Pacific Journal of Mathematics

OPERATORS SATISFYING CONDITION (G_1) LOCALLY

Glenn Richard Luecke

Vol. 40, No. 3 November 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 3, 1972

OPERATORS SATISFYING CONDITION (G_1) LOCALLY

GLENN R. LUECKE

The class of operators that satisfy condition (G_1) locally is studied. For operators in this class, conditions on the spectra which will insure normality are investigated.

An operator (continuous linear transformation from H into H) T on the complex Hilbert space H satisfies condition (G_1) if $\|(T-zI)^{-1}\| = 1/d(z, \sigma(T))$ for all $z \in \rho(T)$, where $\rho(T)$ is the resolvent set of T and $d(z, \sigma(T))$ is the distance from z to $\sigma(T)$, the spectrum of T. T satisfies (G_1) locally if T satisfies (G_1) in an open neighborhood of $\sigma(T)$, i.e. $\|(T - zI)^{-1}\| = 1/d(z, \sigma(T))$ for all $z \in U - \sigma(T)$ where U is some open set containing $\sigma(T)$. Let $\mathscr{G}$ and $\mathscr{G}_{loc}$ be all operators on H satisfying (G_1) and (G_1) locally, respectively. First it is shown how to construct nontrivial examples of operators in $\mathscr{G}$ and $\mathscr{G}_{loc}$. When $\dim H < \infty$, it is well-known that $\mathscr{G}_{loc} = \mathscr{G} = \mathscr{N}$, the set of all normal operators on H. However, when $\dim H = \infty$ then $\mathscr{N}$ is a proper subset of $\mathscr{G}$ and $\mathscr{G}$ is a proper subset of $\mathscr{G}_{loc}$. Next, for $T \in \mathscr{G}_{loc}$ having $\sigma(T)$ countable, conditions on $\sigma(T)$ are investigated to guarantee that T be normal.

1. **Properties of $\mathscr{G}$ and $\mathscr{G}_{loc}$.** First we show how to construct nontrivial operators in $\mathscr{G}$ and $\mathscr{G}_{loc}$. Let A be any operator on H. Then $A \oplus N \in \mathscr{G}$ on the Hilbert space $H \oplus K$ (the orthogonal direct sum of H and K), whenever N is a normal operator on K with $\sigma(N) \supseteqq W(A)$, the numerical range of A [see 8]. The following is an analogous way to construct operators in $\mathscr{G}_{loc}$.

THEOREM 1. *If A is an operator on H, then $A \oplus N \in \mathscr{G}_{loc}$ on $H \oplus K$ whenever N is a normal operator on K with $\sigma(N) \supseteqq U$, where U is an open set containing $\sigma(A)$.*

Proof. Let $T = A \oplus N$ where A and N are as above. Then $\sigma(T) = \sigma(A) \cup \sigma(N) = \sigma(N)$. Let $R(S, z) = (S - zI)^{-1}$ denote the resolvent of S at z. Then for $z \in \rho(T)$ [see 11],

$$\begin{aligned} \|R(T, z)\| &= \operatorname{Max}\{\|R(A, z)\|, \|R(N, z)\|\} \\ &= \operatorname{Max}\{\|R(A, z)\|, 1/d(z, \sigma(T))\}\,. \end{aligned}$$

The last equality holds since N is a normal operator and thus $\|R(N, z)\| = 1/d(z, \sigma(N)) = 1/d(z, \sigma(T))$. Since there is an open set U such that $\sigma(N) \supseteqq U \supseteqq \sigma(A)$, there exists an open set $V \supseteqq \sigma(N) = \sigma(T)$

such that for each $z \in V - \sigma(T)$, $\|R(A, z)\| \leqq 1/d(z, \sigma(T))$. Thus $\|R(T, z)\| = 1/d(z, \sigma(T))$ for all $z \in V - \sigma(T)$, and hence $T \in \mathscr{G}_{loc}$.

