CANONICAL EXTENSIONS OF MEASURES AND THE EXTENSION OF REGULARITY OF CONDITIONAL PROBABILITIES

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Let $(\Omega, \mathcal{A}, P)$ be a probability space with $\mathcal{B}$ a sub $\sigma$-field of $\mathcal{A}$. Let $\mathcal{A}' = \sigma(\mathcal{A}, H)$, the $\sigma$-field generated by $\mathcal{A}$ and $H$, where $H$ is a subset of $\Omega$ not in $\mathcal{A}$. $P_\varepsilon$ will be called a simple extension of $P$ to $\mathcal{A}'$ if $P_\varepsilon$ is a probability measure on $\mathcal{A}'$ which agrees with $P$ on $\mathcal{A}$.

The purpose of this paper is to use a particular type of simple extension called a canonical extension, denoted as $P_c$, to examine under what conditions the regularity of the conditional probability $P^\varepsilon$ will extend the regularity of $P^\varepsilon_c$. Also, if $\mathcal{A}$ is countably generated and $P^\varepsilon_c$ is regular, a characterization of $P^\varepsilon_c$ in terms of $P^\varepsilon$ will be given.

The terminology in the following definitions will be used throughout this paper.

**Definition.** The conditional probability of a set $A \in \mathcal{A}$ given the $\sigma$-field $\mathcal{B}$ is a $\mathcal{B}$-measurable function denoted by $P^\varepsilon(\cdot, A)$ such that for every $B \in \mathcal{B}$

$$
\int_B P^\varepsilon(\cdot, A) \, dP_\varepsilon = P(AB).
$$

**Definition.** The conditional probability (given $\mathcal{B}$) is the collection of functions

$$
\{P^\varepsilon(\cdot, A) | A \in \mathcal{A}\}.
$$

This collection is denoted by $P^\varepsilon$.

**Definition.** For $A \in \mathcal{A}$, a version of $P^\varepsilon(\cdot, A)$ is a selection from the equivalence class of $P^\varepsilon(\cdot, A)$ which will be denoted by $p(\cdot, A | \mathcal{B})$.

**Definition.** A version of the conditional probability $P^\varepsilon$ is a function $p(\cdot, \cdot | \mathcal{B})$ on $X \times \mathcal{A}$ such that for each $A \in \mathcal{A}$, $p(\cdot, A | \mathcal{B})$ is a version of $P^\varepsilon(\cdot, A)$. Also $p(w, \cdot | \mathcal{B})$ will denote a section of $p(\cdot, \cdot | \mathcal{B})$ at $w \in X$.

**Definition.** A conditional probability $P^\varepsilon$ is called regular if there exists a version, $p(\cdot, \cdot | \mathcal{B})$, such that $p(w, \cdot | \mathcal{B})$ is a measure on $\mathcal{A} P_\varepsilon$ a.e.

Before the main body of the paper is presented, it should be
observed that the regularity of $P^*$ itself is not in general sufficient to insure the regularity of $P^*$; for example, see [2], p. 210.

Finally, the scope of this paper is limited to results on canonical extensions. A forthcoming paper will deal with the preservation of regularity for simple extensions.

**The main results.** Observe that the $\sigma$-field

$$\mathcal{A}' = \{A, H + A_1 H^c | A_1, A_2 \in \mathcal{A}\},$$

and make

**DEFINITION 1.** Let $A'$ be any element of $\mathcal{A}'$ with $A' = A, H + A_2 H^c$ for some $A_1$ and $A_2$ in $\mathcal{A}$. A simple extension will be called a canonical extension, $P_c$, if there exists a number $\alpha$ between zero and one with $\beta = 1 - \alpha$ and $K \in \mathcal{A}$ so that

\begin{align}
(1.1) \quad & (a) \quad A'K^c \in \mathcal{A}
\end{align}

\begin{align}
& (b) \quad P_c(A') = P(A'K^c) + \alpha P(A_1 K) + \beta P(A_2 K)
\end{align}

with $P_c$ a well defined probability measure on $\mathcal{A}'$.

Marczewski and Los have shown, [4], that for any subset of $X$ not in $\mathcal{A}$, say $H$, there always exists a canonical extension $P_c$ on $\mathcal{A}'$. (It has been shown by the author in [1] that there exist many simple extensions which are not canonical.)

