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MAXIMINIMAX, MINIMAX, AND ANTIMINIMAX THEOREMS AND A RESULT OF R. C. JAMES

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This paper contains a number of minimax theorems in various topological and non-topological situations. Probably the most interesting is the following: if X is a nonempty bounded convex subset of a real Hausdorff locally convex space E with dual E' and each $\varphi \in E'$ attains its supremum on X then

$$\left.\begin{array}{l}\text{for all nonempty convex equicontinuous } Y \subset E' \\ \inf_{y \in Y} \sup \langle X, y\rangle \leqq \sup_{x \in X} \inf \langle x, Y\rangle\end{array}\right\}(*).$$

It is also proved that if (*) is true and X is complete then X is $w(E, E')$-compact. Combining these results, a proof of a well known result of R. C. James is obtained.

We suppose throughout that $X \neq \phi$, $Y \neq \phi$, and $f: X \times Y \to R$. We write $\mathscr{F}(X)$ for $\{F: \phi \neq F \subset X, F \text{ is finite}\}$ and define $\mathscr{F}(Y)$ similarly. The *maximinimax inequality* is the relation

$$(1) \qquad \inf_{G \in \mathscr{F}(Y)} \sup_{x \in X} \inf f(x, G) \leqq \sup_{F \in \mathscr{F}(X)} \inf_{y \in Y} \sup f(F, y)$$

and the *minimax inequality* is the relation

$$(2) \qquad \inf_{y \in Y} \sup f(X, y) \leqq \sup_{x \in X} \inf f(x, Y) .$$

The main result of this paper is Theorem 5, which gives some conditions under which (1) holds. These conditions are completely non-topological and depend only on the fact that certain functions attain their suprema on X. We prove Theorem 5 by defining a "remoteness" relation on the subsets of Y, but we point out that Theorem 5 can also be proved by first reducing the problem to the "iterated limits unequal" situation (by using the technique of Remark 8 and then the diagonal process) and then going through the same steps as in [6], Lemmas 1–7. The approach adopted here embodies a new type of diagonal argument (Lemmas 2 and 3) which might find applications elsewhere, and an argument similar to but subtler than that used in [9], Lemma 2. There is another proof of Theorem 5 that is "frontended" in the sense that we can choose the functions $k_1, k_2, \cdots$ of Theorem 5 by a purely inductive process without having first to choose a sequence $\{y_n\}_{n \geqq 1}$. The price one pays for the "frontendedness" is that the induction is more complicated and that is why we have avoided the alternative approach.

Remarks 6, 7, and 8 give certain topological conditions under which (1) holds; the result in Remark 8 uses Theorem 5 and is considerably deeper than the other two.

In Lemma 11 we give some conditions (far from the best, but adequate for our purposes) under which (2) follows from (1) and thus derive some minimax Theorems in Theorem 12, Remark 13 and Theorem 14.

Theorem 15 is a converse minimax theorem and leads immediately to Theorem 16, which contains the result of James referred to in the title. Those sections of this paper that are needed for Theorem 16 are entitled "Lemma" and "Theorem", while those not so needed are entitled "Remark".

1. A maximinimax theorem with no topology and hardly any convexity.

NOTATION 1. If $Z \neq \phi$ we write S_Z for "supremum on Z". We write "conv" for "convex hull of".

LEMMA 2. *We suppose that $f(X, Y)$ is bounded and $A \in R$. If $\phi \neq W \subset Y$ and $Z \subset Y$ we write $W \mathscr{R} Z$ if there exists $h \in l_\infty(X)$ such that, whenever $G \in \mathscr{F}(W \cup Z)$,*

$$S_X(\inf f(\cdot, G) - h) - \sup_{F \in \mathscr{F}(X)} \inf_{y \in W} S_F(f(\cdot, y) - h) > A .$$

If $Y_0 \mathscr{R} Z$ we write $t(Y_0, Z) = \inf_{\emptyset \neq W \subset Y_0, W \mathscr{R} Z} \sup f(\cdot, W) \in l_\infty(X)$.

