FUNCTIONS WHICH OPERATE ON $\mathcal{F}L_p(T), 1 < p < 2$

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$\mathcal{F}L_p(T)$ is the algebra of Fourier transforms of functions in $L_p$ of the circle. It is shown that if $F$ is defined on the plane and the composition $F \circ \phi \in \mathcal{F}L_1$ whenever $\phi \in \mathcal{F}L_p$ then for all $\varepsilon > 0, F(z) = P(z, z) + O(|z|^{1/2-\varepsilon})$ where $P$ is a polynomial in $z$ and $\bar{z}$ and $p^{-1} + q^{-1} = 1$ ($1 < p < 2$).

1. Introduction. Throughout, $L_p = L_p(T)$ will denote the usual space of functions on $T$, the unit circle, normed by

$$||f||_p = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{it})|^p dt \right\}^{1/p}.$$  

For $f \in L_1$ the Fourier transform is given by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} f(e^{it}) dt \quad (n = 0, \pm 1, \pm 2, \ldots).$$

$\mathcal{F}L_q$ is the algebra of Fourier transforms of functions in $L_q$ for $q \geq 1$ and $\mathcal{F}C$ is the algebra of transforms of the continuous functions.

Let $F$ be a complex function defined on the plane. $F$ is said to operate from $\mathcal{F}L_q$ to $\mathcal{F}L_r$ provided the composition $F \circ \phi$ belongs to $\mathcal{F}L_r$ whenever $\phi \in \mathcal{F}L_p$.

We shall write $F(z) = O(G(z))$ to mean $F(z)/G(z)$ is bounded near the origin. It is an immediate consequence of Parseval's theorem that $F$ operates from $\mathcal{F}L_q$ to $\mathcal{F}L_q$ if and only if $F(z) = O(z)$. On the other hand it was shown by Helson and Kahane [2] that $F$ operates from $\mathcal{F}L_1$ to $\mathcal{F}L_1$ if and only if $F$ is real analytic in a neighbourhood of the origin and, of course, $F(0) = 0$ (cf. [6, chapter 6]).

For $2 < q \leq \infty$ it was shown by the author [3] that the functions operating from $\mathcal{F}L_q$ to $\mathcal{F}L_q$ and from $\mathcal{F}C$ to $\mathcal{F}L_q$ are the same and combine the types of behavior of the examples above. We state the result for completeness.

**Theorem 1.1.** Let $2 < q \leq \infty$ and $p^{-1} + q^{-1} = 1$. The following are equivalent.

(a) $F$ operates from $\mathcal{F}L_q$ to $\mathcal{F}L_q$.

(b) $F$ operates from $\mathcal{F}C$ to $\mathcal{F}L_q$.

(c) $F(z) = c_1 z + c_2 \bar{z} + O(|z|^{3/2}).$

Half of the Hausdorff-Young theorem [8, Theorem 2.3 ii] was used to show that (c) implies (a) in the above. In fact, it is not difficult to see that $F$ operates from $\mathcal{F}L_q$ to $\mathcal{F}L_q$ if and only if $F(z) =$
The other half of the Hausdorff-Young theorem [8, Theorem 2.3 i] shows that if $1 < p < 2$, $p^{-1} + q^{-1} = 1$ and $F(z) = O(|z|^{\beta})$, then $F$ operates from $\mathcal{F}L_p$ to $\mathcal{F}L_q$. It is also easy to see that this is a necessary condition. Since polynomials operate from $\mathcal{F}L_p$ to $\mathcal{F}L_q$, we then have

**Theorem 1.2.** Let $1 < p < 2$ and $p^{-1} + q^{-1} = 1$. If $F(z) = P(z, \overline{z}) + O(|z|^{\frac{q}{2}})$, where $P$ is a polynomial in $z$ and $\overline{z}$ ($P(0) = 0$), then $F$ operates from $\mathcal{F}L_p$ to $\mathcal{F}L_q$ and thus also from $\mathcal{F}L_p$ to $\mathcal{F}L_q$.

We can assume the polynomial $P$ has order less than $q/2$, for higher order terms can be absorbed into $O(|z|^{\frac{q}{2}})$.

The main result of this paper is the following partial converse to Theorem 1.2.

