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# CANONICAL EXTENSIONS OF MEASURES AND THE EXTENSION OF REGULARITY OF CONDITIONAL PROBABILITIES*

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**Let $(\Omega, \mathfrak{A}, P)$ be a probability space with $\mathfrak{B}$ a sub $\sigma$-field of $\mathfrak{A}$. Let $\mathfrak{A}' \equiv \sigma(\mathfrak{A}, H)$, the $\sigma$-field generated by $\mathfrak{A}$ and $H$, where $H$ is a subset of $\Omega$ not in $\mathfrak{A}$. $P_e$ will be called a simple extension of $P$ to $\mathfrak{A}'$ if $P_e$ is a probability measure on $\mathfrak{A}'$ which agrees with $P$ on $\mathfrak{A}$.**

**The purpose of this paper is to use a particular type of simple extension called a canonical extension, denoted as $P_c$ to examine under what conditions the regularity of the conditional probability $P^{\mathfrak{B}}$ will extend to the regularity of $P_c^{\mathfrak{B}}$. Also, if $\mathfrak{A}$ is countably generated and $P_c^{\mathfrak{B}}$ is regular, a characterization of $P_c^{\mathfrak{B}}$ in terms of $P^{\mathfrak{B}}$ will be given.**

The terminology in the following definitions will be used throughout this paper.

DEFINITION. The conditional probability of a set $A \in \mathfrak{A}$ given the $\sigma$-field $\mathfrak{B}$ is a $\mathfrak{B}$-measurable function denoted by $P^{\mathfrak{B}}(\cdot, A)$ such that for every $B \in \mathfrak{B}$

$$\int_B P^{\mathfrak{B}}(\cdot, A) dP_{\mathfrak{B}} = P(AB).$$

DEFINITION. The conditional probability (given $\mathfrak{B}$) is the collection of functions

$$\{P^{\mathfrak{B}}(\cdot, A) \mid A \in \mathfrak{A}\}.$$

This collection is denoted by $P^{\mathfrak{B}}$.

DEFINITION. For $A \in \mathfrak{A}$, a version of $P^{\mathfrak{B}}(\cdot, A)$ is a selection from the equivalence class of $P^{\mathfrak{B}}(\cdot, A)$ which will be denoted by $p(\cdot, A \mid \mathfrak{B})$.

DEFINITION. A version of the conditional probability $P^{\mathfrak{B}}$ is a function $p(\cdot, \cdot \mid \mathfrak{B})$ on $X \times \mathfrak{A}$ such that for each $A \in \mathfrak{A}$ $p(\cdot, A \mid \mathfrak{B})$ is a version of $P^{\mathfrak{B}}(\cdot, A)$. Also $p(w, \cdot \mid \mathfrak{B})$ will denote a section of $p(\cdot, \cdot \mid \mathfrak{B})$ at $w \in X$.

DEFINITION. A conditional probability $P^{\mathfrak{B}}$ is called regular if there exists a version, $p(\cdot, \cdot \mid \mathfrak{B})$, such that $p(w, \cdot \mid \mathfrak{B})$ is a measure on $\mathfrak{A}$ $P_{\mathfrak{B}}$ a.e.

Before the main body of the paper is presented, it should be observed that the regularity of $P^{\mathfrak{B}}$ itself is not in general sufficient to insure the regularity of $P_c^{\mathfrak{B}}$; for example, see [2], p. 210.

Finally, the scope of this paper is limited to results on canonical extensions. A forthcoming paper will deal with the preservation of regularity for simple extensions.

**The main results.** Observe that the $\sigma$-field

$$\mathfrak{A}' = \{A_1H + A_2H^c \,|\, A_1, A_2 \in \mathfrak{A}\} ,$$

and make

DEFINITION 1. Let $A'$ be any element of $\mathfrak{A}'$ with $A' = A_1H + A_2H^c$ for some $A_1$ and $A_2$ in $\mathfrak{A}$. A simple extension will be called a canonical extension, $P_c$, if there exists a number $\alpha$ between zero and one with $\beta = 1 - \alpha$ and $K \in \mathfrak{A}$ so that

$$\text{(1.1)} \qquad \text{(a)} \quad A'K^c \in \mathfrak{A}$$

$$\text{(b)} \quad P_c(A') = P(A'K^c) + \alpha P(A_1K) + \beta P(A_2K)$$

with $P_c$ a well defined probability measure on $\mathfrak{A}'$.