We suppose that $Y_0 \mathscr{R} \phi$. Then we can choose $y_1, Y_1, y_2, Y_2, \cdots$ such that, for all $p \geqq 1$,

(3, p) $$y_p \in Y_{p-1} \quad \text{and} \quad Z_p = \{y_1, \cdots, y_p\} ,$$

(4, p) $$\emptyset \neq Y_p \subset Y_{p-1},\ Y_p \mathscr{R} Z_p,\ S_X(\inf f(\cdot, Z_p) - \sup f(\cdot, Y_p)) > A ,$$

and

(5, p) $$\begin{cases} \text{for all} \quad g \in \operatorname{conv} f(\cdot, Z_p) \\ \qquad S_X(g - \sup f(\cdot, Y_p)) > S_X(g - t(Y_p, Z_p)) - \dfrac{1}{p} . \end{cases}$$

Proof. We suppose inductively that $p \geqq 1$ and that, for all $n < p$, y_n and Y_n have been chosen so that (3, n), (4, n), and (5, n) are satisfied. We choose $y_p \in Y_{p-1}$ arbitrarily and define Z_p so that (3, p) is satisfied. Since $Y_{p-1} \mathscr{R} Z_{p-1}$ and $Y_{p-1} \cup Z_p = Y_{p-1} \cup Z_{p-1}$ (with the convention that $Z_0 = \phi$), it follows that $Y_{p-1} \mathscr{R} Z_p$. Hence there exists $h \in l_\infty(X)$ such that (taking $G = Z_p \in \mathscr{F}(Y_{p-1} \cup Z_p)$)

$$S_X(\inf f(\cdot, Z_p) - h) - \alpha > A ,$$

where $\alpha = \sup_{F \in \mathscr{F}(X)} \inf_{y \in Y_{p-1}} S_F(f(\cdot, y) - h)$. Consequently there exist $x_0 \in X$ and $\eta > 0$ such that

$$\inf f(x_0, Z_p) - h(x_0) - \alpha > A + \eta . \tag{6}$$

We write $W_0 = \{y: y \in Y_{p-1}, f(x_0, y) - h(x_0) \leqq \alpha + \eta\}$. If $F \in \mathscr{F}(X)$ then $F \cup \{x_0\} \in \mathscr{F}(X)$ hence

$$\inf_{y \in Y_{p-1}} \{S_F(f(\cdot, y) - h) \vee (f(x_0, y) - h(x_0))\} \leqq \alpha$$

from which $\inf_{y \in W_0} S_F(f(\cdot, y) - h) \leqq \alpha$. We have proved that

$$\sup_{F \in \mathscr{F}(X)} \inf_{y \in W_0} S_F(f(\cdot, y) - h) \leqq \alpha$$

and it follows easily from this and $Y_{p-1} \mathscr{R} Z_p$ that $W_0 \mathscr{R} Z_p$. We next choose $g_1, \cdots, g_s \in \operatorname{conv} f(\cdot, Z_p)$ so that, for all $g \in \operatorname{conv} f(\cdot, Z_p)$, there exists $r \in \{1, \cdots, s\}$ such that $S_X(|g - g_r|) < 1/3p$. We choose $W_1, \cdots, W_s$ in sequence as follows: we choose W_r, $\phi \neq W_r \subset W_{r-1}$ such that $W_r \mathscr{R} Z_p$ and

$$S_X(g_r - \sup f(\cdot, W_r)) > S_X(g_r - t(W_{r-1}, Z_p)) - \frac{1}{3p} .$$

We write $Y_p = W_s$. It is immediate that $\phi \neq Y_p \subset Y_{p-1}$ and $Y_p \mathscr{R} Z_p$. Further

$$\begin{aligned} S_X(\inf f(\cdot, Z_p) - \sup f(\cdot, Y_p)) &\geqq S_X(\inf f(\cdot, Z_p) - \sup f(\cdot, W_0)) \\ &\geqq \inf f(x_0, Z_p) - \sup f(x_0, W_0) \\ &\geqq \inf f(x_0, Z_p) - h(x_0) - \alpha - \eta > A \end{aligned}$$

from the definition of W_0 and (6). Hence (4, p) is satisfied. Finally, if $g \in \operatorname{conv} f(\cdot, Z_p)$ we choose $r \in \{1, \cdots, s\}$ such that $S_X(|g - g_r|) < 1/3p$. Then

$$\begin{aligned} S_X(g - \sup f(\cdot, Y_p)) &\geqq S_X(g - \sup f(\cdot, W_r)) \\ &> S_X(g_r - \sup f(\cdot, W_r)) - \frac{1}{3p} \\ &> S_X(g_r - t(W_{r-1}, Z_p)) - \frac{2}{3p} \\ &\geqq S_X(g - t(W_{r-1}, Z_p)) - \frac{1}{p} \\ &\geqq S_X(g - t(Y_p, Z_p)) - \frac{1}{p} . \end{aligned}$$

