

Pacific Journal of Mathematics

**COMPLETIONS OF BOOLEAN ALGEBRAS WITH PARTIALLY
ADDITIVE OPERATORS**

YEN-YI WU

COMPLETIONS OF BOOLEAN ALGEBRAS WITH PARTIALLY ADDITIVE OPERATORS

YEN-YI WU

To generalize a result of Jónsson and Tarski on perfect extensions of Boolean algebras with operators, L. Henkin has introduced the notion of p -additive operation for p a positive integer. Here we use this notion to extend the analogous result of D. Monk which states that each equation without occurrences of the complementation sign has its validity preserved when passing from a Boolean algebra with operators to its completion.

We first point out very briefly the basic notions and results from [1], [2], or [3] needed in the sequel. Then the theory of completions of Boolean algebras with p_i -additive operators, f_i , is developed following the pattern of [3].

1. A Boolean algebra $\mathfrak{B} = \langle B, +, \circ, -, 0, 1 \rangle$ is a completion of a Boolean algebra $\mathfrak{A} = \langle A, +, \circ, -, 0, 1 \rangle$ if (i) \mathfrak{A} is a subalgebra of \mathfrak{B} , (ii) for each subset X of A such that $\Sigma_{x \in X}^{\mathfrak{A}} x$ exists in A , $\Sigma_{x \in X}^{\mathfrak{B}} x$ exists in B and $\Sigma_{x \in X}^{\mathfrak{B}} x = \Sigma_{x \in X}^{\mathfrak{A}} x$, (iii) \mathfrak{B} is the least complete Boolean algebra having \mathfrak{A} as a subalgebra. It is well known that every Boolean algebra \mathfrak{A} has such a completion \mathfrak{B} and that for every element x in B , $x = \Sigma_{x \geq y \in A} y$.

${}^n A$ denotes the set of all n -termed sequences $x = \langle x_0, \dots, x_{n-1} \rangle$ of elements of A . We write, for $x, y \in {}^n A$, $x \leq y$ if $x_i \leq y_i$ for each $i < n$. Furthermore, if $j < n$ and $x, y \in {}^n A$, $x = {}_j y$ means that $x_k = y_k$ for all $k < n$ and $k \neq j$. For p a positive integer and $X \subseteq {}^n A$, $\sigma_p X$ denotes $\{y \in {}^n A: y = x^0 + \dots + x^{p-1} \text{ for some } x^0, \dots, x^{p-1} \in X\}$.

An operation f on ${}^n A$ to A is (i) monotonic if, given any $x, y \in {}^n A$ such that $x \leq y$, we always have $fx \leq fy$, (ii) p -additive if, whenever $X \subseteq {}^n A$ has cardinal number $\leq p + 1$ and there is some $j < n$ such that $x = {}_j y$ for all $x, y \in X$, we always have

$$f(\Sigma X) = \Sigma \{fz: z \in \sigma_p X\},$$

(iii) completely p -additive if, whenever $X \subseteq {}^n A$, ΣX exists in ${}^n A$ and there is some $j < n$ such that $x = {}_j y$ for all $x, y \in X$, then $\Sigma \{fz: z \in \sigma_p X\}$ exists and equals $f(\Sigma X)$. $\Phi_p(\mathfrak{A})$ (or Φ_p if no confusion occurs) denotes the set of all p -additive operations on \mathfrak{A} , $\Phi_p^c(\mathfrak{A})$ that of all completely p -additive operations on \mathfrak{A} , we write Φ_ω for $\bigcup_{p \geq 1} \Phi_p$ and Φ_ω^c for $\bigcup_{p \geq 1} \Phi_p^c$. It is clear from the definition that $\Phi_p^c \subseteq \Phi_p$ for each positive integer

p. The basic result that if $f \in \Phi_\omega$ then f is monotonic is proved in [1] (Theorem 2.3).

Except when stated otherwise we assume hereafter that \mathfrak{A} and \mathfrak{B} are Boolean algebras, \mathfrak{B} is a completion of \mathfrak{A} and f is an n -ary operation on \mathfrak{A} . An operation g on ${}^n B$ to B is said to be an extension of f if for all $x \in {}^n A$ $fx = gx$. $g \upharpoonright {}^n A$ denotes the restriction of g to ${}^n A$. Given an operation f on \mathfrak{A} , Monk has defined in [3] an n -ary operation f^+ on \mathfrak{B} by

$$f^+x = \Sigma \{fy : x \geq y \in {}^n A\}$$

for any $x \in B$. It is obvious from this definition that f^+ is monotonic, and that f^+ is an extension of f if f is monotonic.

