ON SPACES WITH REGULAR $G_δ$-DIAGONALS

PHILLIP LEE ZENOR
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It is the purpose of this note to investigate spaces with regular $G_δ$-diagonals. Among other things, it is shown that if $X$ is $T_1$-space, then 1. $X$ admits a development satisfying the 3-link property if and only if $X$ is a $ωJ$-space with a regular $G_δ$-diagonal and 2. $X$ is metrizable if and only if $X$ is an $iF$-space with a regular $G_δ$-diagonal.

Recall that a subset $H$ of the space $X$ is a regular $G_δ$-set if there is a sequence $\{U_n\}$ of open sets in $X$ such that $H = \bigcap_{n=1}^{∞} U_i = \bigcap_{r=1}^{∞} U_r$. We will say that $X$ has a regular $G_δ$-diagonal if $ΔX = \{(x, x) : x ∈ X\}$ is a regular $G_δ$-set in $X^2$.

In [4], Ceder shows that $X$ has a $G_δ$-diagonal if and only if there is a sequence $\{G_n\}$ of open covers of $X$ such that if $x$ is a point of $X$, then $x = \bigcap_{i=1}^{∞} \text{st}(x, G_i)$. In Theorem 1, we show that there is a similar characterizing property for spaces with regular $G_δ$-diagonals.

**Theorem 1.** The topological space $X$ has a regular $G_δ$-diagonal if and only if there is a sequence $\{G_n\}$ of open covers of $X$ such that if $x$ and $y$ are distinct points of $X$, then there are an integer $n$ and open sets $u$ and $v$ containing $x$ and $y$ respectively such that no member of $G_n$ intersects both $u$ and $v$.

**Proof.** Suppose that $X$ has a regular $G_δ$-diagonal. Let $\{U_n\}$ be a sequence of open sets in $X^2$ such that $ΔX = \bigcap_{i=1}^{∞} U_i = \bigcap_{r=1}^{∞} U_r$. For each $n$, let $G_n = \{g : g$ is an open subset of $X$ such that $g × g ⊆ U_n\}$. Let $x$ and $y$ be distinct points of $X$. Let $n$ be an integer such that $(x, y)$ is not in $U_n$. Let $u$ and $v$ be open sets in $X$ that contain $x$ and $y$ respectively such that $u × v$ does not intersect $U_n$. To see that no member of $G_n$ intersects both $u$ and $v$, suppose otherwise; that is, suppose that $g$ is a member of $G_n$, $p$ is a point of $g$ in $u$ and $q$ is a point of $g$ in $v$. Then $(p, q)$ is a point of $U_n \cap (u × v)$ which is a contradiction.

Now, suppose that $\{G_n\}$ is a sequence of open covers of $X$ as described in the theorem. For each $n$, let $U_n = \bigcup \{(g × g) : g ∈ G_n\}$. Clearly, $ΔX ⊆ \bigcap_{i=1}^{∞} U_i$. To see that $ΔX = \bigcap_{i=1}^{∞} U_i$, let $x$ and $y$ be distinct points of $X$. Then there are an integer $n$ and open sets $u$ and $v$ containing $x$ and $y$ respectively such that no member of $G_n$ intersects both $u$ and $v$. It must be the case that $U_n$ does not intersect $u × v$.

**Corollary.** If $X$ has a regular $G_δ$-diagonal, then $X$ is Hausdorff.
A development \( \{G_n\} \) for the space \( X \) is said to satisfy the 3-link property if it is true that if \( p \) and \( q \) are distinct points of \( X \), then there is an integer \( n \) such that no member of \( G_n \) intersects both \( \text{st}(x, G_n) \) and \( \text{st}(y, G_n) \) (Heath \[6\]). According to Borges \[3\], the space \( X \) is a \( \omega \Delta \)-space if there is a sequence \( \{U_n\} \) of open covers of \( X \) such that if \( x \) is a point and if, for each \( n \), \( x_n \) is a point of \( \text{st}(x, U_n) \), then the sequence \( \{x_n\} \) has a cluster point. Clearly, the class of \( \omega \Delta \)-spaces includes the class of strict \( p \)-spaces, the class of \( M \)-spaces, and the class of developable spaces. It is easy to see that the Niemytski plane (page 100 of \[11\]) is a non-metrizable Moore space that admits a development satisfying the 3-link property. In \[6\], Heath establishes the existence of Moore spaces that do not admit developments that satisfy the 3-link property. In \[5\], Cook asserts that a continuously semi-metrizable space is a Moore space that admits a development that satisfies the 3-link property. Cook’s result follows as a corollary to the following theorem:

**Theorem 2.** Let \( X \) be a topological space. Then the following conditions are equivalent:

1. \( X \) admits a development satisfying the 3-link property.
2. \( X \) is a \( \omega \Delta \)-space with a regular \( G_\delta \)-diagonal. And
3. There is a semi-metric \( d \) on \( X \) such that:
   a. If \( \{x_n\} \) and \( \{y_n\} \) are sequences both converging to \( x \), then \( \lim_{n \to \infty} d(x_n, y_n) = 0 \), and
   b. If \( x \) and \( y \) are distinct points of \( X \) and \( \{x_n\} \) and \( \{y_n\} \) are sequences converging to \( x \) and \( y \) respectively, then there are integers \( N \) and \( M \) such that if \( n > N \), then \( d(x_n, y_n) > 1/M \).