Hence (5, p) is satisfied and the Lemma is proved by induction.

LEMMA 3. *We use the notation of Lemma* 2 *and suppose* $p \geqq 1$.
(a) *If* $\{z_n\}_{n\geqq 1}$ *is a subsequence of* $\{y_n\}_{n>p}$ *then* $\{z_n : n \geqq 1\} \mathscr{R} Z_p$.
(b) *We suppose that* $g \in \operatorname{conv} f(\cdot, Z_p)$. *Then*

$$S_X(g - \limsup_{n\to\infty} f(\cdot, y_n)) \geqq S_X(g - \liminf_{n\to\infty} f(\cdot, y_n)) - \frac{1}{p} .$$

Proof. (a) We write $W = \{z_n : n \geqq 1\}$ and $h = \limsup_{n\to\infty} f(\cdot, z_n)$. If $F \in \mathscr{F}(X)$ then $\inf_{y\in W} S_F(f(\cdot, y) - h) = \inf_{n\geqq 1} S_F(f(\cdot, z_n) - h) \leqq \limsup_{n\to\infty} S_F(f(\cdot, z_n) - h) = S_F(\limsup_{n\to\infty} f(\cdot, z_n) - h) = S_F(0) = 0$. On the other hand if $G \in \mathscr{F}(W \cup Z_p)$ then there exists $q \geqq p$ such that $\inf f(\cdot, G) \geqq \inf f(\cdot, Z_q)$ and since $h \leqq \sup_{n>q} f(\cdot, y_n) \leqq \sup f(\cdot\, Y_q)$

$$S_X(\inf f(\cdot, G) - h) \geqq S_X(\inf f(\cdot, Z_q) - \sup f(\cdot, Y_q)) > A$$

from (4, p). It follows that $W \mathscr{R} Z_p$, as required.

(b) If $\{z_n\}_{n\geqq 1}$ is as in (a) then, from (a) and (5, p),

$$\begin{cases} S_X(g - \limsup_{n\to\infty} f(\cdot, y_n)) \geqq S_X(g - \sup f(\cdot, Y_p)) \\ \qquad\qquad \geqq S_X(g - \sup_{n\geqq 1} f(\cdot, z_n)) - \frac{1}{p} . \end{cases} \tag{7}$$

Now the lim inf of a real sequence is the infimum of the suprema of all its subsequences (this is the crux of [6], Lemma 2) so, taking the supremum of the right-hand side of (7) over all subsequences $\{z_n\}_{n\geqq 1}$ of $\{y_n\}_{n>p}$ we get the required result.

LEMMA 4. *We use the notation of Lemma* 2. *If* $g \in \operatorname{conv}\{f(\cdot, y_n) : n \geqq 1\}$ *then*

$$S_X(g - \limsup_{n\to\infty} f(\cdot, y_n)) = S_X(g - \liminf_{n\to\infty} f(\cdot, y_n)) > A . \tag{8}$$

Proof. If p is sufficiently large then $g \in \operatorname{conv} f(\cdot, Z_p)$. The equality follows from Lemma 3(b) by letting $p \to \infty$ and the inequality follows from (4, p).