**Theorem 1.3.** Let $1 < p < 2$ and $p^{-1} + q^{-1} = 1$. If $F$ operates from $\mathcal{F}L_p$ to $\mathcal{F}L_q$, then, for all $\varepsilon > 0$,

\begin{equation}
F(z) = P(z, \overline{z}) + O(|z|^{\beta - \varepsilon})
\end{equation}

where $P$ is a polynomial in $z$ and $\overline{z}$.

I have not been able to remove the $\varepsilon$ in (1.4). In fact, I have not been able to show whether or not $z^{\frac{q}{2}} \log |z|$ operates from $\mathcal{F}L_p$ to $\mathcal{F}L_q$. However, as a corollary to Theorems 1.2 and 1.3 we can state the following complete result.

**Corollary 1.5.** Let $1 < p < 2$ and $p^{-1} + q^{-1} = 1$. The following are equivalent.

(a) $F$ operates from $\bigcup_{r>p} \mathcal{F}L_r$ to $\mathcal{F}L_q$.
(b) $F$ operates from $\bigcup_{r>p} \mathcal{F}L_r$ to $\bigcup_{r<p} \mathcal{F}L_r$.
(c) $F(z) = P(z, \overline{z}) + O(|z|^{\beta - \varepsilon})$ for all $\varepsilon > 0$.

The proof of Theorem 1.3 uses a factorization of the Rudin-Shapiro polynomials. The idea is to construct polynomials, $P$, with few coefficients so that small changes in $P$ cause large changes in the norms of $P$. This is done in §2.

In §3 these polynomials are used to show that if $F$ operates then, for all complex $w$, all integers $k$ and certain $\beta$,

\begin{equation}
\sum_{j=0}^{k} (-1)^j F((w + j)z) = O(|z|^{\beta})
\end{equation}

Now any polynomial in $z$ and $\overline{z}$ of degree less than $k$ satisfies (1.6). In §4 it is shown that, except for a $O(|z|^{\beta})$ term, these are the only functions which satisfy 1.6, at least if $\beta$ is not an integer and $F(z) = O(|z|^{\beta})$. 


FUNCTIONS WHICH OPERATE ON \( L_p(T), 1 < p < 2 \) O(1). This is then used to obtain a proof of Theorem 1.3.

2. The Rudin-Shapiro polynomials. The Rudin-Shapiro polynomials are defined as follows: let \( P_0(x) = Q_0(x) = 1 \) and define inductively

\[
P_{k+1}(x) = P_k(x) + x^k Q_k(x)
\]
\[
Q_{k+1}(x) = P_k(x) - x^k Q_k(x).
\]

Then

\[
(2.1) \quad P_k(x) = \sum_{\varepsilon(n)} \varepsilon(n)x^n
\]

where \( \varepsilon(n) = \pm 1 \) is independent of \( k \). As shown in [5] and [7],

\[
(2.2) \quad \left| \sum_{n=0}^{N} \varepsilon(n)e^{int} \right| < 5(N + 1)^{1/2} \quad (0 \leq t < 2\pi; N = 1, 2, \cdots).
\]

This definition differs slightly from that given in [5] and [7]. It has also been given by Brillhart and Carlitz [1].

We have the following explicit representation for \( \varepsilon(n) \) (cf. [1] and [4, Lemma 2]).

**Lemma 2.3.** If \( n \) has a binary expansion

\[
n = \delta_0 + 2\delta_1 + 2^2\delta_2 + \cdots + 2^k\delta_k \quad (\delta_i = 1 \text{ or } 0)
\]

then

\[
\varepsilon(n) = \Pi_{i=1}^{k} (1 - 2\delta_i \delta_{i-1}).
\]

In the following we will factor \( \varepsilon(n) \) in various ways as was done in [4]. Fix positive integers \( N \) and \( k \) and let \( 0 \leq n < 2^{Nk+1} \) so that \( n \) has a binary expansion

\[
n = \delta_0 + 2\delta_1 + \cdots + 2^k\delta_N.
\]

Define

\[
(2.4) \quad \rho_j(n) = \Pi_{(j-1)N+1}^{jN} (1 - 2\delta_i \delta_{i-1}) \quad (j = 1, 2, \cdots k).
\]

Note also that \( n \) can be written in a unique way as

\[
(2.5) \quad n = n_1 + n_22^{N+1} + n_32^{N(j-1)}
\]

where
and, by Lemma 2.3, \( \rho_j(n) = \varepsilon(n) \). It also follows from Lemma 2.3 that

\[
\varepsilon(n) = \prod_{j=1}^{k} \rho_j(n) .
\]

Define

\[
R_j(t) = \sum \rho_j(n) e^{int} \quad (j = 1, 2, \ldots, k) ,
\]

the sum being from 0 to \( 2^{Nk+1} - 1 \).