2. First of all we modify an example in 2.6 of [1] so that it will later be clear that our main theorem is indeed an extension of Theorem 1.9 of [3]. Let A be the set of all finite or cofinite subsets of ${}^2\omega$. Define f on A by $fx = x; x$ for all $x \in A$ (here $x; x$ is the relative product of the relation x with itself, so that for any $i, j \in \omega$, we have $\langle i, j \rangle \in fx$ if and only if there is some k such that $\langle i, k \rangle \in x$ and $\langle k, j \rangle \in x$). f is then an operation on A since fx is finite when x is finite and $fx = {}^2\omega$ when x is cofinite. We claim that $f \in \Phi_2^c$: Let $X \subseteq A$ and $\bigcup X$ exist in A . Then $f(\bigcup X) \supseteq fy$ for each $y \in \sigma_2 X$ since f is obviously monotonic, so $f(\bigcup X) \supseteq \bigcup_{y \in \sigma_2 X} fy$. But also if $\langle i, j \rangle \in f(\bigcup X)$, then there is a $k \in \omega$ such that $\langle i, k \rangle \in \bigcup X$ and $\langle k, j \rangle \in \bigcup X$, hence $\langle i, k \rangle \in x$ for some $x \in X$ and $\langle k, j \rangle \in x'$ for some $x' \in X$, and therefore $\langle i, j \rangle \in (x \cup x')$; $(x \cup x')$, hence $\langle i, j \rangle \in \bigcup_{y \in \sigma_2 X} fy$, so that $f(\bigcup X) \subseteq \bigcup_{y \in \sigma_2 X} fy$. However, f is not in Φ_1 , for let $x = \{\langle 0, 1 \rangle\}$ and $y = \{\langle 1, 2 \rangle\}$; then $fx = fy = \phi$, but $f(x \cup y) = \{\langle 0, 2 \rangle\}$.

THEOREM 1. *If $f \in \Phi_p^c(\mathfrak{A})$ then $f^+ \in \Phi_p^c(\mathfrak{B})$.*

Proof. Suppose $f \in \Phi_p^c(\mathfrak{A})$ and $X \subseteq {}^n B$ such that for some $j < n$ we have $x =_j y$ for all $x, y \in X$. We must show that

$$f^+(\Sigma X) = \Sigma \{f^+z : z \in \sigma_p X\}.$$

Since f^+ is monotonic we have, obviously,

$$(1) \quad f^+(\Sigma X) \supseteq \Sigma \{f^+z : z \in \sigma_p X\}.$$

Let $v \in {}^n A$ be such that $v \leq \Sigma X$. Then $v_j \leq (\Sigma X)_j = \Sigma_{x \in X} x_j = \Sigma_{x \in X} \Sigma_{x_i \geq w \in A} w$. For each $x \in X$ and $w \in A$ with $w \leq x_j$, we now define an n -sequence $v^{xw} \in {}^n A$ by $v_k^{xw} = v_k$ if $k \neq j$ and $v_j^{xw} = v_j \cdot w$, and note that $v^{xw} \leq x$. Then we have $v = \{v^{xw} : x \in X \text{ and } w \leq x_j\}$, hence by the complete p -additivity of f , we get

$$fv = \Sigma \{fy: y \in \sigma_p\{v^{xw}: x \in X \text{ and } w \leq x_j\}\} .$$

Let now $y \in \sigma_p\{v^{xw}: x \in X \text{ and } w \leq x_j\}$. Then we have $y \in {}^nA$ and

$$y = v^{x^0w^0} + \dots + v^{x^{p-1}w^{p-1}}$$

for some $x^0, \dots, x^{p-1} \in X$ and $w^0, \dots, w^{p-1} \in A$, where for each $i < p$, $w^i \leq x_j^i$. Therefore $y \leq x^0 + \dots + x^{p-1}$, and hence

$$fy = f^+y \leq f^+(x^0 + \dots + x^{p-1}) \leq \Sigma \{f^+z: z \in \sigma_p X\} .$$

Since this holds for each $y \in \sigma_p\{v^{xw}: x \in X \text{ and } w \leq x_j\}$, we have $fv \leq \Sigma \{f^+z: z \in \sigma_p X\}$, and since this inclusion holds for each $v \in {}^nA$ such that $v \leq \Sigma X$, we get

$$(2) \quad \Sigma \{f^+z: z \in \sigma_p X\} \geq \Sigma \{fv: \Sigma X \geq v \in {}^nA\} = f^+(\Sigma X) .$$

With (1) and (2) the proof is completed.

