Pacific Journal of Mathematics

ON PTAK'S COMBINATORIAL LEMMA

STEPHEN SIMONS

Vol. 40, No. 3 November 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 3, 1972

ON PTAK'S COMBINATORIAL LEMMA

S. SIMONS

A new proof of Ptak's combinatorial lemma on the existence of convex means, is given.

The purpose of this note is to show how the "getting near the inf of S" technique used in]3], Lemma 2 can be used to prove, and indeed generalize, Ptak's combinatorial lemma ([1], (1.3)). (We note, in passing, that [1], (2.1) is an easy consequence of [3], Lemma 2 and that [1], (3.3), Krein's theorem, is proved in [4], Theorem 16). We have already given a proof of Ptak's lemma in [2] using lattice theory and the Hahn-Banach theorem. The proof given here is elementary—as was Ptak's original proof.

1. NOTATION. If $X \neq \emptyset$ we write $l_u(X)$ for the set of all functions from X into $[-\infty, \infty)$. Even though $l_u(X)$ is not a vector space, any *convex* combination of elements of $l_u(X)$ is well defined. We write "conv" for "convex hull of". We write "S_X" for "supremum on X".

2. LEMMA. *We suppose $X \neq \emptyset$. If G is a nonempty convex subset of $l_u(X)$, $A, B, C \in R$ and, for all $g \in G$, $A < B \leqq S_X(g) \leqq C$ then there exists $h \in G$ such that, if $X' = \{x: x \in X, h(x) > A\}$ then* $\inf S_{X'}(G) \geqq A$.

Proof. We choose $\lambda > 0$ so that $\lambda(C - A) < B - A$ and then $h \in G$ so that $S_X(h) < \inf S_X(G) + (B-A)\lambda/(1+\lambda)$. If $g \in G$ then, since $(h+\lambda g)/(1+\lambda) \in G$, $(1+\lambda)S_X(h) < S_X(h+\lambda g) + (B-A)\lambda$ hence there exists $x \in X$ (depending on g) such that

$$h(x) + \lambda g(x) > (1+\lambda)S_X(h) - (B-A)\lambda . \tag{1}$$

We first deduce from (1) that $h(x) > (1+\lambda)S_X(h) - \lambda C - (B-A)\lambda \geqq (1+\lambda)B - \lambda C - (B-A)\lambda > A$, from the choice of λ; hence $x \in X'$. Again from (1), $\lambda g(x) > \lambda S_X(h) - (B-A)\lambda \geqq \lambda A$ from which $g(x) > A$. We have proved that $\inf S_{X'}(G) \geqq A$, as required.

3. THEOREM. *We suppose $X \neq \emptyset$, Y is infinite,*

$$f: X \times Y \to [-\infty, \infty) ,$$

$B, C \in R$, $\delta > 0$ and, for all $g \in \operatorname{conv} f(\cdot, Y)$, $B \leqq S_X(g) \leqq C$. Then there exist $x_1, x_2, \cdots \in X$ and distinct $y_1, y_2, \cdots \in Y$ such that $f(x_p, y_m) \geqq$

$B - S$ *whenever* $1 \leqq m \leqq p$.

Proof. From Lemma 2, there exists $g_1 \in \operatorname{conv} f(\cdot, Y)$ such that if $X_1 = \{x: x \in X, g_1(x) > B - \delta/2\}$ then $\inf S_{X_1}(\operatorname{conv} f(\cdot, Y)) \geqq B - \delta/2$. There exists finite $F_1 \subset Y$ such that $g_1 \in \operatorname{conv} f(\cdot, F_1)$. Clearly $\inf S_{X_1}(\operatorname{conv} f(\cdot, Y \backslash F_1)) \geqq B - \delta/2$. Proceeding inductively we find $g_n \in \operatorname{conv} f(\cdot, Y \backslash F_1 \backslash \cdots \backslash F_{n-1})$ such that, if