The usefulness of the \( R_j \) comes about because if \( S \) is the convolution product \( S = R_1 * R_2 * \cdots * R_k \), then by (2.6)

\[
S = \sum_{n=0}^{2^{Nk+1}-1} \varepsilon(n) e^{int} .
\]

Now, by (2.2), \( \| S \|_\infty \leq 5 \cdot 2^{Nk+1} \) and since \( \| S \|_2 = 2^{(Nk+1)/2} \) it follows that

\[
\frac{1}{5} 2^{(Nk+1)/2} \leq \| S \|_1 \leq \prod_{j=1}^{k} \| R_j \|_1 .
\]

Thus, very roughly, \( \| R_j \|_1 \) must be as large as \( 2^{N/2} \). The following shows that \( \| R_j \|_1 \) is not much larger than this.

**Proposition 2.8.**

\[
\| R_j \|_1 \leq 2^{N/2} N^2 k^2 C
\]

where \( C \) is an absolute constant.

**Proof.** \( R_j \) can be written

\[
R_j = F_1 F_2 F_3
\]

where

\[
F_1(t) = \sum_{0}^{2^{N(j-1)-1}} e^{int}
\]

\[
F_2(t) = \sum_{0}^{2^{N(k-j)-1}} \exp (in 2^{Nj+1}t)
\]

\[
F_3(t) = \sum_{0}^{2^{N+1}-1} \varepsilon(n) \exp (in 2^{N(j-1)}t) .
\]

To see that (2.9) holds, note that the product \( F_1 F_2 F_3 \) consists of \( 2^{Nk+1} \) distinct exponentials between 0 and \( 2^{Nk+1} - 1 \). Also the coefficient
of $e^{int}$ where $n$ is given as in (2.5) is $\varepsilon(n_3) = \rho_j(n)$ so that $F_jF_2F_3 = R_j$.

It is not difficult to see that $\|F_jF_2\|_1 \leq Ck^2N^2$ and since, by (2.2), $\|F_j\|_\infty \leq 52^{(N+1)/2}$ the proposition follows.

**Proposition 2.10.** For $1 < p \leq 2$ and $p^{-1} + q^{-1} = 1$

$$\| R_j \|_p \leq C2^{N(1/p + (k-1)/q)}N^2k^2.$$  

**Proof.** Since $\| R_j \|_2 = 2^{(Nk+1)/2}$ this follows from H"older's inequality and (2.8).

**Lemma 2.11.** For $N$ and $k$ positive integers there is a decomposition of \{0, 1, 2, \cdots, 2^{Nk+1} - 1\} into $k + 1$ sets $A_0, A_1, \cdots, A_k$ such that if

$$T_{N,k}(t) = \sum_{j=0}^{k} j \sum_{A_j} e^{int}$$

and

$$R_{N,k}(t) = \sum_{j=0}^{k} j^k \sum_{A_j} e^{int}$$

then

(a) $\| T_{N,k} \|_1 \leq C(k)N^22^{N/2}$

(b) $\| T_{N,k} \|_p \leq C(k)N^22^{(Nk+1)/2}N^2k^{k-1}$

(c) $\| S_{N,k} \|_1 \geq C(k)2^{Nk/2}$

(d) $\| R_{N,k} \|_1 \geq C(k)2^{Nk/2}$

(e) $\left\| \sum_{A_j} e^{int} \right\|_1 \geq C(k)2^{Nk/2}$

where the $C(k)$ are (different) positive constants depending only on $k$.

For $k = 2$ this has been done in [4].