THEOREM 5. *We suppose that* $f(X, Y)$ *is bounded and, whenever* $0 < \lambda < 1$ *and* $\{k_n : n \geqq 1\} \subset \operatorname{conv} f(\cdot, Y)$ *then there exists* $k \in l_\infty(X)$ *such that* $\liminf_{n\to\infty} k_n \leqq k \leqq \limsup_{n\to\infty} k_n$ and $\sum_{n\geqq 1} \lambda^{n-1}(k_n - k)$ *attains its supremum on* X. *Then* (1) *holds.*

Proof. If (1) does not hold then, for an appropriate $A > 0$ (taking $h = 0$) $Y \mathscr{R} \phi$. From Lemmas 2 and 4, there exists $\{y_n : n \geqq 1\} \subset Y$ such that, whenever $g \in \operatorname{conv}\{f(\cdot, y) : n \geqq 1\}$, (8) holds. For $m \geqq 1$ we write $f_m = f(\cdot, y_m) - \limsup_{n\to\infty} f(\cdot, y_n)$ and $B = \sup_{n\geqq 1} S_X(f_n)$. We

write $\delta = A/3 > 0$ and choose $\lambda > 0, g_1, g_2, \cdots$ as in [9], Lemma 2. Proceeding as therein we obtain ([9], relation (3)) for all $m \geqq 1$

$$(9) \qquad S_X(\sum_{n \leqq m} \lambda^{n-1} g_n) - S_X(\sum_{n \leqq m-1} \lambda^{n-1} g_n) \geqq [A - \delta(1 + \lambda)]\lambda^{m-1} .$$

For all $m \geqq 1$ there exists $k_m \in \operatorname{conv}\{f(\cdot, y_n): n \geqq m\}$ such that $g_m = k_m - \limsup_{n\to\infty} f(\cdot, y_n)$. We choose $k \in l_\infty(X)$ as in the statement of the theorem and write $g'_m = k_m - k$. Since $\liminf_{n\to\infty} f(\cdot, y_n) \leqq k \leqq \limsup_{n\to\infty} f(\cdot, y_n)$, it follows from (8) that, for all $m \geqq 1$,

$$S_X(\sum_{n \leqq m} \lambda^{n-1} g'_n) = S_X(\sum_{n \leqq m} \lambda^{n-1} g_n) .$$

Thus (9) is true with g_n replaced by g'_n. Since $\sum_{n \leqq 1} \lambda^{n-1} g'_n$ attains its supremum on X we can continue as in [9], Lemma 2 and find $x \in X$ such that, for all $m \geqq 1$, $g'_m(x) \geqq A - 2\delta = A/3$, i.e., $k_m(x) \geqq k(x) + A/3$. This contradicts our assumption that $k \geqq \liminf_{n\to\infty} k_n$.

2. Topological conditions implying the maximinimax relation.

REMARK 6. If X is a compact topological space and, for all $y \in Y$, $f(\cdot, y)$ is uppersemicontinuous on X then (1) holds.

Proof. If $\alpha < \inf_{G \in \mathscr{F}(Y)} \sup_{x \in X} \inf f(x, G)$ then, for all $G \in \mathscr{F}(Y)$, $\bigcap_{y \in G} \{x: x \in X, f(x, y) \geqq \alpha\} \neq \phi$. From compactness, $\bigcap_{y \in Y} \{x: x \in X, f(x, y) \geqq \alpha\} \neq \phi$ hence $\sup_{x \in X} \inf f(x, Y) \geqq \alpha$. It follows that

$$\inf_{G \in \mathscr{F}(Y)} \sup_{x \in X} \inf f(x, G) \leqq \sup_{x \in X} \inf f(x, Y)$$

and so (1) certainly holds.

REMARK 7. If X and Y are countably compact topological spaces and f is separately continuous then (1) holds.

Proof. If

$$\alpha < \inf_{G \in \mathscr{F}(Y)} \sup_{x \in X} \inf f(x, G) \quad \text{and} \quad \sup_{F \in \mathscr{F}(X)} \inf_{y \in Y} \sup f(F, y) < \beta$$

then we can construct inductively $\{x_n: n \geqq 1\} \subset X$ and $\{y_n: n \geqq 1\} \subset Y$ such that $f(x_m, y_n) \leqq \beta$ when $1 \leqq m \leqq n$ and $f(x_m, y_n) \geqq \alpha$ when $1 \leqq n < m$. We suppose that $x_0 \in X$ is a cluster point of $\{x_n\}_{n \geqq 1}$ and $y_0 \in Y$ is a cluster point of $\{y_n\}_{n \geqq 1}$. Then simultaneously $f(x_0, y_0) \leqq \beta$ and $f(x_0, y_0) \geqq \alpha$, hence $\alpha \leqq \beta$. Thus (1) holds.

