

# Pacific Journal of Mathematics

**COMPLETIONS OF BOOLEAN ALGEBRAS WITH PARTIALLY  
ADDITIVE OPERATORS**

YEN-YI WU

## COMPLETIONS OF BOOLEAN ALGEBRAS WITH PARTIALLY ADDITIVE OPERATORS

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**To generalize a result of Jónsson and Tarski on perfect extensions of Boolean algebras with operators, L. Henkin has introduced the notion of  $p$ -additive operation for  $p$  a positive integer. Here we use this notion to extend the analogous result of D. Monk which states that each equation without occurrences of the complementation sign has its validity preserved when passing from a Boolean algebra with operators to its completion.**

We first point out very briefly the basic notions and results from [1], [2], or [3] needed in the sequel. Then the theory of completions of Boolean algebras with  $p_i$ -additive operators,  $f_i$ , is developed following the pattern of [3].

1. A Boolean algebra  $\mathfrak{B} = \langle B, +, \circ, -, 0, 1 \rangle$  is a completion of a Boolean algebra  $\mathfrak{A} = \langle A, +, \circ, -, 0, 1 \rangle$  if (i)  $\mathfrak{A}$  is a subalgebra of  $\mathfrak{B}$ , (ii) for each subset  $X$  of  $A$  such that  $\Sigma_{x \in X}^{\mathfrak{A}} x$  exists in  $A$ ,  $\Sigma_{x \in X}^{\mathfrak{B}} x$  exists in  $B$  and  $\Sigma_{x \in X}^{\mathfrak{B}} x = \Sigma_{x \in X}^{\mathfrak{A}} x$ , (iii)  $\mathfrak{B}$  is the least complete Boolean algebra having  $\mathfrak{A}$  as a subalgebra. It is well known that every Boolean algebra  $\mathfrak{A}$  has such a completion  $\mathfrak{B}$  and that for every element  $x$  in  $B$ ,  $x = \Sigma_{x \geq y \in A} y$ .

${}^n A$  denotes the set of all  $n$ -termed sequences  $x = \langle x_0, \dots, x_{n-1} \rangle$  of elements of  $A$ . We write, for  $x, y \in {}^n A$ ,  $x \leq y$  if  $x_i \leq y_i$  for each  $i < n$ . Furthermore, if  $j < n$  and  $x, y \in {}^n A$ ,  $x = {}_j y$  means that  $x_k = y_k$  for all  $k < n$  and  $k \neq j$ . For  $p$  a positive integer and  $X \subseteq {}^n A$ ,  $\sigma_p X$  denotes  $\{y \in {}^n A: y = x^0 + \dots + x^{p-1} \text{ for some } x^0, \dots, x^{p-1} \in X\}$ .

An operation  $f$  on  ${}^n A$  to  $A$  is (i) monotonic if, given any  $x, y \in {}^n A$  such that  $x \leq y$ , we always have  $fx \leq fy$ , (ii)  $p$ -additive if, whenever  $X \subseteq {}^n A$  has cardinal number  $\leq p + 1$  and there is some  $j < n$  such that  $x = {}_j y$  for all  $x, y \in X$ , we always have

$$f(\Sigma X) = \Sigma \{fz: z \in \sigma_p X\},$$

(iii) completely  $p$ -additive if, whenever  $X \subseteq {}^n A$ ,  $\Sigma X$  exists in  ${}^n A$  and there is some  $j < n$  such that  $x = {}_j y$  for all  $x, y \in X$ , then  $\Sigma \{fz: z \in \sigma_p X\}$  exists and equals  $f(\Sigma X)$ .  $\Phi_p(\mathfrak{A})$  (or  $\Phi_p$  if no confusion occurs) denotes the set of all  $p$ -additive operations on  $\mathfrak{A}$ ,  $\Phi_p^c(\mathfrak{A})$  that of all completely  $p$ -additive operations on  $\mathfrak{A}$ , we write  $\Phi_\omega$  for  $\bigcup_{p \geq 1} \Phi_p$  and  $\Phi_\omega^c$  for  $\bigcup_{p \geq 1} \Phi_p^c$ . It is clear from the definition that  $\Phi_p^c \subseteq \Phi_p$  for each positive integer

$p$ . The basic result that if  $f \in \Phi_\omega$  then  $f$  is monotonic is proved in [1] (Theorem 2.3).

