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ON SPACES WITH REGULAR G_δ -DIAGONALS

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It is the purpose of this note to investigate spaces with regular G_δ -diagonals. Among other things, it is shown that if X is T_1 -space, then 1. X admits a development satisfying the 3-link property if and only if X is a ω -space with a regular G_δ -diagonal and 2. X is metrizable if and only if X is an M -space with a regular G_δ -diagonal.

Recall that a subset H of the space X is a regular G_δ -set if there is a sequence $\{U_n\}$ of open sets in X such that $H = \bigcap_{i=1}^\infty U_i = \bigcap_{i=1}^\infty U_i^-$. We will say that X has a regular G_δ -diagonal if $\Delta X = \{(x, x): x \in X\}$ is a regular G_δ -set in X^2 .

In [4], Ceder shows that X has a G_δ -diagonal if and only if there is a sequence $\{G_n\}$ of open covers of X such that if x is a point of X , then $x = \bigcap_{i=1}^\infty \text{st}(x, G_i)$. In Theorem 1, we show that there is a similar characterizing property for spaces with regular G_δ -diagonals.

THEOREM 1. *The topological space X has a regular G_δ -diagonal if and only if there is a sequence $\{G_n\}$ of open covers of X such that if x and y are distinct points of X , then there are an integer n and open sets u and v containing x and y respectively such that no member of G_n intersects both u and v .*

Proof. Suppose that X has a regular G_δ -diagonal. Let $\{U_n\}$ be a sequence of open sets in X^2 such that $\Delta X = \bigcap_{i=1}^\infty U_i = \bigcap_{i=1}^\infty U_i^-$. For each n , let $G_n = \{g: g \text{ is an open subset of } X \text{ such that } g \times g \subset U_n\}$. Let x and y be distinct points of X . Let n be an integer such that (x, y) is not in U_n^- . Let u and v be open sets in X that contain x and y respectively such that $u \times v$ does not intersect U_n . To see that no member of G_n intersects both u and v , suppose otherwise; that is, suppose that g is a member of G_n , p is a point of $g \cap u$ and q is a point of $g \cap v$. Then (p, q) is a point of $U_n \cap (u \times v)$ which is a contradiction.

Now, suppose that $\{G_n\}$ is a sequence of open covers of X as described in the theorem. For each n , let $U_n = \bigcup \{(g \times g): g \in G_n\}$. Clearly, $\Delta X \subset \bigcap_{i=1}^\infty U_i$. To see that $\Delta X = \bigcap_{i=1}^\infty U_i^-$, let x and y be distinct points of X . Then there are an integer n and open sets u and v containing x and y respectively such that no member of G_n intersects both u and v . It must be the case that U_n does not intersect $u \times v$.

COROLLARY. *If X has a regular G_δ -diagonal, then X is Hausdorff.*

A development $\{G_n\}$ for the space X is said to satisfy the *3-link property* if it is true that if p and q are distinct points of X , then there is an integer n such that no member of G_n intersects both $\text{st}(x, G_n)$ and $\text{st}(y, G_n)$ (Heath [6]). According to Borges [3], the space X is a $\omega\Delta$ -space if there is a sequence $\{U_n\}$ of open covers of X such that if x is a point and if, for each n , x_n is a point of $\text{st}(x, U_n)$, then the sequence $\{x_n\}$ has a cluster point. Clearly, the class of $\omega\Delta$ -spaces includes the class of strict p -spaces, the class of M -spaces, and the class of developable spaces. It is easy to see that the Niemytski plane (page 100 of [11]) is a non-metrizable Moore space that admits a development satisfying the 3-link property. In [6], Heath establishes the existence of Moore spaces that do not admit developments that satisfy the 3-link property. In [5], Cook asserts that a continuously semi-metrizable space is a Moore space that admits a development that satisfies the 3-link property. Cook's result follows as a corollary to the following theorem:

THEOREM 2. *Let X be a topological space. Then the following conditions are equivalent:*

1. X admits a development satisfying the 3-link property.
2. X is a $\omega\Delta$ -space with a regular G_δ -diagonal. And
3. There is a semi-metric d on X such that:
 - a. If $\{x_n\}$ and $\{y_n\}$ are sequences both converging to x , then $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$, and
 - b. If x and y are distinct points of X and $\{x_n\}$ and $\{y_n\}$ are sequences converging to x and y respectively, then there are integers N and M such that if $n > N$, then $d(x_n, y_n) > 1/M$.