**Proof.** Define

$$2T_{N,k}(t) = \sum_{i=1}^{k} R_j(t) + \sum_{0}^{2^{Nk+1}-1} e^{int}.$$  

Now

$$T_{N,k}(t) = \sum_{0}^{2^{Nk+1}-1} \phi(n)e^{int}$$

where

$$\phi(n) = \sum_{i=1}^{k} \frac{\rho_j(n) + 1}{2}.$$  

Since $\rho_j(n) = \pm 1$, $\phi(n)$ assumes only the values $0, 1, \cdots, k$ so that
if \( A_j \) consists of the \( n \) with \( \phi(n) = j \) then \( T_{N,k} \) is as in (2.12). (a) then follows from (2.8) and (b) from (2.10).

Now if \( \phi(n) = j \), then precisely \( k - j \) of the \( \rho_i(n) = -1 \), so that, by (2.6), \( \varepsilon(n) = (-1)^{k-j} \). Hence

\[
S_{N,k}(t) = (-1)^k \sum_{0}^{\phi(n) - 1} \varepsilon(n)e^{i\pi t}
\]

so that (e) follows from (2.7).

Define \( T_{N,k} = \sum_{1}^{N^{k+1-1}} e^{i\pi t} \), and inductively

\[
T_{N,k}^{j+1} = T_{N,k}^j \cdot T_{N,k}.
\]

Then \( \{T_{N,k}^j\} (s = 0, 1, \ldots, k) \) are \( k+1 \) linearly independent polynomials which span the space of polynomials of the form \( \sum_0^k c_j \sum_{A_j} e^{i\pi t} \). In particular,

\[
(2.13) \quad S_{N,k} = \sum_{s=0}^k b_s T_{N,k}^s
\]

where the \( b_s \) depend on \( k \) but not on \( N \).

Now it follows from (a) that

\[
|| T_{N,k}^s \||_1 \leq C(k)N^{2s}2^{N\pi/2} \quad (s = 1, 2, \ldots).
\]

Also

\[
|| T_{N,k}^s \||_1 \leq C(k)N
\]

so that

\[
(2.15) \quad || S_{N,k} \||_1 \leq \sum_{0}^{k} || T_{N,k}^s \||_1 \leq C(k)N^{2(k-1)}2^{N(k-1)/2} + || b_k || \cdot || T_{N,k}^k \||_1.
\]

(d) then follows from (2.15) and (c) since \( T_{N,k}^k = R_{N,k} \). (e) holds for the same reasons since, for each \( j \), \( \sum_{A_j} e^{i\pi t} \) and \( \{T_{N,k}^s\} (s = 0, \ldots, k-1) \) are linearly independent.

REMARK. Because \( T_{N,k}^k = R_{N,k} \) we must have \( || T_{N,k} \||_1 \geq C(k)2^{N/2} \). It would be useful to know if the \( N^2 \) in (a) can be removed. Also, by the Hausdorff-Young theorem, \( || T_{N,k} \||_p \geq C(k)2^{N/kq} \). If the right side of (b) could be replaced by \( C(k)2^{N/kq} \), then the \( \varepsilon \) in Theorem 1.3 could be removed.

3. The main lemma. The purpose of this section is to use the polynomials of Lemma 2.11 to prove the following.

**Lemma 3.1.** Let \( F \) operate from \( \mathcal{L}_p \) to \( \mathcal{L}_q \) \( (1 < p \leq 2; p^{-1} + q^{-1} = 1) \). Assume that \( F(z) = O(|z|^\beta) \) for some \( \beta > 0 \). Then for each positive integer \( k \) and each complex \( w \)
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(3.2) $\sum_{c} (-1)^{c} F((w + j)z) = O(|z|^{\beta'})$

where

$$\beta' = \min \left( \beta + \frac{q}{4(k + q)}, \frac{qk}{2(k + q)} \right).$$

Before proving this we need the following lemma. If $F$ operates from $\mathcal{L}_p$ to $\mathcal{L}_1$ then, for $f \in L_p$, $F \circ f$ will denote the function in $L_1$ such that $(F \circ f)^\ast(n) = F(f(n))$.

**Lemma 3.3.** Let $F$ operate from $\mathcal{L}_p$ to $\mathcal{L}_1$.

(a) There are constants $M$ and $\hat{\delta}$ such that $\|f\|_p < \hat{\delta}$ implies $\|F \circ f\|_1 < M$.

(b) $F(z) = O(z)$.

(c) $F(0) = 0$.

**Proof.** The proof of (a) is the same as that of Lemma 1 of [3]. By considering Sidon sets, it is easily seen that $F$ must operate from $\mathcal{L}_2$ to $\mathcal{L}_2$ and this gives (b). (c) is obvious.