The assumption of Theorem 1 that f is completely p -additive cannot be weakened to $f \in \Phi_p$:

THEOREM 2. *If $f \in \Phi_p^c(\mathfrak{B})$ and $f \upharpoonright {}^nA$ is an operation on A then $f \upharpoonright {}^nA \in \Phi_p^c(\mathfrak{A})$.*

Proof. This is immediate from the definition of complete p -additivity and the fact that the sum is preserved from \mathfrak{A} to \mathfrak{B} .

LEMMA 3. *If p is any positive integer and $x \in {}^nB$, then*

$$\sigma_p\{y \in {}^nA: y \leq x\} = \{y \in {}^nA: y \leq x\} .$$

Proof. Obvious.

THEOREM 4. *If $f \in \Phi_p^c(\mathfrak{B})$ and $f \upharpoonright {}^nA$ is an operation on A , then $f = (f \upharpoonright {}^nA)^+$.*

Proof. For any $x \in {}^nB$, we have

$$fx = f(x_0, \dots, x_{n-1}) = f(\Sigma_{x_0 \geq y_0 \in A} y_0, \dots, \Sigma_{x_{n-1} \geq y_{n-1} \in A} y_{n-1}) .$$

Using repeatedly the fact that f is completely p -additive, we get

$$fx = \Sigma_{y_0 \in \sigma_p\{y_0 \in A: y_0 \leq x_0\}, \dots, \Sigma_{y_{n-1} \in \sigma_p\{y_{n-1} \in A: y_{n-1} \leq x_{n-1}\}} fy$$

and then, by Lemma 3,

$$\begin{aligned} fx &= \Sigma_{x_0 \geq y_0 \in A, \dots, \Sigma_{x_{n-1} \geq y_{n-1} \in A} fy = \Sigma_{x \geq y \in {}^nA} fy \\ &= \Sigma_{x \geq y \in {}^nA} (f \upharpoonright {}^nA)y = (f \upharpoonright {}^nA)^+x \end{aligned}$$

as desired.

As in [3] it follows now that each completely p -additive operation on \mathfrak{A} has exactly one extension which is a completely p -additive operation on \mathfrak{B} , and so there is a one-one correspondence between the set of completely p -additive operations on \mathfrak{A} and the set of the ones on \mathfrak{B} which extend those on \mathfrak{A} .

Also established as in [3] is:

THEOREM 5.

- (i) $+^+ = +$.
- (ii) $\cdot^+ = \cdot$.
- (iii) If $f = A \times \{a\}$, then $f^+ = B \times \{a\}$.
- (iv) If $fx = x_i$ for each $x \in {}^nA$ (where $i < n$), then $f^+x = x_i$ for each $x \in {}^nB$.

If f is any m -ary operation and g_0, \dots, g_{m-1} are n -ary operations on A , one composes them to obtain the operation $f[g_0, \dots, g_{m-1}]$, i.e., the n -ary operation h such that $hx = f(g_0x, \dots, g_{m-1}x)$ for every $x \in {}^nA$.

THEOREM 6. If f is m -ary, $f \in \Phi_p^c(\mathfrak{A})$ and g_0, \dots, g_{m-1} are n -ary monotonic operations on A , then

$$(f[g_0, \dots, g_{m-1}])^+ = f^+[g_0^+, \dots, g_{m-1}^+].$$

Proof. Assume that the conditions of the theorem hold. If $x \in {}^nB$, we then have, as in the proof of Theorem 1.8 of [3],

$$f^+[g_0^+, \dots, g_{m-1}^+]x \geq (f[g_0, \dots, g_{m-1}])^+x.$$

Also

$$f^+[g_0^+, \dots, g_{m-1}^+]x = f^+(\sum_{x \geq y^0 \in {}^nA} g_0y^0, \dots, \sum_{x \geq y^{m-1} \in {}^nA} g_{m-1}y^{m-1}).$$

By Theorem 1 we have $f^+ \in \Phi_p^c(\mathfrak{B})$ and using repeatedly this fact, we get

$$f^+[g_0^+, \dots, g_{m-1}^+]x = \sum_{u_0 \in \sigma_p\{g_0y^0: x \geq y^0 \in {}^nA\}, \dots, \sum_{u_{m-1} \in \sigma_p\{g_{m-1}y^{m-1}: x \geq y^{m-1} \in {}^nA\}} f^+u.$$