Proof. It is obvious that a developable space is a $\omega\Delta$ -space; thus, that (1) implies (2) is a corollary to Theorem 1.

To see that (2) implies (1), let X be a $\omega\Delta$ -space with a regular G_δ -diagonal. Let $\{U_n\}$ be a sequence of open covers of X as given by the fact that X is a $\omega\Delta$ -space. According to Theorem 1, there is a sequence $\{V_n\}$ of open covers of X such that if p and q are distinct points, then there are an integer n and open sets u and v containing p and q respectively such that no member of V_n intersects both u and v . For each integer n , let G_n be an open cover of X such that (i) G_n refines both U_n and V_n and (ii) G_{n+1} refines G_n . We will show that $\{G_n\}$ is a development for X that satisfies the 3-link property. First, to see that $\{G_n\}$ is a development, suppose the contrary; that is, suppose that there are a point x and an open set u containing x such that, for each n , there is a point p_n in $\text{st}(x, G_n) - u$. Then, for each n , p_n is in $\text{st}(x, U_n)$. Thus, $\{p_n\}$ has a cluster point p . Since for each n , G_n refines each of V_1, \dots, V_n , it follows that there are an

integer N and open sets v and w containing x and p respectively such that if $j > N$, then no member of G_j intersects both v and w . But this is a contradiction since there is a $j < N$ such that p_j is in w . Thus, $\{G_n\}$ is a development for X . To see that G_n satisfies the 3-link property, let p and q be distinct points, u and v open sets containing p and q respectively, and N an integer such that if $n > N$, then no member of G_n meets both u and v . Let S and T be integers such that $\text{st}(p, G_S) \subset u$ and $\text{st}(q, G_T) \subset v$. Let $M = \max\{N, S, T\}$. Then no member of G_M meets both $\text{st}(p, G_M)$ and $\text{st}(q, G_M)$.

(1) implies (3): Let $\{G_n\}$ be a development satisfying the 3-link property. Assume that for each n , G_{n+1} refines G_n . If x and y are distinct points, define $d(x, y) = 1/N$, where N is the first integer such that y is not in $\text{st}(x, G_N)$. Define $d(x, x) = 0$. It is a standard argument to see that d is a semi-metric on X . To show that (a) is satisfied, suppose that $\{x_n\}$ and $\{y_n\}$ are sequences converging to x . Let N be an integer and let g be a member of G_N that contains x . There is an integer $M > 0$ such that if $n > M$, then both x_n and y_n are in g . It follows that if $n > M$, then $d(x_n, y_n) < 1/N$; and so, $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. To see that (b) is satisfied, let x and y be distinct points of X and suppose that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y . Let M denote an integer such that if $n \geq M$, then no member of G_n intersects both $\text{st}(x, G_n)$ and $\text{st}(y, G_n)$. There is an integer N such that if $n > N$, then x_n is in $\text{st}(x, G_M)$ and y_n is in $\text{st}(y, G_M)$. Thus, if $n > \max\{N, M\}$, then $d(x_n, y_n) > 1/M$.