Marczewski and Loś have shown, [4], that for any subset of $X$ not in $\mathfrak{A}$, say $H$, there always exists a canonical extension $P_c$ on $\mathfrak{A}'$. (It has been shown by the author in [1] that there exist many simple extensions which are not canonical.)

REMARK 2. One way of obtaining the set $K$ of Definition 1 is by letting $K_1$ be an element of $\mathfrak{A}$ such that $(PK_1) = P_*(H)$ and $K_2$ be an element of $\mathfrak{A}$ such that $P(K_2) = P^*(H)$ with $K_1 \subset H \subset K_2$. Then, simply define $K = K_2 \backslash K_1$. (See [2], P. 71). Observe that there exists another $K' \in \mathfrak{A}$ which will extend $P$ canonically to $\mathfrak{A}'$ as in Definition 1 if and only if $P(K \Delta K') = 0$.

LEMMA 3. *Let $(X, \mathfrak{A}, P)$, $\mathfrak{B} \subset \mathfrak{A}$ and $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$ be given. Let $p(\cdot, \cdot|\mathfrak{B})$ be a version of $P^{\mathfrak{B}}$ which makes $P^{\mathfrak{B}}$ regular. Let $P_c$ be a canonical extension of $P$ to $\mathfrak{A}'$ with $\alpha$, $\beta$ and $K$ as in Definition* 1. *Suppose for $w$, $P_{\mathfrak{B}}$ a.e., $p_c(w, \cdot|\mathfrak{B})$ is a canonical extension of $p(w, \cdot|\mathfrak{B})$ to $\mathfrak{A}'$ with the same $\alpha$ and $\beta$ and $K$ as $P_c$. Then, $P_c^{\mathfrak{B}}$ is regular.*

*Proof.* It will suffice to produce a version of $P_c^{\mathfrak{B}}$ which makes $P_c^{\mathfrak{B}}$ regular.

Let $A' \in \mathfrak{A}'$ with $A' = A_1H + A_2H^c$ for some $A_1$ and $A_2$ in $\mathfrak{A}$. For $w$, $P_{\mathfrak{B}}$ a.e.,

$$\text{(3.1)} \qquad P_c(w, A'|\mathfrak{B}) = p(w, A'K^c|\mathfrak{B}) + \alpha p(w, A_1K\,|\,\mathfrak{B}) + \beta p(w, A_2K|\mathfrak{B}) \; .$$

Thus it is immediate from (3.1) that $p_c(\cdot, A'|\mathfrak{B})$ is a $\mathfrak{B}$-measurable function for all $A' \in \mathfrak{A}'$ and for $w$, $P_{\mathfrak{B}}$ a.e., $p_c(w, \cdot|\mathfrak{B})$ is a measure on $\mathfrak{A}'$. It is also clear that for $A' \in \mathfrak{A}'$ and $B \in \mathfrak{B}$

$$\int_B P_c(\cdot, A'|\mathfrak{B})dP_c = P_c(A'B) . \tag{3.2}$$

For, integrating the right side of (3.1) with respect to $P$ gives

$$P(A'K^cB) + \alpha P(A_1KB) + \beta P(A_2KB) = P_c(A'B) .$$

But $P_c = P$ on $\mathfrak{B}$ and so the integral of the right side of (3.1) is exactly the left side of (3.2) .

Hence, $p_c(\cdot, \cdot|\mathfrak{B})$ is the desired version.