Now if $u \in {}^nA$ is such that for each $k < m$, $u_k \in \sigma_p\{g_ky^k: x \geq y^k \in {}^nA\}$, then $u_k = \sum_{i < p} g_ky^{k,i}$ where for each $i < p$, $y^{k,i} \in {}^nA$ and $y^{k,i} \leq x$. If $z = \sum\{y^{k,i}: k < m \text{ and } i < p\}$ then $z \in {}^nA$ and $z \leq x$. For $k < m$ we have $g_kz \geq g_ky^{k,i}$ for all $i < p$, hence $g_kz \geq \sum_{i < p} g_ky^{k,i} = u_k$ by monotonicity of g_k . Thus

$$\begin{aligned} f^+u &\leq f^+(g_0z, \dots, g_{m-1}z) \leq \sum\{f(g_0z, \dots, g_{m-1}z): x \geq z \in {}^nA\} \\ &= (f[g_0, \dots, g_{m-1}])^+x. \end{aligned}$$

Since this inclusion holds for each u with $u_k \in \sigma_p\{g_k y^k: x \geq y^k \in {}^n A\}$ for each $k < m$, we have $f^+[g_0^+, \dots, g_{m-1}^+]x \leq (f[g_0, \dots, g_{m-1}]^+)^+x$, and this completes our proof.

In Theorem 6 the condition that $f \in \Phi_p^c$ cannot be replaced by $f \in \Phi_p$, as the example following Theorem 1.7 of [3] shows.

THEOREM 7. *Let $f_0, \dots, f_{k-1} \in \Phi_\omega^c(\mathfrak{A})$ and let $\tau(f_0, \dots, f_{k-1}) = \rho(f_0, \dots, f_{k-1})$ be an equation which holds for all $x \in {}^n A$. Then the corresponding equation $\tau(f_0^+, \dots, f_{k-1}^+) = \rho(f_0^+, \dots, f_{k-1}^+)$ holds for all $x \in {}^n B$.*

The proof of Theorem 7 is similar to that of 3.8 of [1] except we use Theorems 5 and 6 here.

We adopt terminology slightly different from that in [1] and say that a system $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, f_i \rangle_{i \in I}$ is a *Boolean algebra with partially additive operators* if $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra and $f_i \in \Phi_\omega$ for each $i \in I$, that \mathfrak{A} is *completely partially additive* if $f_i \in \Phi_\omega^c$ for each $i \in I$, and that \mathfrak{A} is *complete* if \mathfrak{A} is completely partially additive and $BL\mathfrak{A}$ (the Boolean part of \mathfrak{A}) is complete. We may now extend the notion of completion to Boolean algebras with partially additive operators and call a system

$$\mathfrak{B} = \langle B, +, \cdot, -, 0, 1, g_i \rangle_{i \in I}$$

a *completion* of a Boolean algebra with partially additive operators $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, f_i \rangle_{i \in I}$ in case $BL\mathfrak{B}$ is completion of $BI\mathfrak{A}$ and for each $i \in I$, $g_i = f_i^+$. Theorem 2 then yields:

THEOREM 8. *If \mathfrak{A} is a Boolean algebra with partially additive operators which is completely partially additive, then there is a completion of \mathfrak{A} which is complete.*

If we associate an equational logic $L_{\mathfrak{A}}$ with a class to which a given Boolean algebra with partially additive operators \mathfrak{A} belongs, and call a term σ of $L_{\mathfrak{A}}$ positive if the complementation sign does not occur in σ , and an equation $\tau = \rho$ positive, if both τ and ρ are positive, then we immediately obtain the following extensions of other of Monk's theorems:

THEOREM 9. *If \mathfrak{B} is a completion of a completely partially additive Boolean algebra \mathfrak{A} , then a positive equation $\tau = \rho$ holds in \mathfrak{A} if and only if it holds in \mathfrak{B} .*

THEOREM 10. *With \mathfrak{A} and \mathfrak{B} as in Theorem 9, if Γ is a conjunction or disjunction of formulas of the form $\sigma = 0$ or $\sigma \neq 0$ where*

σ is positive, and if τ and ρ are positive, then $\Gamma \rightarrow \tau = \rho$ holds in \mathfrak{A} if and only if it holds in \mathfrak{B} .