Except when stated otherwise we assume hereafter that  $\mathfrak{A}$  and  $\mathfrak{B}$  are Boolean algebras,  $\mathfrak{B}$  is a completion of  $\mathfrak{A}$  and  $f$  is an  $n$ -ary operation on  $\mathfrak{A}$ . An operation  $g$  on  ${}^n B$  to  $B$  is said to be an extension of  $f$  if for all  $x \in {}^n A$   $fx = gx$ .  $g \upharpoonright {}^n A$  denotes the restriction of  $g$  to  ${}^n A$ . Given an operation  $f$  on  $\mathfrak{A}$ , Monk has defined in [3] an  $n$ -ary operation  $f^+$  on  $\mathfrak{B}$  by

$$f^+x = \Sigma \{fy: x \geq y \in {}^n A\}$$

for any  $x \in B$ . It is obvious from this definition that  $f^+$  is monotonic, and that  $f^+$  is an extension of  $f$  if  $f$  is monotonic.

2. First of all we modify an example in 2.6 of [1] so that it will later be clear that our main theorem is indeed an extension of Theorem 1.9 of [3]. Let  $A$  be the set of all finite or cofinite subsets of  ${}^2\omega$ . Define  $f$  on  $A$  by  $fx = x; x$  for all  $x \in A$  (here  $x; x$  is the relative product of the relation  $x$  with itself, so that for any  $i, j \in \omega$ , we have  $\langle i, j \rangle \in fx$  if and only if there is some  $k$  such that  $\langle i, k \rangle \in x$  and  $\langle k, j \rangle \in x$ ).  $f$  is then an operation on  $A$  since  $fx$  is finite when  $x$  is finite and  $fx = {}^2\omega$  when  $x$  is cofinite. We claim that  $f \in \Phi_2^c$ : Let  $X \subseteq A$  and  $\bigcup X$  exist in  $A$ . Then  $f(\bigcup X) \supseteq fy$  for each  $y \in \sigma_2 X$  since  $f$  is obviously monotonic, so  $f(\bigcup X) \supseteq \bigcup_{y \in \sigma_2 X} fy$ . But also if  $\langle i, j \rangle \in f(\bigcup X)$ , then there is a  $k \in \omega$  such that  $\langle i, k \rangle \in \bigcup X$  and  $\langle k, j \rangle \in \bigcup X$ , hence  $\langle i, k \rangle \in x$  for some  $x \in X$  and  $\langle k, j \rangle \in x'$  for some  $x' \in X$ , and therefore  $\langle i, j \rangle \in (x \cup x')$ ;  $(x \cup x')$ , hence  $\langle i, j \rangle \in \bigcup_{y \in \sigma_2 X} fy$ , so that  $f(\bigcup X) \subseteq \bigcup_{y \in \sigma_2 X} fy$ . However,  $f$  is not in  $\Phi_1$ , for let  $x = \{\langle 0, 1 \rangle\}$  and  $y = \{\langle 1, 2 \rangle\}$ ; then  $fx = fy = \phi$ , but  $f(x \cup y) = \{\langle 0, 2 \rangle\}$ .

**THEOREM 1.** *If  $f \in \Phi_p^c(\mathfrak{A})$  then  $f^+ \in \Phi_p^c(\mathfrak{B})$ .*

*Proof.* Suppose  $f \in \Phi_p^c(\mathfrak{A})$  and  $X \subseteq {}^n B$  such that for some  $j < n$  we have  $x = {}_j y$  for all  $x, y \in X$ . We must show that