THEOREM 4. *Let $(X, \mathfrak{A}, P)$, $\mathfrak{B}$, and $\mathfrak{A}'$ be as in Lemma* 3. *Suppose $P^{\mathfrak{B}}$ is regular and $p(\cdot, \cdot|\mathfrak{B})$ is a version such that*

(4.1) *$p(w, \cdot|\mathfrak{B})$ is a measure $P_{\mathfrak{B}}$ a.e.*

(4.2) *$p(w, \cdot|\mathfrak{B}) \ll Q(P_{\mathfrak{B}}$ a.e.) where $Q$ is a probability measure on $\mathfrak{A}$.*

*Let $P_c$ be a canonical extension of $P$ to $\mathfrak{A}'$ with respect to $\alpha$, $\beta$ and $K$ as in* (1.1). *Then, $P_c^{\mathfrak{B}}$ is regular.*

*Proof*. Suppose $K' = K_2 \backslash K_1$, where $K_1 \subset H \subset K_2$, $Q_*(H) = Q(K_1)$ and $Q^*(H) = Q(K_2)$. Consider any set $A \subset K_2 \backslash H$ where $A \in \mathfrak{A}$. $Q(A) = 0$. By (4.2) $p(w, A|\mathfrak{B}) = 0$ ($P_{\mathfrak{B}}$ a.e.) and so therefore $P(A) = 0$ also. Similarly, if $B \subset H \backslash K_1$, where $B \in \mathfrak{A}$, then $Q(B) = 0$ and hence $p(w, B|\mathfrak{B}) = 0$ and so $P(B) = 0$ also. Thus $p^*(w, H|\mathfrak{B}) = p(w, K_2|\mathfrak{B})$ ($P^{\mathfrak{B}}$ a.e.) and $p(w, K_1|\mathfrak{B}) = p_*(w, H|\mathfrak{B})$ ($P_{\mathfrak{B}}$ a.e.). Also, $P(K_1) = P_*(H)$ and $P^*(H) = P(K_2)$. According to Remark 2, $p(w, \cdot|\mathfrak{B})$ can be extended canonically to $\mathfrak{A}'$ with respect to $\alpha$, $\beta$ and $K'$ and by Lemma 3 the proof is complete.

The following result is a consequence of Theorem 4.

THEOREM 5. *Let $(X, \mathfrak{A}, P)$, $\mathfrak{B}$ and $\mathfrak{A}'$ be as in Lemma* 3. *Suppose $P^{\mathfrak{B}}$ is regular and $p(\cdot, \cdot|\mathfrak{B})$ is a version such that*

(5.1) *$p(w, \cdot|\mathfrak{B})$ is a measue $P_{\mathfrak{B}}$ a.e.*

(5.2) *there exists a sequence $\{w_n\}_{n=1}$ such that for every $\varepsilon > 0$ and any $w(P_{\mathfrak{B}}$ a.e.) there is an $w_n$ with*

$$\sup_{A \in \mathfrak{A}} |p(w, A|\mathfrak{B}) - p(w_n, A|\mathfrak{B})| < \varepsilon .$$

*Let $P_c$ be a canonical extension of $P$ to $\mathfrak{A}'$ with $\alpha$, $\beta$ and $K$ as in* (1.1). *Then, $P_c^{\mathfrak{B}}$ is regular.*

*Proof.* Let $Q$ be a probability measure defined as

$$\sum_{n=1}^{\infty} \frac{1}{2^n} p(w_n, \cdot|\mathfrak{B}) .$$

Condition (5.2) insures that $p(w, \cdot|\mathfrak{B}) \ll Q$ $P_{\mathfrak{B}}$ a.e. and the result follows from Theorem 4.

The following proposition is presented for the sake of completeness.

Let $(X, \mathfrak{A}, P)$ be a probability space with $(X, \bar{\mathfrak{A}}, \bar{P})$ denoting the completion. Suppose $H$ is in $\bar{\mathfrak{A}}$ but not in $\mathfrak{A}$. Let $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$.

PROPOSITION 6. *Let $(X, \mathfrak{A}, P)$, $\mathfrak{B} \subset \mathfrak{A}$, and $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$ with $H \in \bar{\mathfrak{A}} \backslash \mathfrak{A}$ be given. Let $P_1$ denote the restriction of $\bar{P}$ to $\mathfrak{A}'$. If $P^{\mathfrak{B}}$ is regular then so is $P_1^{\mathfrak{B}}$.*

The proof can be viewed as an easy consequence of Lemma 3 and is therefore omitted.