**Proof.** It is obvious that a developable space is a \( \omega \Delta \)-space; thus, that (1) implies (2) is a corollary to Theorem 1.

To see that (2) implies (1), let \( X \) be a \( \omega \Delta \)-space with a regular \( G_\delta \)-diagonal. Let \( \{U_n\} \) be a sequence of open covers of \( X \) as given by the fact that \( X \) is a \( \omega \Delta \)-space. According to Theorem 1, there is a sequence \( \{V_n\} \) of open covers of \( X \) such that if \( p \) and \( q \) are distinct points, then there are an integer \( n \) and open sets \( u \) and \( v \) containing \( p \) and \( q \) respectively such that no member of \( V_n \) intersects both \( u \) and \( v \). For each integer \( n \), let \( G_n \) be an open cover of \( X \) such that (i) \( G_n \) refines both \( U_n \) and \( V_n \) and (ii) \( G_{n+1} \) refines \( G_n \). We will show that \( \{G_n\} \) is a development for \( X \) that satisfies the 3-link property. First, to see that \( \{G_n\} \) is a development, suppose the contrary; that is, suppose that there are a point \( x \) and an open set \( u \) containing \( x \) such that, for each \( n \), there is a point \( p_n \) in \( \text{st}(x, G_n) \) — \( u \). Then, for each \( n \), \( p_n \) is in \( \text{st}(x, U_n) \). Thus, \( \{p_n\} \) has a cluster point \( p \). Since for each \( n \), \( G_n \) refines each of \( V_1, \ldots, V_n \), it follows that there are an
integer \( N \) and open sets \( v \) and \( w \) containing \( x \) and \( p \) respectively such that if \( j > N \), then no member of \( G_j \) intersects both \( v \) and \( w \). But this is a contradiction since there is a \( j < N \) such that \( p_j \) is in \( w \). Thus, \( \{G_j\} \) is a development for \( X \). To see that \( G_n \) satisfies the 3-link property, let \( p \) and \( q \) be distinct points, \( u \) and \( v \) open sets containing \( p \) and \( q \) respectively, and \( N \) an integer such that if \( n > N \), then no member of \( G_n \) meets both \( u \) and \( v \). Let \( S \) and \( T \) be integers such that \( \text{st}(p, G_S) \subset u \) and \( \text{st}(q, G_T) \subset v \). Let \( M = \max\{N, S, T\} \). Then no member of \( G_M \) meets both \( \text{st}(p, G_M) \) and \( \text{st}(q, G_M) \).

(1) implies (3): Let \( \{G_n\} \) be a development satisfying the 3-link property. Assume that for each \( n, G_{n+1} \) refines \( G_n \). If \( x \) and \( y \) are distinct points, define \( d(x, y) = 1/N \), where \( N \) is the first integer such that \( y \) is not in \( \text{st}(x, G_N) \). Define \( d(x, x) = 0 \). It is a standard argument to see that \( d \) is a semi-metric on \( X \). To show that (a) is satisfied, suppose that \( \{x_n\} \) and \( \{y_n\} \) are sequences converging to \( x \). Let \( N \) be an integer and let \( g \) be a member of \( G_N \) that contains \( x \). There is an integer \( M > 0 \) such that if \( n > M \), then both \( x_n \) and \( y_n \) are in \( g \). It follows that if \( n > M \), then \( d(x_n, y_n) < 1/N \); and so, \( \lim_{n \to \infty} d(x_n, y_n) = 0 \).

To see that (b) is satisfied, let \( x \) and \( y \) be distinct points of \( X \) and suppose that \( \{x_n\} \) converges to \( x \) and \( \{y_n\} \) converges to \( y \). Let \( M \) denote an integer such that if \( n \geq M \), then no member of \( G_n \) intersects both \( \text{st}(x, G_n) \) and \( \text{st}(y, G_n) \). There is an integer \( N \) such that if \( n > N \), then \( x_n \) is in \( \text{st}(x, G_M) \) and \( y_n \) is in \( \text{st}(y, G_M) \). Thus, if \( n > \max\{N, M\} \), then \( d(x_n, y_n) > 1/M \).