$$f^+(\Sigma X) = \Sigma \{f^+z: z \in \sigma_p X\}.$$

Since  $f^+$  is monotonic we have, obviously,

$$(1) \quad f^+(\Sigma X) \supseteq \Sigma \{f^+z: z \in \sigma_p X\}.$$

Let  $v \in {}^n A$  be such that  $v \leq \Sigma X$ . Then  $v_j \leq (\Sigma X)_j = \Sigma_{x \in X} x_j = \Sigma_{x \in X} \Sigma_{x_i \geq w \in A} w$ . For each  $x \in X$  and  $w \in A$  with  $w \leq x_j$ , we now define an  $n$ -sequence  $v^{xw} \in {}^n A$  by  $v_k^{xw} = v_k$  if  $k \neq j$  and  $v_j^{xw} = v_j \cdot w$ , and note that  $v^{xw} \leq x$ . Then we have  $v = \{v^{xw}: x \in X \text{ and } w \leq x_j\}$ , hence by the complete  $p$ -additivity of  $f$ , we get

$$fv = \Sigma \{fy: y \in \sigma_p\{v^{xw}: x \in X \text{ and } w \leq x_j\}\} .$$

Let now  $y \in \sigma_p\{v^{xw}: x \in X \text{ and } w \leq x_j\}$ . Then we have  $y \in {}^nA$  and

$$y = v^{x^0w^0} + \dots + v^{x^{p-1}w^{p-1}}$$

for some  $x^0, \dots, x^{p-1} \in X$  and  $w^0, \dots, w^{p-1} \in A$ , where for each  $i < p$ ,  $w^i \leq x_j^i$ . Therefore  $y \leq x^0 + \dots + x^{p-1}$ , and hence

$$fy = f^+y \leq f^+(x^0 + \dots + x^{p-1}) \leq \Sigma \{f^+z: z \in \sigma_p X\} .$$

Since this holds for each  $y \in \sigma_p\{v^{xw}: x \in X \text{ and } w \leq x_j\}$ , we have  $fv \leq \Sigma \{f^+z: z \in \sigma_p X\}$ , and since this inclusion holds for each  $v \in {}^nA$  such that  $v \leq \Sigma X$ , we get

$$(2) \quad \Sigma \{f^+z: z \in \sigma_p X\} \geq \Sigma \{fv: \Sigma X \geq v \in {}^nA\} = f^+(\Sigma X) .$$

With (1) and (2) the proof is completed.

The assumption of Theorem 1 that  $f$  is completely  $p$ -additive cannot be weakened to  $f \in \Phi_p$ :

**THEOREM 2.** *If  $f \in \Phi_p^c(\mathfrak{B})$  and  $f \upharpoonright {}^nA$  is an operation on  $A$  then  $f \upharpoonright {}^nA \in \Phi_p^c(\mathfrak{A})$ .*

*Proof.* This is immediate from the definition of complete  $p$ -additivity and the fact that the sum is preserved from  $\mathfrak{A}$  to  $\mathfrak{B}$ .

**LEMMA 3.** *If  $p$  is any positive integer and  $x \in {}^nB$ , then*

$$\sigma_p\{y \in {}^nA: y \leq x\} = \{y \in {}^nA: y \leq x\} .$$

*Proof.* Obvious.

**THEOREM 4.** *If  $f \in \Phi_p^c(\mathfrak{B})$  and  $f \upharpoonright {}^nA$  is an operation on  $A$ , then  $f = (f \upharpoonright {}^nA)^+$ .*

*Proof.* For any  $x \in {}^nB$ , we have

$$fx = f(x_0, \dots, x_{n-1}) = f(\Sigma_{x_0 \geq y_0 \in A} y_0, \dots, \Sigma_{x_{n-1} \geq y_{n-1} \in A} y_{n-1}) .$$

Using repeatedly the fact that  $f$  is completely  $p$ -additive, we get

$$fx = \Sigma_{y_0 \in \sigma_p\{y_0 \in A: y_0 \leq x_0\}} \dots \Sigma_{y_{n-1} \in \sigma_p\{y_{n-1} \in A: y_{n-1} \leq x_{n-1}\}} f y$$

and then, by Lemma 3,

$$\begin{aligned} fx &= \Sigma_{x_0 \geq y_0 \in A} \dots \Sigma_{x_{n-1} \geq y_{n-1} \in A} f y = \Sigma_{x \geq y \in {}^nA} f y \\ &= \Sigma_{x \geq y \in {}^nA} (f \upharpoonright {}^nA) y = (f \upharpoonright {}^nA)^+ x \end{aligned}$$

as desired.