It is well-known [13, Th. 1] that $\mathscr{G}$ contains $\mathscr{N}$, the set of all normal operators on H. It is immediate that $\mathscr{G} \subseteqq \mathscr{G}_{loc}$. Putnam [10] has shown that for $T \in \mathscr{G}_{loc}$ the isolated points of $\sigma(T)$ are normal eigenvalues ($z \in \sigma(T)$ is a normal eigenvalue of T if z is an eigenvalue of T and $\{x \in H: Tx = zx\} = \{x \in H: T^*x = z^*x\}$ where z^* is the complex conjugate of z). Thus for $\dim H < \infty$, $\mathscr{G}_{loc} = \mathscr{N}$; and consequently $\mathscr{G}_{loc} = \mathscr{G} = \mathscr{N}$ [see 7].

THEOREM 2. *$\mathscr{G} \neq \mathscr{G}_{loc}$ when* $\dim H = \infty$.

Proof. Let M be a two dimensional subspace of H and let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ on M. Let N be normal operator on $M^{\perp}$ with $\sigma(N)$ equal to the closed disc of radius 1/2 about the origin (this requires that $\dim M^{\perp} = \infty$). From Theorem 1, $T = A \oplus N \in \mathscr{G}_{loc}$. However $T \notin \mathscr{G}$ since upon calculation one finds that $\|R(T, z)\| > 1/d(z, \sigma(T))$ when, for example, $z = 1$.

From [9] we know that for each $T \in \mathscr{G}$, $\operatorname{co} \sigma(T) = \operatorname{Cl} W(T)$, where $\operatorname{co} \sigma(T)$ denotes the convex hull of $\sigma(T)$ However, from the example in the proof of Theorem 2 we see that not all $T \in \mathscr{G}_{loc}$ satisfy $\operatorname{co} \sigma(T) = \operatorname{Cl} W(T)$.

Let $B(H)$ denote the set of all operators on H and give $B(H)$ the norm topology. When $\dim H < \infty$, then $\mathscr{G}_{loc} = \mathscr{G} = \mathscr{N}$ is a closed subset of $B(H)$. When $\dim H = \infty$, then $\mathscr{G}$ and $\mathscr{N}$ are closed subsets of $B(H)$ [8].

THEOREM 3. *$\mathscr{G}_{loc}$ is niether an open nor closed subset of $B(H)$ when* $\dim H = \infty$.

Proof. To see that $\mathscr{G}_{loc}$ is not open, it suffices to observe that (1) the zero operator is in $\mathscr{G}_{loc}$, (2) $T \in \mathscr{G}_{loc}$ and α a complex number implies $\alpha T \in \mathscr{G}_{loc}$, and (3) $\mathscr{G}_{loc} \neq B(H)$.

Let H, M, and A be as in the proof of Theorem 2. Let N_n be a normal operator on $M^{\perp}$ whose spectrum is the closed disc of radius $1/n$ about the origin. Let $T_n = A \oplus N_n$. By Theorem 1, $T_n \in \mathscr{G}_{loc}$. Let Z be the zero operator on $M^{\perp}$. Then $T_n \rightarrow A \oplus Z$ in norm and since $A \oplus Z \notin \mathscr{G}_{loc}$, $\mathscr{G}_{loc}$ is not closed.

For a detailed discussion of the topological properties of $\mathscr{G}$ see

[8].

II. Operators in $\mathscr{G}_{loc}$ with countable spectra. In general an operator $T \in \mathscr{G}_{loc}$ with countable spectrum need not be normal. However, such a non-normal operator can always be decomposed as the orthogonal direct sum of a normal operator and another operator:

THEOREM 4. *If $T \in \mathscr{G}_{loc}$ has countable spectrum, then either T is normal or $T = A \oplus N$ where N is a normal operator with $\sigma(N) = \sigma(T)$ and A is an operator with $\sigma(A)$ a subset of the derived set of $\sigma(T)$.*

Proof. If z is an isolated point of $\sigma(T)$, then by [10] z is a normal eigenvalue of T; let $E(z)$ be the eigenspace of z. Let $\sigma_0(T)$ denote the isolated points of $\sigma(T)$ and let

$$M = \text{closed span} \bigcup_{z \in \sigma_0(T)} E(z) \,.$$

Since each $E(z)$, $z \in \sigma_0(T)$, reduces T, T is normal on $E(z)$; and consequently M reduces T and T is normal on M. Since $\sigma(T)$ must have at least one isolated point, $M \neq (0)$. If $M = H$, then T is normal.