$$X_n = \{x: x \in X_{n-1}, g_n(x) > B - \delta/2 - \cdots - \delta/2^n\}$$

then $\inf S_{X_n}(\operatorname{conv} f(\cdot, Y \backslash F_1 \backslash \cdots F_{n-1})) \geqq B - \delta/2 - \cdots - \delta/2^n$ and finite $F_n \subset Y \backslash F_1 \backslash \cdots \backslash F_{n-1}$ such that $g_n \in \operatorname{conv} f(\cdot, F_n)$. In this way we obtain a family $[F_n: n \geqq 1\}$ of disjoint finite subsets of Y, $g_n \in \operatorname{conv} f(\cdot, F_n)$ such that, for all $n \geqq 1$,

$$\bigcap_{m \leqq n} \{x: x \in X, g_m(x) \geqq B - \delta\} (\supset X_n) \neq \emptyset .$$

Since $g_1 \in \operatorname{conv} f(\cdot, F_1)$,

$$\{x: x \in X, g_1(x) \geqq B - \delta\} \subset \bigcup_{y \in F_1} \{x: x \in X, f(x, y) \geqq B - \delta\}$$

hence there exists $y_1 \in F_1$ such that, for arbitrarily large $n \geqq 2$,

$$\{x: x \in X, f(x, y_1) \geqq B - \delta\} \cap \bigcap_{2 \leqq m \leqq n} \{x: x \in X, g_m(x) \geqq B - \delta\} \neq \emptyset .$$

This relationship must clearly then hold for all $n \geqq 2$. Proceeding inductively we find $y_n \in F_n$ such that, for all $1 \leqq p < n$,

$$\bigcap_{m \leqq p} \{x: x \in X, f(x, y_m) \geqq B - \delta\} \cap \bigcap_{p+1 \leqq m \leqq n} \{x: x \in X, g_m(x) \geqq B - \delta\} \neq \emptyset$$

and so, in particular,

$$\bigcap_{m \leqq p} \{x: x \in X, f(x, y_m) \geqq B - \delta\} \neq \emptyset$$

from which the required result follows ($y_1, y_2, \cdots$ are distinct because $F_1, F_2, \cdots$ disjoint).

Ptak's Lemma. We suppose that Y is an infinite set and that X is a nonvoid family of subsets of Y. We write $P(Y)$ for the collection of all positive, real valued functions λ on Y such that $\{y: y \in Y, \lambda(y) > 0\}$ is finite and $\sum_{y \in Y} \lambda(y) = 1$; for $x \subset Y$ we write $\lambda(x) = \sum_{y \in x} \lambda(y)$. If

$$B = \inf_{\lambda \in P(Y)} \sup_{x \in X} \lambda(x) > 0$$

then there exist $x_1, x_2, \cdots \in X$ and distinct $y_1, y_2, \cdots \in Y$ such that, for each $p \geqq 1$, $\{y_1, \cdots, y_p\} \subset x_p$.

Proof. We define $f: X \times Y \to R$ by $f(x, y) = 1$ if, and only if, $y \in x$. By hypothesis, $\inf S_X(\mathrm{conv} f(\cdot, Y)) = B$. From Theorem 3 with $\delta = B/2$, there exist $x_1, x_2, \cdots \in X$ and distinct $y_1, y_2, \cdots \in Y$ such that $f(x_p, y_m) \geqq B/2$, hence $y_m \in x_p$, whenever $1 \leqq m \leqq p$.

REFERENCES

1. V. Ptak, *A combinatorial lemma on the existence of convex means and its application to weak compactness*, Proc. Symp. Pure Math., **VII** (1963), 437-450.
2. S. Simons, *A theorem on lattice ordered groups, results of Ptak, Namioka and Banach, and a frontended proof of Lebesgue's theorem*, Pacific J. Math., **20** (1967), 149-153.
3. ———, *A convergence theorem with boundary*, To precede this paper.
4. ———, *Maximinimax, minimax and antiminimax theorems and a result of R.C. James*, To precede this paper.

Received January 28, 1971. This research was supported in part by NSF grant number 20632.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index. to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics
Vol. 40, No. 3 November, 1972