As in [3] it follows now that each completely  $p$ -additive operation on  $\mathfrak{A}$  has exactly one extension which is a completely  $p$ -additive operation on  $\mathfrak{B}$ , and so there is a one-one correspondence between the set of completely  $p$ -additive operations on  $\mathfrak{A}$  and the set of the ones on  $\mathfrak{B}$  which extend those on  $\mathfrak{A}$ .

Also established as in [3] is:

**THEOREM 5.**

- (i)  $+^+ = +$ .
- (ii)  $\cdot^+ = \cdot$ .
- (iii) If  $f = A \times \{a\}$ , then  $f^+ = B \times \{a\}$ .
- (iv) If  $fx = x_i$  for each  $x \in {}^nA$  (where  $i < n$ ), then  $f^+x = x_i$  for each  $x \in {}^nB$ .

If  $f$  is any  $m$ -ary operation and  $g_0, \dots, g_{m-1}$  are  $n$ -ary operations on  $A$ , one composes them to obtain the operation  $f[g_0, \dots, g_{m-1}]$ , i.e., the  $n$ -ary operation  $h$  such that  $hx = f(g_0x, \dots, g_{m-1}x)$  for every  $x \in {}^nA$ .

**THEOREM 6.** *If  $f$  is  $m$ -ary,  $f \in \Phi_p^c(\mathfrak{A})$  and  $g_0, \dots, g_{m-1}$  are  $n$ -ary monotonic operations on  $A$ , then*

$$(f[g_0, \dots, g_{m-1}])^+ = f^+[g_0^+, \dots, g_{m-1}^+].$$

*Proof.* Assume that the conditions of the theorem hold. If  $x \in {}^nB$ , we then have, as in the proof of Theorem 1.8 of [3],

$$f^+[g_0^+, \dots, g_{m-1}^+]x \geq (f[g_0, \dots, g_{m-1}])^+x.$$

Also

$$f^+[g_0^+, \dots, g_{m-1}^+]x = f^+(\sum_{x \geq y^0 \in {}^nA} g_0y^0, \dots, \sum_{x \geq y^{m-1} \in {}^nA} g_{m-1}y^{m-1}).$$

By Theorem 1 we have  $f^+ \in \Phi_p^c(\mathfrak{B})$  and using repeatedly this fact, we get

$$f^+[g_0^+, \dots, g_{m-1}^+]x = \sum_{u_0 \in \sigma_p\{g_0y^0: x \geq y^0 \in {}^nA\}, \dots, \sum_{u_{m-1} \in \sigma_p\{g_{m-1}y^{m-1}: x \geq y^{m-1} \in {}^nA\}} f^+u.$$

Now if  $u \in {}^m A$  is such that for each  $k < m$ ,  $u_k \in \sigma_p\{g_k y^k: x \geq y^k \in {}^n A\}$ , then  $u_k = \sum_{i < p} g_k y^{k,i}$  where for each  $i < p$ ,  $y^{k,i} \in {}^n A$  and  $y^{k,i} \leq x$ . If  $z = \sum\{y^{k,i}: k < m \text{ and } i < p\}$  then  $z \in {}^n A$  and  $z \leq x$ . For  $k < m$  we have  $g_k z \geq g_k y^{k,i}$  for all  $i < p$ , hence  $g_k z \geq \sum_{i < p} g_k y^{k,i} = u_k$  by monotonicity of  $g_k$ . Thus

$$\begin{aligned} f^+u &\leq f^+(g_0z, \dots, g_{m-1}z) \leq \sum\{f(g_0z, \dots, g_{m-1}z): x \geq z \in {}^n A\} \\ &= (f[g_0, \dots, g_{m-1}])^+x. \end{aligned}$$

Since this inclusion holds for each  $u$  with  $u_k \in \sigma_p\{g_k y^k: x \geq y^k \in {}^n A\}$  for each  $k < m$ , we have  $f^+[g_0^+, \dots, g_{m-1}^+]x \subseteq (f[g_0, \dots, g_{m-1}]^+x$ , and this completes our proof.

