

# Pacific Journal of Mathematics

**ON PTAK'S COMBINATORIAL LEMMA**

STEPHEN SIMONS

## ON PTAK'S COMBINATORIAL LEMMA

S. SIMONS

**A new proof of Ptak's combinatorial lemma on the existence of convex means, is given.**

The purpose of this note is to show how the "getting near the inf of  $S$ " technique used in [3], Lemma 2 can be used to prove, and indeed generalize, Ptak's combinatorial lemma ([1], (1.3)). (We note, in passing, that [1], (2.1) is an easy consequence of [3], Lemma 2 and that [1], (3.3), Krein's theorem, is proved in [4], Theorem 16). We have already given a proof of Ptak's lemma in [2] using lattice theory and the Hahn-Banach theorem. The proof given here is elementary — as was Ptak's original proof.

1. NOTATION. If  $X \neq \emptyset$  we write  $l_u(X)$  for the set of all functions from  $X$  into  $[-\infty, \infty)$ . Even though  $l_u(X)$  is not a vector space, any convex combination of elements of  $l_u(X)$  is well defined. We write "conv" for "convex hull of". We write " $S_X$ " for "supremum on  $X$ ".

2. LEMMA. *We suppose  $X \neq \emptyset$ . If  $G$  is a nonempty convex subset of  $l_u(X)$ ,  $A, B, C \in R$  and, for all  $g \in G$ ,  $A < B \leq S_X(g) \leq C$  then there exists  $h \in G$  such that, if  $X' = \{x: x \in X, h(x) > A\}$  then  $\inf S_{X'}(G) \geq A$ .*

*Proof.* We choose  $\lambda > 0$  so that  $\lambda(C - A) < B - A$  and then  $h \in G$  so that  $S_X(h) < \inf S_X(G) + (B - A)\lambda/(1 + \lambda)$ . If  $g \in G$  then, since  $(h + \lambda g)/(1 + \lambda) \in G$ ,  $(1 + \lambda)S_X(h) < S_X(h + \lambda g) + (B - A)\lambda$  hence there exists  $x \in X$  (depending on  $g$ ) such that

$$(1) \quad h(x) + \lambda g(x) > (1 + \lambda)S_X(h) - (B - A)\lambda.$$

We first deduce from (1) that  $h(x) > (1 + \lambda)S_X(h) - \lambda C - (B - A)\lambda \geq (1 + \lambda)B - \lambda C - (B - A)\lambda > A$ , from the choice of  $\lambda$ ; hence  $x \in X'$ . Again from (1),  $\lambda g(x) > \lambda S_X(h) - (B - A)\lambda \geq \lambda A$  from which  $g(x) > A$ . We have proved that  $\inf S_{X'}(G) \geq A$ , as required.

3. THEOREM. *We suppose  $X \neq \emptyset$ ,  $Y$  is infinite,*

$$f: X \times Y \rightarrow [-\infty, \infty),$$

*$B, C \in R, \delta > 0$  and, for all  $g \in \text{conv } f(\cdot, Y)$ ,  $B \leq S_X(g) \leq C$ . Then there exist  $x_1, x_2, \dots \in X$  and distinct  $y_1, y_2, \dots \in Y$  such that  $f(x_p, y_m) \geq$*

$B - S$  whenever  $1 \leq m \leq p$ .

*Proof.* From Lemma 2, there exists  $g_1 \in \text{conv } f(\cdot, Y)$  such that if  $X_1 = \{x: x \in X, g_1(x) > B - \delta/2\}$  then  $\inf S_{X_1}(\text{conv } f(\cdot, Y)) \geq B - \delta/2$ . There exists finite  $F_1 \subset Y$  such that  $g_1 \in \text{conv } f(\cdot, F_1)$ . Clearly  $\inf S_{X_1}(\text{conv } f(\cdot, Y \setminus F_1)) \geq B - \delta/2$ . Proceeding inductively we find  $g_n \in \text{conv } f(\cdot, Y \setminus F_1 \setminus \dots \setminus F_{n-1})$  such that, if

$$X_n = \{x: x \in X_{n-1}, g_n(x) > B - \delta/2 - \dots - \delta/2^n\}$$

then  $\inf S_{X_n}(\text{conv } f(\cdot, Y \setminus F_1 \setminus \dots \setminus F_{n-1})) \geq B - \delta/2 - \dots - \delta/2^n$  and finite  $F_n \subset Y \setminus F_1 \setminus \dots \setminus F_{n-1}$  such that  $g_n \in \text{conv } f(\cdot, F_n)$ . In this way we obtain a family  $\{F_n: n \geq 1\}$  of disjoint finite subsets of  $Y$ ,  $g_n \in \text{conv } f(\cdot, F_n)$  such that, for all  $n \geq 1$ ,

$$\bigcap_{m \leq n} \{x: x \in X, g_m(x) \geq B - \delta\} (\supset X_n) \neq \emptyset .$$

Since  $g_1 \in \text{conv } f(\cdot, F_1)$ ,

$$\{x: x \in X, g_1(x) \geq B - \delta\} \subset \bigcup_{y \in F_1} \{x: x \in X, f(x, y) \geq B - \delta\}$$

hence there exists  $y_1 \in F_1$  such that, for arbitrarily large  $n \geq 2$ ,

$$\{x: x \in X, f(x, y_1) \geq B - \delta\} \cap \bigcap_{2 \leq m \leq n} \{x: x \in X, g_m(x) \geq B - \delta\} \neq \emptyset .$$

This relationship must clearly then hold for all  $n \geq 2$ . Proceeding inductively we find  $y_n \in F_n$  such that, for all  $1 \leq p < n$ ,

$$\bigcap_{m \leq p} \{x: x \in X, f(x, y_m) \geq B - \delta\} \cap \bigcap_{p+1 \leq m \leq n} \{x: x \in X, g_m(x) \geq B - \delta\} \neq \emptyset$$

and so, in particular,

$$\bigcap_{m \leq p} \{x: x \in X, f(x, y_m) \geq B - \delta\} \neq \emptyset$$

from which the required result follows ( $y_1, y_2, \dots$  are distinct because  $F_1, F_2, \dots$  disjoint).

*Ptak's Lemma.* We suppose that  $Y$  is an infinite set and that  $X$  is a nonvoid family of subsets of  $Y$ . We write  $P(Y)$  for the collection of all positive, real valued functions  $\lambda$  on  $Y$  such that  $\{y: y \in Y, \lambda(y) > 0\}$  is finite and  $\sum_{y \in Y} \lambda(y) = 1$ ; for  $x \subset Y$  we write  $\lambda(x) = \sum_{y \in x} \lambda(y)$ . If

$$B = \inf_{\lambda \in P(Y)} \sup_{x \in X} \lambda(x) > 0$$

then there exist  $x_1, x_2, \dots \in X$  and distinct  $y_1, y_2, \dots \in Y$  such that, for each  $p \geq 1$ ,  $\{y_1, \dots, y_p\} \subset x_p$ .

*Proof.* We define  $f: X \times Y \rightarrow R$  by  $f(x, y) = 1$  if, and only if,  $y \in x$ . By hypothesis,  $\inf S_x(\text{conv } f(\cdot, Y)) = B$ . From Theorem 3 with  $\delta = B/2$ , there exist  $x_1, x_2, \dots \in X$  and distinct  $y_1, y_2, \dots \in Y$  such that  $f(x_p, y_m) \geq B/2$ , hence  $y_m \in x_p$ , whenever  $1 \leq m \leq p$ .

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