**REMARK 2.** One way of obtaining the set $K$ of Definition 1 is by letting $K_1$ be an element of $\mathcal{A}$ such that $(PK_1) = P_*(H)$ and $K_2$ be an element of $\mathcal{A}$ such that $P(K_2) = P^*(H)$ with $K_1 \subset H \subset K_2$. Then, simply define $K = K_1 \cup K_2$. (See [2], P. 71). Observe that there exists another $K' \in \mathcal{A}$ which will extend $P$ canonically to $\mathcal{A}'$ as in Definition 1 if and only if $P(K \Delta K') = 0$.

**LEMMA 3.** Let $(X, \mathcal{A}, P), \mathcal{B} \subset \mathcal{A}$ and $\mathcal{A}' = \sigma(\mathcal{A}, H)$ be given. Let $p(\cdot, \cdot | \mathcal{B})$ be a version of $P^*$ which makes $P^*$ regular. Let $P_\alpha$ be a canonical extension of $P$ to $\mathcal{A}'$ with $\alpha, \beta$ and $K$ as in Definition 1. Suppose for $w, P_\alpha$ a.e., $p_\alpha(w, \cdot | \mathcal{B})$ is a canonical extension of $p(w, \cdot | \mathcal{B})$ to $\mathcal{A}'$ with the same $\alpha$ and $\beta$ and $K$ as $P_\alpha$. Then, $P_\alpha$ is regular.

**Proof.** It will suffice to produce a version of $P_\alpha$ which makes $P_\alpha$ regular.

Let $A' \in \mathcal{A}'$ with $A' = A_1 H + A_2 H^c$ for some $A_1$ and $A_2$ in $\mathcal{A}$. For $w, P_\alpha$ a.e.,

\begin{align}
P_\alpha(w, A' | \mathcal{B}) = p(w, A'K^c | \mathcal{B}) + \alpha p(w, A_1 K | \mathcal{B})
\end{align}

\begin{align}
& + \beta p(w, A_2 K | \mathcal{B}).
\end{align}
Thus it is immediate from (3.1) that $p_c(\cdot, A'|\mathcal{B})$ is a $\mathcal{B}$-measurable function for all $A' \in \mathcal{W}$ and for $w, P_5$ a.e., $p_c(w, \cdot|\mathcal{B})$ is a measure on $\mathcal{W}$. It is also clear that for $A' \in \mathcal{W}$ and $B \in \mathcal{B}$

$$
(3.2) \quad \int_B p_c(\cdot, A'|\mathcal{B})dP_c = P_c(A'B).
$$

For, integrating the right side of (3.1) with respect to $P$ gives

$$
P(A'K'B) + \alpha P(A_5KB) + \beta P(A_5KB) = P_c(A'B).
$$

But $P_c = P$ on $\mathcal{B}$ and so the integral of the right side of (3.1) is exactly the left side of (3.2).

Hence, $p_c(\cdot, \cdot|\mathcal{B})$ is the desired version.

**Theorem 4.** Let $(X, \mathcal{A}, P), \mathcal{B},$ and $\mathcal{W}$ be as in Lemma 3. Suppose $P^\alpha$ is regular and $p(\cdot, \cdot|\mathcal{B})$ is a version such that

$$
(4.1) \quad p(w, \cdot|\mathcal{B}) \text{ is a measure } P^\alpha \text{ a.e.}
$$

$$
(4.2) \quad p(w, \cdot|\mathcal{B}) \ll Q(P_5 \text{ a.e.}) \text{ where } Q \text{ is a probability measure on } \mathcal{A}.
$$

Let $P_c$ be a canonical extension of $P$ to $\mathcal{W}$ with respect to $\alpha, \beta$ and $K$ as in (1.1). Then, $P_c^\alpha$ is regular.