If $M \neq H$, then write $H = K \oplus M$ and $T = A \oplus N$ where A is T restricted to K and N is T restricted to M. Clearly $\sigma(N) = \sigma(T)$ and N is normal. Suppose to the contrary that $\sigma(A)$ is not a subset of the derived set of $\sigma(T)$. Then there exists $w \in \sigma(A)$ such that w is an isolated point of $\sigma(T)$. Therefore w is an isolated point of $\sigma(A)$, so there exists a circle C about w such that if $z \in C$, then $|z - w| = d(z, \sigma(T)) = d(z, \sigma(A))$. Then for $z \in C$

$$\begin{aligned} \|R(A, z)\| &\leqq \operatorname{Max}\{\|R(A, z)\|, \|R(N, z)\|\} = \|R(T, z)\| \\ &= 1/d(z, \sigma(T)) = 1/d(z, \sigma(A)) \,. \end{aligned}$$

Then since $\|(z - w)R(A, z)\| \leqq 1$ as $z \to w$, $(z - w)R(A, z)$ is a vector-valued analytic function of z at $z = w$. Therefore $(z - w)R(A, z)$ is analytic on an open disc containing C. Let

$$P = -\frac{1}{2\pi i}\int_C R(A, z)dz \,.$$

then

$$AP - wP = -\frac{1}{2\pi i}\int_C (z - w)R(A, z)dz = 0$$

so that $AP = wP$. Since $P \neq 0$ [12, p. 421], w is an eigenvalue of A

and hence of T. Since w is isolated point of $\sigma(T)$, w is a normal eigenvalue of T. Hence $K \cap M \neq (0)$. Contradiction.

With Theorem 4 we can easily classify all compact operators in $\mathscr{S}_{loc}$.

COROLLARY. *If $T \in \mathscr{S}_{loc}$ is compact, then either T is normal or $T = A \oplus N$ where N is compact and normal, and A is compact and quasi-nilpotent.*

Proof. The spectrum of a compact operator is countable with zero the only possible point of accumulation.

The existence of a non-normal $T \in \mathscr{S}_{loc}$ follows immediately from the following:

THEOREM 5. *If A is any operator, then there exists a normal operator N such that*

1. $A \oplus N \in \mathscr{S}_{loc}$
2. $\sigma(N) \supseteqq \sigma(A)$, *and*
3. $\sigma(N) - \sigma(A)$ *is a countable set whose points of accumulation are contained in* $\sigma(A)$.

Proof. Assume $\|A\| = 1$. We would like to find a normal operator N so that $\sigma(N)$ is the disjoint union of $\sigma(A)$ and some countable set $X \subseteqq \{z: |z| \leqq 2\}$ such that the following properties hold:

(i) the accumulation points of X are contained in $\sigma(A)$,

(ii) for $|z| \geqq 2$, $d(z, \sigma(N) \leqq d(z, W(A))$, and

(iii) for $|z| < 2$ and $z \in \rho(N)$, $\|R(A, z)\| \leqq 1/d(z, \sigma(N))$.

Property (i) guarantees that $\sigma(A) \cup X$ is a compact set so that there does exist a normal operator N with $\sigma(N) = \sigma(A) \cup X$. Let $T = A \oplus N$. Then for $|z| > 2$ property (ii) implies

$$\|R(A, z)\| \leqq 1/d(z, W(A)) = 1/d(z, \sigma(T)) .$$

Combining this with property (iii) we see that for every $z \in \rho(T)$, $\|R(A, z)\| \leqq 1/d(z, \sigma(T))$. Consequently $T = A \oplus N \in \mathscr{S} \subseteqq \mathscr{S}_{loc}$. Thus, it sufficies to construct such a set X.