In Theorem 6 the condition that  $f \in \Phi_p^c$  cannot be replaced by  $f \in \Phi_p$ , as the example following Theorem 1.7 of [3] shows.

**THEOREM 7.** *Let  $f_0, \dots, f_{k-1} \in \Phi_\omega^c(\mathfrak{A})$  and let  $\tau(f_0, \dots, f_{k-1}) = \rho(f_0, \dots, f_{k-1})$  be an equation which holds for all  $x \in {}^n A$ . Then the corresponding equation  $\tau(f_0^+, \dots, f_{k-1}^+) = \rho(f_0^+, \dots, f_{k-1}^+)$  holds for all  $x \in {}^n B$ .*

The proof of Theorem 7 is similar to that of 3.8 of [1] except we use Theorems 5 and 6 here.

We adopt terminology slightly different from that in [1] and say that a system  $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, f_i \rangle_{i \in I}$  is a *Boolean algebra with partially additive operators* if  $\langle A, +, \cdot, -, 0, 1 \rangle$  is a Boolean algebra and  $f_i \in \Phi_\omega$  for each  $i \in I$ , that  $\mathfrak{A}$  is *completely partially additive* if  $f_i \in \Phi_\omega^c$  for each  $i \in I$ , and that  $\mathfrak{A}$  is *complete* if  $\mathfrak{A}$  is completely partially additive and  $BL\mathfrak{A}$  (the Boolean part of  $\mathfrak{A}$ ) is complete. We may now extend the notion of completion to Boolean algebras with partially additive operators and call a system

$$\mathfrak{B} = \langle B, +, \cdot, -, 0, 1, g_i \rangle_{i \in I}$$

a *completion* of a Boolean algebra with partially additive operators  $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, f_i \rangle_{i \in I}$  in case  $BL\mathfrak{B}$  is completion of  $BI\mathfrak{A}$  and for each  $i \in I$ ,  $g_i = f_i^+$ . Theorem 2 then yields:

**THEOREM 8.** *If  $\mathfrak{A}$  is a Boolean algebra with partially additive operators which is completely partially additive, then there is a completion of  $\mathfrak{A}$  which is complete.*

If we associate an equational logic  $L_{\mathfrak{A}}$  with a class to which a given Boolean algebra with partially additive operators  $\mathfrak{A}$  belongs, and call a term  $\sigma$  of  $L_{\mathfrak{A}}$  positive if the complementation sign does not occur in  $\sigma$ , and an equation  $\tau = \rho$  positive, if both  $\tau$  and  $\rho$  are positive, then we immediately obtain the following extensions of other of Monk's theorems:

**THEOREM 9.** *If  $\mathfrak{B}$  is a completion of a completely partially additive Boolean algebra  $\mathfrak{A}$ , then a positive equation  $\tau = \rho$  holds in  $\mathfrak{A}$  if and only if it holds in  $\mathfrak{B}$ .*

**THEOREM 10.** *With  $\mathfrak{A}$  and  $\mathfrak{B}$  as in Theorem 9, if  $\Gamma$  is a conjunction or disjunction of formulas of the form  $\sigma = 0$  or  $\sigma \neq 0$  where*

$\sigma$  is positive, and if  $\tau$  and  $\rho$  are positive, then  $\Gamma \rightarrow \tau = \rho$  holds in  $\mathfrak{A}$  if and only if it holds in  $\mathfrak{B}$ .