(3) implies (1): Let $G = \{\text{int. } D_\varepsilon(x) : \varepsilon > 0, x \in X\}$ where $D_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$. For each N , let $G_N = \{g \in G : \text{diam. } g < 1/N\}$ where $\text{diam. } g = \text{lub}\{d(x, y) : (x, y) \in g \times g\}$. Clearly, if for each n , G_n covers X , then $\{G_n\}$ is a development for X . Suppose that $x \in X$ and N is an integer such that no member of G_N contains x . Then for each integer j there are points x_j and y_j such that $d(x, x_j) \leq 1/j$ and $d(x, y_j) \leq 1/j$ and such that $d(x_j, y_j) > 1/N$. But this says that $\{x_j\}$ and $\{y_j\}$ are sequences converging to x such that the sequence $\{d(x_j, y_j)\}$ does not converge to zero. This is a contradiction from which it follows that $\{G_n\}$ is a development for X .

Now, suppose that x and y are distinct points of X such that for each n there is a member of G_n intersecting both $\text{st}(x, G_n)$ and $\text{st}(y, G_n)$. Then for each n , there are points x_n and y_n in $\text{st}(x, G_n)$ and $\text{st}(y, G_n)$ respectively such that x_n and y_n are in a common member of G_n . But, this means that $\{x_n\}$ converges to x , $\{y_n\}$ converges to y , and $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ which is a contradiction.

Note. The argument that (3) implies (1) is essentially the argument that H. Cook used when he showed the author how to prove that a continuously semi-metrizable space admits a development satisfy-

ing the 3-link property. Also, recall that in [1] it is shown that X is developable if and only if there is a semi-metric satisfying condition (a) and in [7], Hodel defines the notion of a G_δ^* -diagonal and he shows that the space X is a Hausdorff developable space if and only if X is a $\omega\Delta$ -space with a G_δ^* -diagonal.

A space X is said to be an M -space if there is a normal sequence $\{G_n\}$ of open covers of X such that if x is a point and $\{x_n\}$ is a sequence of points such that, for each n , x_n is in $\text{st}(x, G_n)$, then $\{x_n\}$ has a cluster point (Morita [10]).

LEMMA. *If X is an M -space, then either X is discrete or there is a countable discrete subspace of X that is not closed in X .*

Proof. Suppose that x_0 is a limit point of X . Let $\{G_n\}$ be a normal sequence of open covers of X as given by the fact that X is an M -space. Let x_1 be a point of $\text{st}(x_0, G_1)$ distinct from x_0 and let u_1 be an open set containing x_1 such that x_0 is not in $\text{cl } u_1$. Having x_1, \dots, x_n and u_1, \dots, u_n , let x_{n+1} be a point of $\text{st}(x_0, G_{n+1}) - \bigcup_{i=1}^n \text{cl } u_i$ distinct from x_0 . Let u_{n+1} be an open set containing x_{n+1} such that x_0 is not in $\text{cl } u_{n+1}$. $\{x_1, x_2, \dots\}$ is a countable discrete subspace of X that is not closed in X .

THEOREM 3. *Let X be a topological space. The following statements are equivalent:*

1. X is metrizable.
2. X is a Hausdorff M -space such that X^2 is perfectly normal.
3. X is an M -space with a regular G_δ -diagonal.
4. X is a Hausdorff M -space such that X^3 is hereditarily normal.
5. X is a Hausdorff M -space such that X^3 is hereditarily countable paracompact.

Proof. That (1) implies each of the other conditions is obvious. Also, it is clear that (2) implies (3). That (4) implies (2) follows from our Lemma and Corollary 1 of [8] and that (5) implies (2) follows from our Lemma and Theorem B of [12]. It remains to show that (3) implies (1). To this end, it follows from Theorem 2 that X is developable and Hausdorff. According to Theorem 6.1 of [10], there is a closed mapping f taking X onto a metric space Y such that $f^{-1}(y)$ is countably compact for each y in Y . Since X is developable, $f^{-1}(y)$ is compact for each y in Y ; thus, f is a perfect map. It is a well known consequence of Theorem 1 of [9] that the preimage of a metric space under a perfect map is paracompact. But, it is shown in [2] that a paracompact developable space is metrizable.

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