**Proof of 3.1.** $k$ and $w$ are fixed throughout this proof. If $0 < |z| < 1$, then a positive integer $N$ can be chosen so that

(3.4) $2^{-N((k + \varphi)/q)} \leq |z| < 2^{-(N - 1)((k + \varphi)/q)}$.

Let $T_{N,k}$ be as in Lemma 2.11 and define

$$f(t) = z(T_{N,k}(t) + wT_{N,k}^t(t)).$$

Then by (3.4) and (2.11 (b))

$$\|f\|_p \leq C(k, w)N^22^{-N(1/2+1/q)}.$$

Thus if $M$ and $\hat{\delta}$ are as in Lemma 3.2 and $|z|$ is small enough then $\|f\|_p < \hat{\delta}$ so that

(3.5) $\|F \circ f\|_1 < M$.

Now

$$F \circ f = \sum_c F((w + j)z) \sum_{A_i} e^{int}$$

(3.6) $\sum_c b_c T_{N,k}^c$

where the $b_c$ satisfy
\[ F((w + j)z) = \sum_{j=0}^{k} b_j z^j \quad (j = 0, 1, \ldots, k). \]

Solving for the \( b_s \) and using the assumption that \( F(z) = O(|z|^\beta) \) gives that, for \( |z| \) small enough,

\[ |b_s| \leq C(k) |z|^\beta \quad (s = 0, 1, \ldots) \]

and

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & F(wz) \\
1 & 1 & 1 & \cdots & 1 & F((w + 1)z) \\
1 & 2 & 2^2 & \cdots & 2^{k-1} & F((w + 2)z) \\
1 & 3 & 3^2 & \cdots & 3^{k-1} & F((w + 3)z) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & k & k^2 & \cdots & k^{k-1} & F((w + k)z)
\end{pmatrix}
\]

\[ b_k = \frac{\det}{F(wz)} = A(k) \sum_{i=0}^{k} (-1)^i \binom{k}{i} F((w + j)z) \]

where \( A(k) \neq 0 \) is independent of \( z \). Now by (3.5) and (3.6)

\[ |b_k| \leq T_{N,k}^* \leq M + \sum_{i=0}^{k-1} |b_i| \leq T_{N,k}^*. \]

Lemma 2.11d, (2.14), (3.7), (3.8) and (3.9) then give, if \( |z| \) is small enough,

\[ |\sum_{i=0}^{k} (-1)^i \binom{k}{i} F((w + j)z)| \leq C(k) \left\{ \frac{M}{2^{Nk^2}} + \frac{|z|^\beta N^{2(k-1)}}{2^{N^2}} \right\} \]

\[ \leq C(k) \left\{ \frac{M}{2^{Nk^2}} + \frac{|z|^\beta}{2^{N^2}} \right\}. \]

By (3.4) the right side of (3.10) is bounded by

\[ C(k)\{M |z|^{k/(2(k+q))} + |z|^\beta} \}

and this gives (3.2).

4. Proof of Theorem 1.3. We can now prove Theorem 1.3 provided we have the following theorem.
THEOREM 4.1. Suppose $F$ is bounded near the origin and for some positive integer $k$ and each complex $w$, $F$ satisfies

\[(4.2) \sum_{0}^{k} (-1)^{\binom{k}{j}} F((w + j)z) = O(|z|^\beta)\]

where $\beta > 0$ and is not an integer. Then

$$F(z) = P(z, \bar{z}) + H(z)$$

where $P$ is a polynomial in $z$ and $\bar{z}$ of degree less than $k$,

$$H(z) = O(|z|^\beta) \text{ and } H(0) = 0.$$  

REMARKS. Since $\beta > 0$ and $H(0) = 0$ it follows that $H$ and thus also $F$ is continuous at 0. $F$ need not be continuous anywhere else.

The theorem is false if $\beta$ is an integer as can be seen by letting $\beta = 1$, $k = 2$ and $F(z) = z \log |z|$ for all $z \neq 0$.

It is also false if $F(z) \neq O(1)$. For there are functions defined on the plane which are unbounded near the origin and satisfy $F(z + w) = F(z) + F(w)$ for all $z$ and $w$. The left side of (4.2) is then 0 for all $k > 1$. Being unbounded $F$ cannot satisfy the conclusion of the theorem.