Finally, Theorem 1.12 of [3] can also be extended to

**THEOREM 11.** *Let  $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$  be Boolean algebras with partially additive operators,  $\mathfrak{A}$  completely partially additive,  $\mathfrak{B}$  a completion of  $\mathfrak{A}$ ,  $\mathfrak{A}$  a subalgebra of  $\mathfrak{C}$ ,  $\mathfrak{C}$  complete and  $BL\mathfrak{A}$  a regular subalgebra of  $BL\mathfrak{C}$  (i.e., a subalgebra for which the sum is preserved from  $\mathfrak{A}$  to  $\mathfrak{C}$ ). Then there is an isomorphism  $f$  from  $\mathfrak{B}$  into  $\mathfrak{C}$  such that  $Id \upharpoonright A \subset f$  (where  $Id$  is the identity map).*

*Proof.* As in the proof of Theorem 1.12 of [3], if we define  $fb = \sum_{b \geq a \in A} a$  for any  $b \in B$ , then  $f$  is a complete Boolean isomorphism into, and  $Id \upharpoonright A \subseteq f$ . To show that  $f$  preserves non-Boolean operations, we may then use Theorem 2.8 of [1] and our Lemma 3.

#### REFERENCES

1. L. Henkin, *Extending Boolean operations*, Pacific J. Math., **32**, no. 3, (1970), 723-752.
2. B. Jónsson, A. Tarski, *Boolean algebras with operators I*, Amer. J. Math., **73** (1951), 891-939.
3. D. Monk, *Completions of Boolean algebras with operators*, Mathematische Nachrichten, Bd46, H 1-6 (1970), 47-55.

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Wazir Husan Abdi, <i>A quasi-Kummer function</i> .....	521
Vasily Cateforis, <i>Minimal injective cogenerators for the class of modules of zero singular submodule</i> .....	527
W. Wistar (William) Comfort and Anthony Wood Hager, <i>Cardinality of <math>k</math>-complete Boolean algebras</i> .....	541
Richard Brian Darst and Gene Allen DeBoth, <i>Norm convergence of martingales of Radon-Nikodym derivatives given a <math>\sigma</math>-lattice</i> .....	547
M. Edelstein and Anthony Charles Thompson, <i>Some results on nearest points and support properties of convex sets in <math>c_0</math></i> .....	553
Richard Goodrick, <i>Two bridge knots are alternating knots</i> .....	561
Jean-Pierre Gossez and Enrique José Lami Dozo, <i>Some geometric properties related to the fixed point theory for nonexpansive mappings</i> .....	565
Dang Xuan Hong, <i>Covering relations among lattice varieties</i> .....	575
Carl Groos Jockusch, Jr. and Robert Irving Soare, <i>Degrees of members of <math>\Pi_1^0</math> classes</i> .....	605
Leroy Milton Kelly and R. Rottenberg, <i>Simple points in pseudoline arrangements</i> .....	617
Joe Eckley Kirk, Jr., <i>The uniformizing function for a class of Riemann surfaces</i> .....	623
Glenn Richard Luecke, <i>Operators satisfying condition <math>(G_1)</math> locally</i> .....	629
T. S. Motzkin, <i>On <math>L(S)</math>-tuples and <math>l</math>-pairs of matrices</i> .....	639
Charles Estep Murley, <i>The classification of certain classes of torsion free Abelian groups</i> .....	647
Louis D. Nel, <i>Lattices of lower semi-continuous functions and associated topological spaces</i> .....	667
David Emroy Penney, II, <i>Establishing isomorphism between tame prime knots in <math>E^3</math></i> .....	675
Daniel Rider, <i>Functions which operate on <math>\mathbb{F}L_p(T)</math>, <math>1 &lt; p &lt; 2</math></i> .....	681
Thomas Stephen Shores, <i>Injective modules over duo rings</i> .....	695
Stephen Simons, <i>A convergence theorem with boundary</i> .....	703
Stephen Simons, <i>Maximinimax, minimax, and antiminimax theorems and a result of R. C. James</i> .....	709
Stephen Simons, <i>On Ptak's combinatorial lemma</i> .....	719
Stuart A. Steinberg, <i>Finitely-valued <math>f</math>-modules</i> .....	723
Pui-kei Wong, <i>Integral inequalities of Wirtinger-type and fourth-order elliptic differential inequalities</i> .....	739
Yen-Yi Wu, <i>Completions of Boolean algebras with partially additive operators</i> .....	753
Phillip Lee Zenor, <i>On spaces with regular <math>G_\delta</math>-diagonals</i> .....	759