Let

$$S_n = \{z: |z| \leqq 2 \text{ and } 3/(n + 1) \leqq d(z, \sigma(A)) \leqq 3/n\}$$

for $n = 1, 2, 3, \cdots$. Since $\|R(A, z)\|$ is bounded on each compact set

S_n, there exists a finite set of points $X_n \subseteqq S_n$ such that $d(z, X_n) \leqq \|R(A, z)\|^{-1}$ for all $z \in S_n$. Let

$$X = \bigcup_{n=1}^{\infty} X_n \,.$$

Since $\|R(A, z)\| \geqq 1/d(z, \sigma(A))$ [see 4, p. 566], X has all of its accumulation points in $\sigma(A)$, and hence property (i) is satisfied. To see that property (iii) is satisfied, let $z \in S_n \cap \rho(N)$. Then

$$d(z, \sigma(N)) = d(z, X) \leqq d(z, X_n) \leqq \|R(A, z)\|^{-1} \,.$$

Thus $\|R(A, z)\| \leqq 1/d(z, \sigma(N))$. Since $W(A)$ is a subset of the closed unit disc, property (ii) can be satisfied, for example, by making sure taht X contains the points $2 \exp(n\pi i/4)$, for $n = 0, 1, \cdots, 7$.

One can further require in Theorem 5 that $T = A \oplus N \notin \mathscr{G}$. This can be done, in essentially the same manner as above, by choosing $\sigma(N) = \sigma(A) \cup X$ where X is as above only instead of satisfying properties (ii) and (iii) X satisfies the following: for $x \in \rho(N)$ $\|R(A, z)\| \leqq 1/d(z, \sigma(N)$ only for z contained in a sufficiently small neighborhood of $\sigma(A)$ instead of for all $z \in \{z \in \rho(N) \colon |z| < 2\}$. This can be done by choosing m sufficiently large and then letting

$$X = \bigcup_{n=m}^{\infty} X_n \,.$$

To show that there exists a non-normal $T \in \mathscr{G}_{loc}$ with countable spectrum, let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and choose a normal operator N as in Theorem 5.

Stampfli [17] has shown that if $T \in \mathscr{G}_{loc}$ has $\sigma(T)$ lying on a C^2-smooth rectifiable Jordon curve C, then T is normal. The following question now arises: If $T \in \mathscr{G}_{loc}$ has countable spectrum, then can we weaken the assumption that $\sigma(T) \subseteqq C$ and still conclude that T must be normal? The answer is not fully known, but the following material gives a partial answer.

If S is a countable compact subset of the complex plane, then S satisfies *condition* (A) if for each $p \in S$ there exists $q \notin S$ such that $|q - p| = d(q, S)$.

To show that S satisfying condition (A) is weaker than $S \subseteqq C$, let S be the following countable, compact set of complex numbers:

$$S = \{0\} \cup \{1/n + i(\sin n)/n \colon n = 1, 2, 3, \cdots\} \,.$$

Then S does not lie on a C^2-smooth rectifiable Jordon arc, but S does satisfy condition (A).

THEOREM 6. *If T is a scalar operator in $\mathscr{S}_{loc}$ whose spectrum is countable and satisfies condition (A), then T is normal.*

Proof. Let $u \in \sigma(T)$, then there exists a sequence $\{u_n\} \subseteqq \rho(T)$ such that $u_n \to u$ and $|u_n - u| = d(u_n, \sigma(T))$. Since T is scalar

$$T = \int_{\sigma(T)} z dE_z \ .$$

Therefore

$$(u - u_n)R(T, u_n) = \int_{\sigma(T)} \frac{u - u_n}{z - u_n} dE_z \ .$$

Let $x, y \in H$ be fixed and define m to be the complex Borel measure $m(S) = (E(S)x, y)$ for each Borel set S in $\sigma(T)$. For each $z \in \sigma(T)$ let

$$f_n(z) = \frac{u - u_n}{z - u_n} \text{ and } f(z) = \begin{cases} 1 \text{ if } z = u \\ 0 \text{ if } z \neq u \end{cases} .$$

Then $|f_n(z)| \leqq 1$ and $f_n(z) \to f(z)$. Therefore we may apply the Lebesgue dominated convergence theorem:

$$\begin{aligned} |m(\{u\})| &= \left| \int_{\sigma(T)} f(z) \,\mathrm{dm}\,(z) \right| \\ &= \lim_{n\to\infty} \left| \int_{\sigma(T)} f_n(z) \,\mathrm{dm}\,(z) \right| \\ &= \lim_{n\to\infty} |((u - u_n)R(T, u_n)x, y)| \\ &\leqq |u - u_n| \, ||R(T, u_n)|| \, ||x|| \, ||y|| = ||x|| \, ||y|| \ . \end{aligned}$$

Since $m(\{u\}) = (E(\{u\})x, y)$, we have that

$$|(E(\{u\})x, y)| \leqq ||x|| \, ||y|| \ .$$

Letting $y = E(\{u\})x$, we obtain $||E(\{u\})x|| \leqq ||x||$, and hence $||E(\{u\})|| \leqq 1$. Therefore $E(\{u\})$ is an orthogonal projection for each $u \in \sigma(T)$.