**Proof.** Suppose $K' = K_\mathcal{A} \setminus K_\mathcal{B}$, where $K_\mathcal{A} \subset H \subset K_\mathcal{B}$, $Q_\mathcal{A}(H) = Q(K_\mathcal{A})$ and $Q_\mathcal{B}(H) = Q(K_\mathcal{B})$. Consider any set $A \subset K_\mathcal{A} \setminus H$ where $A \in \mathcal{A}$. $Q(A) = 0.$ By (4.2) $p(w, A|\mathcal{B}) = 0$ ($P_5$ a.e.) and so therefore $P(A) = 0$ also. Similarly, if $B \subset H \setminus K_\mathcal{A}$, then $Q(B) = 0$ and hence $p(w, B|\mathcal{B}) = 0$ and so $P(B) = 0$ also. Thus $p^\alpha(w, H|\mathcal{B}) = p(w, K_\mathcal{A}|\mathcal{B}) (P^\alpha$ a.e.) and $p(w, K_\mathcal{B}|\mathcal{B}) = p_\mathcal{A}(w, H|\mathcal{B}) (P_5$ a.e.) Also, $P(K_\mathcal{A}) = P^\alpha(H)$ and $P^\alpha(H) = P(K_\mathcal{A})$. According to Remark 2, $p(w, \cdot|\mathcal{B})$ can be extended canonically to $\mathcal{W}$ with respect to $\alpha, \beta$ and $K'$ and by Lemma 3 the proof is complete.

The following result is a consequence of Theorem 4.

**Theorem 5.** Let $(X, \mathcal{A}, P), \mathcal{B}$ and $\mathcal{W}$ be as in Lemma 3. Suppose $P^\alpha$ is regular and $p(\cdot, \cdot|\mathcal{B})$ is a version such that

$$
(5.1) \quad p(w, \cdot|\mathcal{B}) \text{ is a measure } P_5 \text{ a.e.}
$$

$$
(5.2) \quad \text{there exists a sequence } \{w_n\}_{n=1}^\infty \text{ such that for every } \varepsilon > 0 \text{ and any } w(P_5 \text{ a.e.) there is an } w_n \text{ with }
$$

$$
\sup_{A \in \mathcal{A}} |p(w, A|\mathcal{B}) - p(w_n, A|\mathcal{B})| < \varepsilon.
$$

Let $P_c$ be a canonical extension of $P$ to $\mathcal{W}$ with $\alpha, \beta$ and $K$ as in (1.1). Then, $P_c^\alpha$ is regular.
Proof. Let \( Q \) be a probability measure defined as
\[
\sum_{n=1}^{\infty} \frac{1}{2^n} p(w_n, \cdot | \mathcal{B}) .
\]
Condition (5.2) insures that \( p(w, \cdot | \mathcal{B}) \ll Q \) \( P_\alpha \) a.e. and the result follows from Theorem 4.

The following proposition is presented for the sake of completeness.

Let \((X, \mathcal{A}, P)\) be a probability space with \((X, \mathcal{B}, \bar{P})\) denoting the completion. Suppose \( H \) is in \( \mathcal{B} \) but not in \( \mathcal{A} \). Let \( \mathcal{B}' = \sigma(\mathcal{A}, H) \).

**Proposition 6.** Let \((X, \mathcal{A}, P)\), \( \mathcal{B} \subset \mathcal{A} \), and \( \mathcal{B}' = \sigma(\mathcal{A}, H) \) with \( H \in \mathcal{B} \). Let \( P_1 \) denote the restriction of \( \bar{P} \) to \( \mathcal{B}' \). If \( P_\alpha \) is regular then so is \( P_1 \).

The proof can be viewed as an easy consequence of Lemma 3 and is therefore omitted.

The remainder of this paper is devoted to the single

**Theorem 7.** Let \((X, \mathcal{A}, P)\) be a probability space with \( \mathcal{A} \) generated by a countable field, \( \mathcal{A} \). Let \( \mathcal{A}' \) be the field generated by \( \mathcal{A} \) and \( H \) and \( \mathcal{B} = \sigma(\mathcal{A}') \). Let \( P_\epsilon \) be a canonical extension of \( P \) to \( \mathcal{B} \) with respect to \( \alpha, \beta \) and \( K \) and suppose \( P_\alpha \) is regular where \( \mathcal{B} \subset \mathcal{A} \). Then, there exists a version \( p'(\cdot, \cdot | \mathcal{B}) \) of \( P_\alpha \) such that \( P_\epsilon \) a.e. \( p'(w, \cdot | \mathcal{B}) \) is a probability measure which is a canonical extension of \( p'(w, \cdot | \mathcal{B}) | \mathcal{A} \) with respect to the same \( \alpha, \beta \) and \( K \) that are associated with \( P_\epsilon \).