Proof of 1.3. $F$ operates from $\mathcal{F}L_p$ to $\mathcal{F}L_1$ where $1 < p < 2$. There is a positive integer $r$ such that $r < q/2 \leq r + 1$. We will prove the theorem by induction on $r$.

First, we can assume that

\[(4.3) \quad F(z) = O(|z|^{-\delta}) \quad \text{ for all } \delta > 0.\]

For if $r = 1$ then, by Lemma 3.3b, (4.3) holds even with $\delta = 0$. On the other hand, suppose $r > 1$ and the theorem holds when $r - 1 < q'/2 \leq r$. Since $F$ operates from $\mathcal{F}L_p$ to $\mathcal{F}L_1$, it operates from $\mathcal{F}L_s$ to $\mathcal{F}L_1$ where $s^{-1} + (2r)^{-1} = 1$. Thus $F(z) = P(z, \bar{z}) + O(|z|^{-\delta})$ for all $\epsilon > 0$. Since polynomials operate we can assume $p = 0$, that is (4.3).

Next choose $k$ so large and then $\delta$ so small that $\beta' = \min (r - \delta + q/4(k + q), q/2(k + q)) > r$ and also so that $\beta'$ is not an integer. Then by (4.3), Lemma 3.1 and Theorem 4.1

$$F(z) = P(z, \bar{z}) + O(|z|^\beta).$$

Thus, by subtracting another polynomial from $F$, we can assume

\[(4.4) \quad F(z) = O(|z|^\beta') \quad \text{ for some } \beta' > r.\]

Finally, let $\gamma = \sup \beta'$ such that (4.4) holds. If $\gamma < q/2$ then we
can choose \( k \) so large and then \( r < \beta' < \gamma \) so that

\[
(4.5) \quad \beta'' = \min \left( \beta' + \frac{q}{4(k + q)}, \frac{qk}{2(k + q)} \right) > \gamma
\]

and \( \beta'' \) is not an integer.

Then by Lemma 3.1 and Theorem 4.1 again

\[
F(z) = P(z, \bar{z}) + O(|z|^\beta').
\]

Since \( F(z) = O(|z|^\beta') \) and \( r < \beta' < \beta'' < r + 1 \) we must have \( P(z, \bar{z}) = O(|z|^\beta'') \) so that \( F(z) = O(|z|^\beta') \). Since \( \beta'' > \gamma \) this is a contradiction. Thus (4.4) holds for all \( \beta' < q/2 \) and this completes the proof of the theorem.

It now remains to give a proof of Theorem 4.1.

**Lemma 4.6.** Suppose \( F \), defined on the plane—\( \{0\} \), satisfies

\[
F(qz) - q^s F(z) = O(|z|^\beta)
\]

where \( q > 1 \).

(a) If \( F = O(1) \) and \( s > \beta > 0 \), then \( F(z) = O(|z|^\beta) \).

(b) If \( \beta > s > 0 \) then \( F(z) = K(z) + O(|z|^\beta) \) where \( K(qz) = q^s K(z) \).

If also \( F(z) = O(1) \) then \( K(s) = O(|z|^s) \).

The proof of (a) is simple and that of (b) is the same as the proof of Lemma 3 of [3].

**Lemma 4.7.** Suppose \( F \) is bounded near the origin and, for some positive integer \( k \) and each nonnegative integer \( p \), \( F \) satisfies

\[
(4.8) \quad \sum_{j=0}^{k} (-1)^j \binom{k}{j} F((p + j)z) = O(|z|^\beta)
\]

where \( \beta > 0 \) and \( \beta \) is not an integer. Then

\[
(4.9) \quad F(z) = F(0) + \sum_{j=1}^{k-1} F_j(z) + O(|z|^\beta)
\]

where

\[
(4.10) \quad F_j(qz) = q^j F_j(z)
\]

for all positive integers \( q \) and \( F_j(z) = O(|z|^j) \).

Note that it follows from the conclusion that \( F \) is continuous at 0.

**Proof.** The lemma is clear if \( k = 1 \), so assume \( k > 1 \) and the lemma holds for \( k - 1 \). Fix \( q > 1 \), an integer and for a nonnegative integer \( p \) consider the polynomial
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$S(\lambda) = \sum_{0}^{k-1} (-1)^i(\lambda_{p+j}^p - q^{k-1}\lambda_{p+j})$.