Finally, Theorem 1.12 of [3] can also be extended to

THEOREM 11. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be Boolean algebras with partially additive operators, \mathfrak{A} completely partially additive, \mathfrak{B} a completion of \mathfrak{A} , \mathfrak{A} a subalgebra of \mathfrak{C} , \mathfrak{C} complete and $BL\mathfrak{A}$ a regular subalgebra of $BL\mathfrak{C}$ (i.e., a subalgebra for which the sum is preserved from \mathfrak{A} to \mathfrak{C}). Then there is an isomorphism f from \mathfrak{B} into \mathfrak{C} such that $Id \upharpoonright A \subset f$ (where Id is the identity map).*

Proof. As in the proof of Theorem 1.12 of [3], if we define $fb = \sum_{b \geq a \in A} a$ for any $b \in B$, then f is a complete Boolean isomorphism into, and $Id \upharpoonright A \subseteq f$. To show that f preserves non-Boolean operations, we may then use Theorem 2.8 of [1] and our Lemma 3.

REFERENCES

1. L. Henkin, *Extending Boolean operations*, Pacific J. Math., **32**, no. 3, (1970), 723-752.
2. B. Jónsson, A. Tarski, *Boolean algebras with operators I*, Amer. J. Math., **73** (1951), 891-939.
3. D. Monk, *Completions of Boolean algebras with operators*, Mathematische Nachrichten, Bd46, H 1-6 (1970), 47-55.

Received January 11, 1971 and in revised form April 12, 1971. The author wishes to express his gratitude to Professor H. Ribeiro for the assistance and advice during work on this paper.

THE PENNSYLVANIA STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON

Stanford University
Stanford, California 94305

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY

University of Washington
Seattle, Washington 98105

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index. to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 40, No. 3

November, 1972

Wazir Husan Abdi, <i>A quasi-Kummer function</i>	521
Vasily Cateforis, <i>Minimal injective cogenerators for the class of modules of zero singular submodule</i>	527
W. Wistar (William) Comfort and Anthony Wood Hager, <i>Cardinality of k-complete Boolean algebras</i>	541
Richard Brian Darst and Gene Allen DeBoth, <i>Norm convergence of martingales of Radon-Nikodym derivatives given a σ-lattice</i>	547
M. Edelstein and Anthony Charles Thompson, <i>Some results on nearest points and support properties of convex sets in c_0</i>	553
Richard Goodrick, <i>Two bridge knots are alternating knots</i>	561
Jean-Pierre Gossez and Enrique José Lami Dozo, <i>Some geometric properties related to the fixed point theory for nonexpansive mappings</i>	565
Dang Xuan Hong, <i>Covering relations among lattice varieties</i>	575
Carl Groos Jockusch, Jr. and Robert Irving Soare, <i>Degrees of members of Π_1^0 classes</i>	605
Leroy Milton Kelly and R. Rottenberg, <i>Simple points in pseudoline arrangements</i>	617
Joe Eckley Kirk, Jr., <i>The uniformizing function for a class of Riemann surfaces</i>	623
Glenn Richard Luecke, <i>Operators satisfying condition (G_1) locally</i>	629
T. S. Motzkin, <i>On $L(S)$-tuples and l-pairs of matrices</i>	639
Charles Estep Murley, <i>The classification of certain classes of torsion free Abelian groups</i>	647
Louis D. Nel, <i>Lattices of lower semi-continuous functions and associated topological spaces</i>	667
David Emroy Penney, II, <i>Establishing isomorphism between tame prime knots in E^3</i>	675
Daniel Rider, <i>Functions which operate on $\mathbb{F}L_p(T)$, $1 < p < 2$</i>	681
Thomas Stephen Shores, <i>Injective modules over duo rings</i>	695
Stephen Simons, <i>A convergence theorem with boundary</i>	703
Stephen Simons, <i>Maximinimax, minimax, and antiminimax theorems and a result of R. C. James</i>	709
Stephen Simons, <i>On Ptak's combinatorial lemma</i>	719
Stuart A. Steinberg, <i>Finitely-valued f-modules</i>	723
Pui-kei Wong, <i>Integral inequalities of Wirtinger-type and fourth-order elliptic differential inequalities</i>	739
Yen-Yi Wu, <i>Completions of Boolean algebras with partially additive operators</i>	753
Phillip Lee Zenor, <i>On spaces with regular G_δ-diagonals</i>	759