The following lemmas are introduced before presenting the main body of the proof.

**Lemma 8.** Let \((X, \mathcal{A}, P)\) be a probability space with \( \mathcal{A}' = \sigma(\mathcal{A}, H) \) and \( P_\epsilon \) an arbitrary simple extension of \( P \) to \( \mathcal{A}' \). Let \( K \) be the set associated with a canonical extension of \( P \) to \( \mathcal{A}' \) as in Remark 2. Then, for each set \( A \in \mathcal{A} \) there exist constants \( \alpha_A \) and \( \beta_A \) with \( 0 \leq \alpha_A \leq 1 \) and \( 0 \leq \beta_A \leq 1 \) and such that \( P_\epsilon(AHK) = \alpha_A P(AK) \) and \( P_\epsilon(AH'K) = \beta_A P(AK) \).

**Proof.** For \( A \in \mathcal{A}, AK \supset AHK \). If \( P(AK) \neq 0 \), then \( \alpha_A = P_\epsilon(AHK)/P(AK) \); otherwise, let \( \alpha_A \) be arbitrary between zero and one. \( \beta_A \) is obtained similarly.

**Lemma 9.** Assume the hypothesis of Lemma 8. Let \( \mathcal{A} \) be a field which generates \( \mathcal{A} \) and \( \mathcal{A}' \) the field generated by \( \mathcal{A} \) and \( H \). Let \( \alpha(\mathcal{A}) = \sup_{A \in \mathcal{A}} \alpha_A \) and \( \beta(\mathcal{A}) = \sup_{A \in \mathcal{A}} \beta_A \). Then, a necessary and sufficient condition that \( P_\epsilon \) be a canonical extension of \( P \) to \( \mathcal{A} \) is that

\[
\alpha(\mathcal{A}) \leq \beta(\mathcal{A}).
\]
$\alpha(\mathcal{A}) = \alpha_x$ or $\beta(\mathcal{A}) = \beta_x$ for some $\mathcal{A}$ which generates $\mathcal{F}$.

**Proof.** Necessity is obvious and only sufficiency is proved. Let $\mathcal{A}$ be some field which generates $\mathcal{F}$ and $\alpha(\mathcal{A}) = \alpha_x$. (For simplicity, write $\alpha(\mathcal{A}) = \alpha$.) By hypothesis,

$$P_e(HK) = \alpha P(K) .$$

For $A \in \mathcal{A}$ it follows by Lemma 8 that

(9.1) $$P_e(AHK) = \alpha_A P(AK)$$

and

(9.2) $$P_e(A^c HK) = \alpha_A P(A^c K) .$$

The following equalities also hold

(9.3) $$\alpha P(K) = \alpha P(AK) + \alpha P(A^c K)$$

(9.4) $$P_e(HK) = P_e(AHK) + P_e(A^c HK) .$$

By (9.1) - (9.4) it follows that

(9.5) $$0 = (\alpha - \alpha_A) P(AK) + (\alpha - \alpha_A^c) P(A^c K) .$$

If $P(AK) = 0$, set $\alpha_A = \alpha$ or if $P(A^c K) = 0$, set $\alpha_A^c = \alpha$ (see Lemma 8). Otherwise, (9.5) forces $\alpha - \alpha_A = \alpha - \alpha_A^c = 0$ and hence for any $A \in \mathcal{A}$, $P_e(AHK) = \alpha P(AK)$.

Next, the fact that $P_e(AH^c K) = \beta P(AK)$, $\beta = 1 - \alpha$, is immediate from the following chain of equalities:

$$P(A) = P_e(AH + AH^c) = P_e((AH + AH^c) K) + P_e(AHK)$$

$$+ P_e(AH^c K) = P(AK) + \alpha P(AK) + P_e(AHK) .$$

Hence, where $\mathcal{A}' = \{AH + A_i^c H | A_i \in \mathcal{A}, i = 1, 2\}$, $A'$ in $\mathcal{A}'$ can be written as $A' = A_i H + A_i^c H^c$ and it follows that

$$P_e(A') = P(A' K) + \alpha P(A, K) + \beta P(A, K) .$$

Finally, let

$$\phi_\alpha = \{A \in \mathcal{F} | P_e(AHK) = \alpha P(AK)\}$$

$$\phi_\beta = \{A \in \mathcal{F} | P_e(AH^c K) = \beta P(AK)\} .$$

Both $\phi_\alpha$ and $\phi_\beta$ are monotone classes containing $\mathcal{A}$; hence, the proof is complete by the monotone class theorem (see [3], p. 60).