Now $S$ has a zero of order $k$ at 1 and thus can be written

$S(\lambda) = (1 - \lambda)^k \sum_{0}^{b} a_{j} \lambda^j \quad (b = (p + k - 1)q - k)$

$= \sum_{0}^{b} a_{j} \sum_{0}^{k} (-1)^i \lambda_{p+j}^i$.

By comparing the coefficients of $\lambda^n$ in the two forms of $S$ it is seen that for any function $F$

$\sum_{0}^{k-1} (-1)^i(\lambda^i)F((p+j)qz) - q^{k-1}F((p+j)z) = \sum_{0}^{b} a_{j} \sum_{0}^{k} (-1)^i \lambda F((s+j)z)$.

Thus if $F$ satisfies the hypotheses of the lemma for $k$ then the function $T(z) = F(qz) - q^{k-1}F(z)$ satisfies them for $k - 1$. Thus

$T(z) = T(0) + \sum_{1}^{k-2} T_{j}(z) + O(|z|^\beta)$

where the $T_{j}$ satisfy (4.10). Let

$(4.11) \quad H(z) = F(z) - F(0) - \sum_{0}^{k-1} T_{j}(z) - q^{k-1}.$

Then $H(qz) - q^{k-1}H(z) = O(|z|^\beta)$. Since $\beta$ is not an integer and $H(z) = O(1)$ one of the two cases of Lemma 4.6 holds so that $H$ can be written

$H(z) = K(z) + O(|z|^\beta)$

where $K(qz) = q^{k-1}K(z)$ and $K(z) = O(|z^{k-1}|)$. If $\beta < k - 1$ then we can assume $K = 0$ and by using any $q$, (4.11) gives the desired form for $F$. If $\beta > k - 1$, then it is easily seen that $F_{j} = T_{j}/(q^{i} - q^{k-1})$ and $F_{k-1} = K$ are independent of the choice of $q$. All the $F_{j}$ then satisfy (4.10), and by (4.11), $F$ is given by (4.9).

Proof of Theorem 4.1. We have that for each complex $w$

$(4.12) \quad \sum_{0}^{k} (-1)^i(\lambda^i)F((w+j)z) = O(|z|^\beta)$.

Because of the previous lemma we need only consider functions of the form

$F(z) = F(0) + \sum_{1}^{k-1} F_{s}(z)$

where the $F_{s}$ satisfy (4.10) and $F_{s} = 0$ if $s > \beta$. Also since constant functions satisfy (4.12) we can assume $F(0) = 0$. If $\beta < 1$ there is
nothing left to prove so assume $\beta > 1$.

Now by (4.10) and (4.12), for each positive integer $q$,

$$\sum_{j=0}^{k-1} q^j \sum_{w=0}^{k-1} (-1)^j F_s((w + j)z) = q \sum_{j=0}^{k-1} (-1)^j F((w + j)z/q) = qO\left(\frac{|z|^\beta}{q^\beta}\right).$$

Fixing $z$ and letting $q \to \infty$ then gives

$$\sum_{j=0}^{k-1} (-1)^j F_i((w + j)z) = 0$$

so that

$$\sum_{j=0}^{k-1} (-1)^j F_i(w + jz) = 0 \quad \text{(4.13)}$$

for all $z$ and $w$. Similarly (4.13) holds for $F_2, F_3, \ldots, F_{k-1}$. Then, for each complex $w$, the function $H(z) = F_s(w + z)$ satisfies the hypotheses of Lemma 4.7, but this implies that $H$ is continuous at 0 so that $F_s$ is continuous everywhere and $F_s(xz) = x^s F_s(z)$ for all $x \geq 0$. Finally, for each integer $n$,

$$K_n(z) = \int_0^{2\pi} F_s(ze^{it}) e^{-i nt} dt$$

satisfies (4.13) and for $x \geq 0$

$$K_n(xe^{it}) = x^s e^{i nt} K_n(1).$$

It can be easily seen directly that $K_n(1)$ must be zero unless $s + n$ is even and $|n| \leq s$ which implies $F_s(z) = \sum c_r z^{2r-s}$ and this completes the proof of the theorem.

REFERENCES


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