REMARK 8. If X is a pseudocompact topological space, $f(X, Y)$ is bounded and each sequence from $\operatorname{conv} f(\cdot, Y)$ has a continuous cluster

point in the topology of pointwise convergence on X then (1) is true.

Proof. This is immediate from Theorem 5.

3. Some old and new minimax theorems for concave-convex functions.

LEMMA 9. *We suppose that X is a nonempty convex subset of a real linear space and $a_1, \cdots, a_m$ are real concave functions on X. Then there exist $\lambda_1, \cdots, \lambda_m \geqq 0$, $\lambda_1 + \cdots + \lambda_m = 1$ such that*

$$S_X(\lambda_1 a_1 + \cdots + \lambda_m a_m) = S_X(a_1 \wedge \cdots \wedge a_m) .$$

Proof. The result is immediate if $S_X(a_1 \wedge \cdots \wedge a_m) = \infty$ so we suppose that $S_X(a_1 \wedge \cdots \wedge a_m) < \infty$. If $f \in R^m$ we write

$$\alpha(f) = S_X[(a_1 - f(1)) \wedge \cdots \wedge (a_m - f(m))]$$

α is a well-defined concave function on R^m. If $f \in R^m$ we write

$$\beta(f) = \inf_{\lambda > 0} \lambda\left[\alpha(0) - \alpha\left(\frac{f}{\lambda}\right)\right] .$$

Then β is a well-defined sublinear functional on R^m. From the Hahn-Banach theorem there exist $\lambda_1, \cdots, \lambda_m$ such that

$$\text{for all } \quad f \in R^m \; \lambda_1 f(1) + \cdots + \lambda_m f(m) \leqq \beta(f) .$$

For all $f \in R^m$, $\beta(f) \leqq f(1) \vee \cdots \vee f(m)$, hence $\lambda_1, \cdots, \lambda_m \geqq 0$ and $\lambda_1 + \cdots + \lambda_m = 1$. Further

$$\begin{aligned} &S_X(\lambda_1 a_1 + \cdots + \lambda_m a_m) \\ &\quad \leqq \sup_{y \in X} \beta(a_1(y), \cdots, a_m(y)) \\ &\quad \leqq \sup_{y \in X} [\alpha(0) - \alpha(a_1(y), \cdots, a_m(y))] \\ &\quad = \alpha(0) - \inf_{y \in X} \sup_{x \in X} [(a_1(x) - a_1(y)) \wedge \cdots \wedge (a_m(x) - a_m(y))] \\ &\quad \leqq \alpha(0) = S_X(a_1 \wedge \cdots \wedge a_m) . \end{aligned}$$

REMARK 10. The axiom of choice is not used in Lemma 9. (The Hahn-Banach theorem is used only in R^m.) Using the axiom of choice one can prove the following extension: if $N \neq \varnothing$, for all $n \in N$, a_n is a real concave function on X and, for all $x \in X$, $a_\cdot(x) \in l_\infty(N)$ then there exists a positive linear functional μ on $l_\infty(N)$ such that $\mu(1) = 1$ and $\sup_{x \in X} \mu(a_\cdot(x)) = S_X(\inf_{n \in N} a_n)$. This is closely related to the following result: if S is a sublinear functional on a real linear space E and D is a nonempty convex subset of E then there exists a linear func-

tional φ on E such that φ is dominated by S on E and $\inf \varphi(D) = \inf S(D)$. This latter result is discussed in [4], [5], and [8], Theorem 28(d) and Remark 29.

We point out that Lemma 9 can be deduced from known minimax theorems. On the other hand Lemma 9 generalizes [2], Theorem 6.

