y, z) | z \leq 1/2\}$, and $E_3 = A_2 - [x_m, x_{m+1}]$. A two valued function g may be defined on $\{b_i\}$ so that $g(b_i) = 0$ if b_i is an over-crossing and $g(b_i) = u$ if b_i is an undercrossing (in the y -direction). Assume that two successive values of g are equal and then there are essentially two cases; i.e., case a , b_i and b_{i+1} both lie above (or below) E_1, E_2 , or E_3 , and case b , b_i lies above (or below) E_i and b_{i+1} lies above (or below) E_k with $l \neq k$.

If case a holds, then there exists t' and t'' such that $\bigcup_{t' \leq t \leq t''} O_t$ contains only b_i and b_{i+1} as crossings of $h_2 h_1(K)$. There is an arc α , such that (1) $\alpha \subset \bigcup_{t' \leq t \leq t''} O_t$ (2) α has endpoints $h_1(B_1) \cap O_{t'}$ and $h_1(B_1) \cap O_{t''}$, (3) α does not cross E_1, E_2 or E_3 and (4) α monotonely approaches the center of I . Let f_i be a space homeomorphism moving $h_1(B_1) \cap \bigcup_{t' \leq t \leq t''} O_t$ to α and leaving $E_1 \cup E_2 \cup E_3$ and $E^3 - [\bigcup_{t \leq t \leq t''} O_t]$ fixed (Fig. 4).

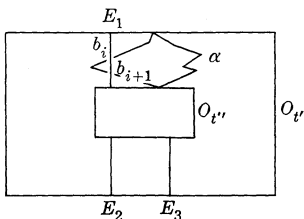


Figure 4.

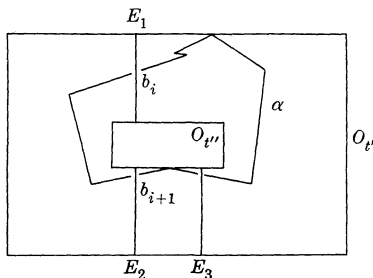


Figure 5.

If case b holds, define t', t'' , and α as above, except α will cross the third E segment once in the same way that $h_1(B_1)$ crosses the other two. Define f_1 as a space homeomorphism taking $h_1(B_1) \cap \bigcup_{t' \leq t \leq t''} O_t$ to α and leaving $E_1 \cup E_2 \cup E_3$ and $E^3 - [\bigcup_{t' \leq t \leq t''} O_t]$ fixed (Fig. 4).

Hence if $h_2 h_1(B_1)$ is not alternating then there exists a sequence of $\{f_i\}$ such that $f_{i_1} f_{i_2} \dots f_{i_k} h_2 h_1(B_1)$ is alternating. Let $h_3 = f_{i_1} f_{i_2} \dots f_{i_k}$. Then $h_3 h_2 h_1(K)$ is alternating by the following lemma.

LEMMA 1. Let K be a knot in regular position with respect to

the $y = 0$ plane, and B a subarc of K such that (1) B does not cross itself, (2) every crossing of K has exactly one crossing point in B , and (3) the crossings of B alternate, then K is an alternating knot.

Proof. It can be assumed that $B = \{(x, y, z) | 0 \leq x \leq 1, y = 0, z = 0\}$ and B satisfies conditions (1) through (3). If K is not an alternating knot, then there are two successive crossings of K , b_1, b_2 , such that both b_1 and b_2 are overcrossings (or undercrossings). Let A be the arc joining b_1 and b_2 which has no crossings in its interior (Fig. 6). As the crossings of B alternate, A cannot lie in B .

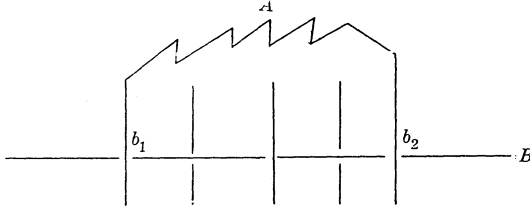


Figure 6.

A cannot contain both endpoints of B . If A contains neither endpoint of B , define C to be the simple closed curve containing A , the subarc B' of B with endpoints below (above) b_1 and b_2 , and the two vertical segments joining b_1 and b_2 to their respective undercrossing (overcrossing) points. If K contains a single endpoint of B , define C to be the simple closed curve containing A , the subarc B' of B containing one of b_1 or b_2 in its interior and having as endpoints the other b_i and the endpoint of B in A , and the vertical segment joining the b_i endpoint of B' to A .

As the crossings of B alternate and b_1 and b_2 are both overcrossing points, there is an odd number of crossings on B' between b_1 and b_2 , and hence an odd number of crossings on C . $C \cup K$ is the union of three simple closed curves, C , C_1 , and C_2 (C_2 is possibly degenerate). But $C_1 \cup C_2$ must cross C an even number of times, contradicting the fact that C is crossed an odd number of times.

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Received August 24, 1970.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 108 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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Wazir Husan Abdi, <i>A quasi-Kummer function</i>	521
Vasily Cateforis, <i>Minimal injective cogenerators for the class of modules of zero singular submodule</i>	527
W. Wistar (William) Comfort and Anthony Wood Hager, <i>Cardinality of k-complete Boolean algebras</i>	541
Richard Brian Darst and Gene Allen DeBoth, <i>Norm convergence of martingales of Radon-Nikodym derivatives given a σ-lattice</i>	547
M. Edelstein and Anthony Charles Thompson, <i>Some results on nearest points and support properties of convex sets in c_0</i>	553
Richard Goodrick, <i>Two bridge knots are alternating knots</i>	561
Jean-Pierre Gossez and Enrique José Lami Dozo, <i>Some geometric properties related to the fixed point theory for nonexpansive mappings</i>	565
Dang Xuan Hong, <i>Covering relations among lattice varieties</i>	575
Carl Groos Jockusch, Jr. and Robert Irving Soare, <i>Degrees of members of Π_1^0 classes</i>	605
Leroy Milton Kelly and R. Rottenberg, <i>Simple points in pseudoline arrangements</i>	617
Joe Eckley Kirk, Jr., <i>The uniformizing function for a class of Riemann surfaces</i>	623
Glenn Richard Luecke, <i>Operators satisfying condition (G_1) locally</i>	629
T. S. Motzkin, <i>On $L(S)$-tuples and l-pairs of matrices</i>	639
Charles Estep Murley, <i>The classification of certain classes of torsion free Abelian groups</i>	647
Louis D. Nel, <i>Lattices of lower semi-continuous functions and associated topological spaces</i>	667
David Emroy Penney, II, <i>Establishing isomorphism between tame prime knots in E^3</i>	675
Daniel Rider, <i>Functions which operate on $\mathbb{F}L_p(T)$, $1 < p < 2$</i>	681
Thomas Stephen Shores, <i>Injective modules over duo rings</i>	695
Stephen Simons, <i>A convergence theorem with boundary</i>	703
Stephen Simons, <i>Maximinimax, minimax, and antiminimax theorems and a result of R. C. James</i>	709
Stephen Simons, <i>On Ptak's combinatorial lemma</i>	719
Stuart A. Steinberg, <i>Finitely-valued f-modules</i>	723
Pui-kei Wong, <i>Integral inequalities of Wirtinger-type and fourth-order elliptic differential inequalities</i>	739
Yen-Yi Wu, <i>Completions of Boolean algebras with partially additive operators</i>	753
Phillip Lee Zenor, <i>On spaces with regular G_δ-diagonals</i>	759