Theorem 7 can now be proved.

**Proof.** For $w \in X$, $P_\beta$ a.e., and $A \in \mathcal{A}$, write
\[ p'(w, \mathcal{AHK}|\mathcal{B}) = \alpha_w, \mathcal{A}p(w, \mathcal{AK}|\mathcal{B}) \]

where \( 0 \leq \alpha_{w, \mathcal{A}} \leq 1 \) as in Lemma 8 and \( p(w, \cdot|\mathcal{B}) \) will be written for \( p'(w, \cdot|\mathcal{B})|_\mathcal{B} \). For fixed \( A \in \mathcal{A} \), \( \alpha_{w, \mathcal{A}} \) is a \( \mathcal{B} \)-measurable function where

\[
\begin{align*}
(7.1) \quad \alpha_{w, \mathcal{A}} &= p'(w, \mathcal{AHK}|\mathcal{B})/p(w, \mathcal{AK}|\mathcal{B}) \quad \text{for} \quad p(w, \mathcal{AK}|\mathcal{B}) \neq 0 \\
\alpha_{w, \mathcal{A}} &= \alpha \quad \text{if} \quad p(w, \mathcal{AK}|\mathcal{B}) = 0 \, . 
\end{align*}
\]

(In (7.1) \( \alpha \) is associated with \( P_e \) and by Lemma 9, \( \alpha = \sup_{\mathcal{A} \in \mathcal{A}} \alpha_{\mathcal{A}} \).

For \( A \in \mathcal{A} \) let

\[
(7.2) \quad U_\mathcal{A} \equiv \{ w | \alpha_{w, \mathcal{A}} > \alpha \} .
\]

Observe that \( U_\mathcal{A} \) is contained in the complement of the set of \( w \)'s where \( p(w, \mathcal{AK}|\mathcal{B}) = 0 \).

Also, \( U_\mathcal{A} \in \mathcal{B} \) (see (7.1)). Hence, since \( P_e \) is a canonical extension, it follows that

\[
(7.3) \quad \alpha P(AU_\mathcal{A}K) = P_e(AU_\mathcal{A}HK) = \int_{U_\mathcal{A}} p'(w, \mathcal{AHK}|\mathcal{B}) dP_e .
\]

Also,

\[
(7.4) \quad \int_{U_\mathcal{A}} \alpha p(w, \mathcal{AK}|\mathcal{B}) dP = \alpha P(AU_\mathcal{A}K) .
\]

Hence, the defining properties of \( U_\mathcal{A} \) together with (7.3) and (7.4) say that \( P(U_\mathcal{A}) = 0 \).

If \( L_\mathcal{A} \equiv \{ w | \alpha_{w, \mathcal{A}} < \alpha \} \), then an argument similar to the preceding one shows \( P(L_\mathcal{A}) = 0 \).

Hence, for each set \( A \in \mathcal{A} \), there exists a \( P_\mathcal{A} \) null set on the complement of which \( \alpha_{w, \mathcal{A}} = \alpha \). But where \( \mathcal{A} \) is countable, it follows that there exists a \( P_\mathcal{A} \) null set, \( N \), on the complement of which \( \alpha_{w, \mathcal{A}} = \alpha \) for all \( A \in \mathcal{A} \). Thus,

\[
(7.5) \quad p'(w, \mathcal{AHK}|\mathcal{B}) = \alpha p(w, \mathcal{AK}|\mathcal{B})
\]

for all \( w \in N^c \) and \( A \in \mathcal{A} \).

Finally, if \( \alpha_w = \sup_{\mathcal{A} \in \mathcal{A}} \alpha_{w, \mathcal{A}} \), then it is immediate from (7.5) that \( P_\mathcal{A} \) a.e. \( \alpha_w = \alpha = \alpha_X \) and by Lemma 9 the theorem is proved.

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Received November 30, 1970. This paper is based in part on the author's doctoral dissertation completed at the University of Maryland under the direction of Professor R. Syski.

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