LEMMA 11. *We suppose that X and Y are nonempty convex subsets of real linear spaces and*

$$\text{for all } y \in Y \quad f(\cdot, y) \text{ is concave on } X, \tag{10}$$

$$\text{for all } x \in X \quad f(x, \cdot) \text{ is convex on } Y, \tag{11}$$

then

$$\inf_{G \in \mathscr{F}(Y)} \sup_{x \in X} \inf f(x, G) = \inf_{y \in Y} \sup f(X, y) \tag{12}$$

and

$$\sup_{F \in \mathscr{F}(X)} \inf_{y \in Y} \sup f(F, y) = \sup_{x \in X} \inf f(x, Y) . \tag{13}$$

Proof. If $G = \{y_1, \cdots, y_m\} \in \mathscr{F}(Y)$ then, from (10) and Lemma 9, there exist $\lambda_1, \cdots, \lambda_m \geqq 0$, $\lambda_1 + \cdots + \lambda_m = 1$ such that

$$S_X(\lambda_1 f(\cdot, y_1) + \cdots + \lambda_m f(\cdot, y_m)) = \sup_{x \in X} \inf f(x, G) .$$

From (11), if

$$y = \lambda_1 y_1 + \cdots + \lambda_m y_m \in Y \text{ then } \sup f(X, y) \leqq \sup_{x \in X} \inf f(x, G) .$$

This establishes "$\geqq$" in (12) and "$\leqq$" is trivial. (13) is proved by an analogous argument. (Compare [1], Theorem 1.)

THEOREM 12. *We suppose that X, Y and f satisfy the conditions of Lemma* 11 *and Theorem* 5. *Then* (2) *holds.*

Proof. This is immediate from Theorem 5 and Lemma 11.

REMARK 13. Arguing as above, we can prove easily that if X, Y and f satisfy the conditions of Lemma 11 and either Remark 6, 7, or 8 then (2) is satisfied. The first of these results is well known; the other two do not seem to be.

THEOREM 14. *We suppose that E is a real Hausdorff locally convex space with dual E' and that X is a nonempty bounded convex subset of E. Then* (14)$\Rightarrow$(15)$\Rightarrow$(16)$\Rightarrow$(17)$\Leftrightarrow$(18).

(14) X *is* $w(E, E')$*-compact*

(15) *There exists* Z *such that* $X = \operatorname{conv}^- Z$ *and* Z *is* $w(E, E')$*-compact*

(16) *For all* $\varphi \in E'$, φ *attains its supremum on* X

(17) *Whenever* Y *is a nonempty convex equicontinuous subset of* E' *then* $\inf_{y \in Y} \sup \langle X, y\rangle \leqq \sup_{x \in X} \inf \langle x, Y\rangle$

(18) *Whenever* Y *is a nonempty convex equicontinuous subset of* E' *and* $d \in Z$ *then* $\inf_{x \in X} S_Y(d - \langle x, \cdot\rangle) \leqq 0$, *where* Z *stands for the closure in* $R^{E'}$ *of* $\{\langle x, \cdot\rangle : x \in X\}$.

Proof. It is trivial that (14)⇒(15) and immediate from the bipolar theorem that (15)⇒(16). It follows from Theorem 12 that (16)⇒(17). (If $\{y_n : n \geqq 1\} \subset Y$ and $y \in E'$ is a $w(E', E)$-cluster point of $\{y_n\}_{n \geqq 1}$ then, for all $x \in X$, $\liminf_{n \to \infty} \langle x, y_n\rangle \leqq \langle x, y\rangle \leqq \limsup_{n \to \infty} \langle x, y_n\rangle$. Consequently the conditions of Theorem 5 are satisfied.)

(17)⇒(18). We suppose that Y, Z and d are as in (18) and $\delta > 0$. Since d is bounded and affine on Y we can write $Y = Y_1 \cup \cdots \cup Y_m$ where, for all $n = 1, \cdots, m$, Y_n is nonempty and convex and $\sup d(Y_n) - \inf d(Y_n) \leqq \delta$. From (17) (with Y replaced by Y_n) there exists $x_n \in X$ such that

$$\begin{aligned} \inf \langle x_n, Y_n\rangle &\geqq \inf_{y \in Y_n} \sup \langle X, y\rangle - \delta \\ &\geqq \inf d(Y_n) - \delta \qquad \text{since} \quad d \in Z \\ &\geqq \sup d(Y_n) - 2\delta \qquad \text{from the choice of } Y_n . \end{aligned}$$

From Lemma 9, there exists $x \in X$ such that

$$\begin{aligned} S_Y(d - \langle x, \cdot\rangle) &= S_Y((d - \langle x_1, \cdot\rangle) \wedge \cdots \wedge (d - \langle x_m, \cdot\rangle)) \\ &\leqq \sup_{1 \leqq n \leqq m} S_{Y_n}(d - \langle x_n, \cdot\rangle) \\ &\leqq \sup_{1 \leqq n \leqq m} [\sup d(Y_n) - \inf \langle x_n, Y_n\rangle] \leqq 2\delta \end{aligned}$$

from the inequalities above. The result follows since $\delta > 0$ is arbitrary.