(3) implies (1): Let \( G = \{\text{int. } D_i(x) : \varepsilon > 0, x \in X\} \) where \( D_i(x) = \{y \in X : d(x, y) < \varepsilon\} \). For each \( N \), let \( G_N = \{g \in G : \text{diam. } g < 1/N\} \) where \( \text{diam. } g = \text{lub}\{d(x, y) : (x, y) \in g \times g\} \). Clearly, if for each \( n, G_n \) converges \( X \), then \( \{G_n\} \) is a development for \( X \). Suppose that \( x \in X \) and \( N \) is an integer such that no member of \( G_N \) contains \( x \). Then for each integer \( j \) there are points \( x_j \) and \( y_j \) such that \( d(x, x_j) \leq 1/j \) and \( d(x, y_j) \leq 1/j \) and such that \( d(x_j, y_j) > 1/N \). But this says that \( \{x_j\} \) and \( \{y_j\} \) are sequences converging to \( x \) such that the sequence \( \{d(x_j, y_j)\} \) does not converge to zero. This is a contradiction from which it follows that \( \{G_n\} \) is a development for \( X \).

Now, suppose that \( x \) and \( y \) are distinct points of \( X \) such that for each \( n \) there is a member of \( G_n \) intersecting both \( \text{st}(x, G_n) \) and \( \text{st}(y, G_n) \). Then for each \( n \), there are points \( x_n \) and \( y_n \) in \( \text{st}(x, G_n) \) and \( \text{st}(y, G_n) \) respectively such that \( x_n \) and \( y_n \) are in a common member of \( G_n \). But this means that \( \{x_n\} \) converges to \( x \), \( \{y_n\} \) converges to \( y \), and \( \lim_{n \to \infty} d(x_n, y_n) = 0 \) which is a contradiction.

Note. The argument that (3) implies (1) is essentially the argument that H. Cook used when he showed the author how to prove that a continuously semi-metrizable space admits a development satisfy-
ing the 3-link property. Also, recall that in [1] it is shown that $X$ is developable if and only if there is a semi-metric satisfying condition (a) and in [7], Hodel defines the notion of a $G^*_d$-diagonal and he shows that the space $X$ is a Hausdorff developable space if and only if $X$ is a $\omega d$-space with a $G^*_d$-diagonal.

A space $X$ is said to be an $M$-space if there is a normal sequence $\{G_n\}$ of open covers of $X$ such that if $x$ is a point and $\{x_n\}$ is a sequence of points such that, for each $n$, $x_n$ is in $st(x, G_n)$, then $\{x_n\}$ has a cluster point (Morita [10]).

**Lemma.** If $X$ is an $M$-space, then either $X$ is discrete or there is a countable discrete subspace of $X$ that is not closed in $X$.

**Proof.** Suppose that $x_0$ is a limit point of $X$. Let $\{G_\lambda\}$ be a normal sequence of open covers of $X$ as given by the fact that $X$ is an $M$-space. Let $x_\lambda$ be a point of $st(x_0, G_\lambda)$ distinct from $x_0$ and let $u_\lambda$ be an open set containing $x_\lambda$ such that $x_0$ is not in $cl\ u_\lambda$. Having $x_1, \cdots, x_\mu$ and $u_1, \cdots, u_\mu$, let $x_{\mu+1}$ be a point of $st(x_0, G_{\mu+1}) - \bigcup_{i=1}^\mu cl\ u_i$ distinct from $x_0$. Let $u_{\mu+1}$ be an open set containing $x_{\mu+1}$ such that $x_0$ is not in $cl\ u_{\mu+1} \cdot \{x_1, x_2, \cdots\}$ is a countable discrete subspace of $X$ that is not closed in $X$.

**Theorem 3.** Let $X$ be a topological space. The following statements are equivalent:

1. $X$ is metrizable.
2. $X$ is a Hausdorff $M$-space such that $X^2$ is perfectly normal.
3. $X$ is an $M$-space with a regular $G_\delta$-diagonal.
4. $X$ is a Hausdorff $M$-space such that $X^3$ is hereditarily normal.
5. $X$ is a Hausdorff $M$-space such that $X^3$ is hereditarily countable paracompact.

**Proof.** That (1) implies each of the other conditions is obvious. Also, it is clear that (2) implies (3). That (4) implies (2) follows from our Lemma and Corollary 1 of [8] and that (5) implies (2) follows from our Lemma and Theorem B of [12]. It remains to show that (3) implies (1). To this end, it follows from Theorem 2 that $X$ is developable and Hausdorff. According to Theorem 6.1 of [10], there is a closed mapping $f$ taking $X$ onto a metric space $Y$ such that $f^{-1}(y)$ is countably compact for each $y$ in $Y$. Since $X$ is developable, $f^{-1}(y)$ is compact for each $y$ in $Y$; thus, $f$ is a perfect map. It is a well known consequence of Theorem 1 of [9] that the preimage of a metric space under a perfect map is paracompact. But, it is shown in [2] that a paracompact developable space is metrizable.
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