Let $S \subseteqq \sigma(T)$ be a Borel set, then S is a countable set so write $S = \{z_1, z_2, z_3, \cdots\}$. Then for each $x, y \in H$, we have

$$\begin{aligned} (E(S)x, y) &= \sum_{n=1}^{\infty} (E(\{z_n\})x, y) = \sum_{n=1}^{\infty} (x, E(\{z_n\})y) \\ &= \operatorname{conj} \sum_{n=1}^{\infty} (E(\{z_n\})y, x) = \operatorname{conj}\,(E(S)y, x) = (x, E(S)y) \ . \end{aligned}$$

Therefore $E(S) = E(S)^*$, the adjoint of $E(S)$, and hence $E(S)$ is an orthogonal projection. Consequently, T is a scalar operator with a resolution of the identity of orthogonal projections; and thus T is normal.

In light of Theorem 6 it seems reasonable to conjecture the following theorem: If $T \in \mathscr{G}_{loc}$ has countable spectrum satisfying condition (A), then T is normal. The following thorem shows that this conjecture is false.

THEOREM 7. *There exists* $T \in \mathscr{G}_{loc}$ *with* $\sigma(T)$ *satisfying condition* A *such that*

(i) $\sigma(T)$ *is countable with zero the only point of accumulation,*

(ii) *if* $z \in \sigma(T)$, *then* $|z - 2| \leqq 2$, *and*

(iii) T *is not normal.*

Proof. Let D_n be the closed disc of radius n about n, for $n = 1, 2$. Let V be the Volterra integration operator. Let $B = (I + V)^{-1}$, and let $A = I - B$. By [6, problem 150], $\sigma(B) = \{1\}$ and $\|B\| = 1$. Hence $\sigma(A) = \{0\}$ and $W(B)$ is contained in the closed disc about the origin of radius $\|B\| = 1$. Therefore $W(A) \subseteqq D_1$. We now proceed to fill up D_2 with enough points, X so that if N is a normal operator with $\sigma(N) = X \cup \{0\}$, then $A \oplus N \in \mathscr{G}_{loc}$ and $\sigma(A \oplus N)$ is a countable set with zero the only point of accumulation. The procedure is similar to that used in the proof of Theorem 5 only the details are a little more involved. For $n = 1, 2, \cdots$, let

1. $F_n = \{z \in D_2 \colon 4/(n+1) \leqq |z| \leqq 4/n\}$.
2. $M_n = \sup\{\|R(A, z)\| \colon z \in F_n\}$,
3. $d_n = \inf\{d(z, W(A)) \colon z \in (\partial D_2) \cap F_n\} > 0$
4. $P_n = \operatorname{Max}\{M_n, 1/d_n\}$, and
5. $B(z, r)$ be the open disc of radius r about z.

Then

$$F_n \subseteqq \bigcup_{z \in F_n} B(z, 1/P_n) .$$

Since F_n is compact, there exists $z_{n_i} \in F_n$, $1 \leqq i \leqq m_n$, such that

$$F_n \subseteqq \bigcup_{i=1}^{m_n} B(z_{u_i}, 1/P_n) .$$

Let N be a normal operator with $\sigma(N) = \{0\} \cup \{z_{n_i} \colon 1 \leqq i \leqq m_n, n = 1, 2, 3, \cdots\}$, then $\sigma(N)$ is a countable set with zero the only point of accumulation. Let $T = A \oplus N$, then $\sigma(T) = \sigma(N)$. We now verify that $T \in \mathscr{G}_{loc}$.