(18)⇒(17). We suppose that Y is as in (17) and we define Z as in (18). We suppose, further, that $\alpha < \inf_{y \in Y} \sup \langle X, y\rangle$. If $G \in \mathscr{F}(Y)$ then, from Lemma 9, there exists $y \in Y$ such that $\sup \langle X, y\rangle = \sup_{x \in X} \inf \langle x, G\rangle$, hence $\sup_{x \in X} \inf \langle x, G\rangle > \alpha$. Consequently

$$\bigcap_{y \in G} \{d : d \in Z, d(y) \geqq \alpha\} \supset \bigcap_{y \in G} \{\langle x, \cdot\rangle : x \in X, \langle x, y\rangle \geqq \alpha\} \neq \varnothing .$$

From Tychonoff's theorem, Z is compact hence $\bigcap_{y \in Y} \{d : d \in Z, d(y) \geqq \alpha\} \neq \varnothing$ from which there exists $d \in Z$ such that $\inf d(Y) \geqq \alpha$. (Compare the proof of Remark 6.) From (18)

$$0 \geqq \inf_{x \in X} S_Y(d - \langle x, \cdot \rangle) \geqq \inf_{x \in X} (\inf d(Y) - \inf \langle x, Y \rangle)$$
$$= \inf d(Y) - \sup_{x \in X} \inf \langle x, Y \rangle \geqq \alpha - \sup_{x \in X} \inf \langle x, Y \rangle$$

i.e.,

$$\alpha \leqq \sup_{x \in X} \inf \langle x, Y \rangle .$$

The result follows since $\alpha < \inf_{y \in Y} \sup \langle X, y \rangle$ is arbitrary.

4. A converse minimax theorem, James's Theorem and Krein's theorem.

THEOREM 15. *We suppose that E, E' and X are as in Theorem* 14 *and that X is complete. If* (17) *is true then X is $w(E, E')$-compact.*

Proof. We suppose that Z, d are as in (18) and we write $\mathscr{U}$ for the family of closed, circled neighborhoods of 0 in E directed by $\subset$. If $U \in \mathscr{U}$ then, since (17) $\Rightarrow$ (18), there exists $\psi(U) \in X$ such that $S_{U^0}(|d - \langle \psi(U), \cdot \rangle|) \leqq 1/2$. (The absolute value can be put in because $d - \langle \psi(U), \cdot \rangle$ is linear on E' and U^0 is symmetric.) If $U, V, W \in \mathscr{U}$ and $V, W \subset U$ then $\psi(V) - \psi(W) \in U^{00} = U$ hence the net ψ is Cauchy in X. Since X is complete, there exists $x \in X$ such that $\psi \to x$. We suppose that $y \in E'$. If $U \in \mathscr{U}$ and $U^0 \ni y$ then $|d(y) - \langle \psi(U), y \rangle| \leqq 1/2$ and, passing to the limit, $|d(y) - \langle x, y \rangle| \leqq 1/2$. By homogeneity, $d = \langle x, \cdot \rangle \in R^{E'}$. We have proved that $\langle X, \cdot \rangle$ is closed in $R^{E'}$ hence, by Tychonoff's theorem, X is $w(E, E')$-compact.

THEOREM 16. *We suppose that E, E' and X are as in Theorem* 15. *Then* (14), (15) *and* (16) *are equivalent.*

Proof. This is immediate from Theorems 14 and 15. (14) $\Leftrightarrow$ (15) is Krein's Theorem and (14) $\Leftrightarrow$ (16) is James's Theorem. (See [3], [6], and [7].)

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