If $z \in D_2$, $z \neq 0$, then there exists n and i such that $z \in F_n \cap B(z_{n_i}, 1/P_n)$. Then

$$\begin{aligned} d(z, \sigma(N))\|R(A, z)\| &\leqq |z - z_{n_i}|\,\|R(A, z)\| \\ &\leqq (1/P_n)\|R(A, z)\| \\ &\leqq (1/M_n)\|R(A, z)\| \leqq 1 . \end{aligned}$$

If z is real and negative, then

$$d(z, \sigma(N))\|R(A, z)\| \leqq |z|/d(z, W(A)) = 1 .$$

Suppose $z \notin D_2$ and that z is not real and negative. Let x be the point of intersection of ∂D_2 with the shortest line segment connecting z and Cl $W(A)$. Observe that $x \neq 0$. Then $d(z, W(A)) = |z - x| + d(x, W(A))$. There exists n and i such that $x \in F_n \cap B(z_{n_i}, 1/P_n)$. Then $|x - z_{n_i}| \leqq 1/P_n$, and so

$$\begin{aligned} |z - z_{n_i}| &\leqq |z - x| + 1/P_n \leqq |z - x| + d_n \\ &\leqq |z - x| + d(x, W(A)) = d(z, W(A)) . \end{aligned}$$

Therefore,

$$\begin{aligned} d(z, \sigma(N))\|R(A, z)\| &\leqq |z - z_{n_i}| \, \|R(A, z)\| \\ &\leqq d(z, W(A))/d(z, W(A)) = 1 . \end{aligned}$$

Therefore, for each complex number $z \neq 0$, $d(z, \sigma(N))\|R(A, z)\| \leqq 1$. Since N is normal, for each $z \in \rho(T) = \rho(N)$,

$$\|R(N, z)\| = 1/d(z, \sigma(N)) = 1/d(z, \sigma(T)) .$$

Hence, for $z \in \rho(T)$

$$\|R(T, z)\| = \mathrm{Max}\,\{\|R(A, z)\|, \|R(N, z)\|\} = 1/d(z, \sigma(T))) .$$

Therefore $T \in \mathscr{G}_{loc}$.

References

1. S. K. Berberian, *Some conditions on an operator implying normality*, Math. Ann., **184** (1970). 188-192.

2. ———, *Some conditions on an operator implying normality II*, Proc. Amer. Math. Soc., (to appear).

3. W. F. Donoghue, *On the numerical range of a bounded operator*, Michigan Math. J., **4** (1957), 261-263.

4. N. Dunford and J. T. Schwartz, *Linear Operators Part I and II*, Interscience Publishers, New York, 1957.

5. N. Dunford, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc., **64** (1958), 217-274.

6. P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand Company, Princeton, New Jersey, 1967.

7. S. Hildebrandt, *Uber den Numerischen Wertebereich eines Operators*, Math. Annalen, **163** (1966), 230-247.

8. G. R. Luecke, *Topological properties of paranormal operators*, Trans. Amer. Math. Soc., (to appear).

9. G. Orland, *On a class of operators*, Proc. Amer. Math. Soc., **15** (1964), 75-79.

10. C. R. Putnam, *Eigenvalues and boundary spectra*, Illinois J. Math., **12** (1968), 278-282.

11. ———, *The spectra of operators having resolvents of first-order growth*, Trans.

Amer. Math. Soc., **133** (1968), 505-510.
12. F. Riesz and B. Sz-Nagy, *Functional Analysis*, F. Ungar Publishing Company, New York, 1965.
13. J. G. Stampfli, *Hyponormal operators*, Pacific J. Math., **12** (1962), 1453-1458.
14. J. F. Stampfli, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc., **117** (1966), 469-476.
15. J. G. Stampfli, *Analytic extensions and spectral localization*, J. Math. Mech., **19** (1966), 287-296.
16. ———, *Minimal range theorems for operators with thin spectra*, Pacific J. Math., **23** (1967), 601-612.
17. ———, *A local spectral theorey for operators*, J. Functional Anal., **4** (1969), 1-10.
18. ———, *A local spectral theory for operators II*, Bull. Amer. Math. Soc., **75** (1969), 803-806.

Received September 21, 1970.

IOWA STATE UNIVERSITY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index. to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 40, No. 3 November, 1972