The remainder of this paper is devoted to the single

THEOREM 7. *Let $(X, \mathfrak{A}, P)$ be a probability space with $\mathfrak{A}$ generated by a countable field, $\mathscr{A}$. Let $\mathscr{A}'$ be the field generated by $\mathscr{A}$ and $H$ and $\mathfrak{A}' = \sigma(\mathscr{A}')$. Let $P_c$ be a canonical extension of $P$ to $\mathfrak{A}'$ with respect to $\alpha$, $\beta$ and $K$ and suppose $P_c^{\mathfrak{B}}$ is regular where $\mathfrak{B} \subset \mathfrak{A}$. Then, there exists a version $p'(\cdot, \cdot|\mathfrak{B})$ of $P_c^{\mathfrak{B}}$ such that $P_{\mathfrak{B}}$ a.e. $p'(w, \cdot|\mathfrak{B})$ is a probability measure which is a canonical extension of $p'(w, \cdot|\mathfrak{B})|\mathfrak{A}$ with respect to the same $\alpha$, $\beta$ and $K$ that are associated with $P_c$.*

The following lemmas are introduced before presenting the main body of the proof.

LEMMA 8. *Let $(X, \mathfrak{A}, P)$ be a probability space with $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$ and $P_e$ an arbitrary simple extension of $P$ to $\mathfrak{A}'$. Let $K$ be the set associated with a canonical extension of $P$ to $\mathfrak{A}'$ as in Remark 2. Then, for each set $A \in \mathfrak{A}$ there exist constants $\alpha_A$ and $\beta_A$ with $0 \leqq \alpha_A \leqq 1$ and $0 \leqq \beta_A \leqq 1$ and such that $P_e(AHK) = \alpha_A P(AK)$ and $P_e(AH^cK) = \beta_A P(AK)$.*

*Proof.* For $A \in \mathfrak{A}, AK \supset AHK$. If $P(AK) \neq 0$, then $\alpha_A = P_e(AHK)/P(AK)$; otherwise, let $\alpha_A$ be arbitrary between zero and one. $\beta_A$ is obtained similarly.

LEMMA 9. *Assume the hypothesis of Lemma 8. Let $\mathscr{A}$ be a field which generates $\mathfrak{A}$ and $\mathscr{A}'$ the field generated by $\mathscr{A}$ and $H$. Let $\alpha(\mathscr{A}) \equiv \sup_{A \in \mathscr{A}} \alpha_A$ and $\beta(\mathscr{A}) \equiv \sup_{A \in \mathscr{A}} \beta_A$. Then, a necessary and sufficient condition that $P_e$ be a canonical extension of $P$ to $\mathfrak{A}'$ is that*

*$\alpha(\mathscr{A}) = \alpha_X$ or $\beta(\mathscr{A}) = \beta_X$ for some $\mathscr{A}$ which generates $\mathfrak{A}$.*

*Proof.* Necessity is obvious and only sufficiency is proved. Let $\mathscr{A}$ be some field which generates $\mathfrak{A}$ and $\alpha(\mathscr{A}) = \alpha_X$. (For simplicity, write $\alpha(\mathscr{A}) = \alpha$.) By hypothesis,

$$P_e(HK) = \alpha P(K) .$$

For $A \in \mathscr{A}$ it follows by Lemma 8 that

$$P_e(AHK) = \alpha_A P(AK) \tag{9.1}$$

and

$$P_e(A^c HK) = \alpha_{A^c} P(A^c K) . \tag{9.2}$$

The following equalities also hold

$$\alpha P(K) = \alpha P(AK) + \alpha P(A^c K) \tag{9.3}$$

$$P_e(HK) = P_e(AHK) + P_e(A^c HK) . \tag{9.4}$$

By (9.1) – (9.4) it follows that

$$0 = (\alpha - \alpha_A)P(AK) + (\alpha - \alpha_{A^c})P(A^c K) . \tag{9.5}$$

If $P(AK) = 0$, set $\alpha_A = \alpha$ or if $P(A^c K) = 0$, set $\alpha_{A^c} = \alpha$ (see Lemma 8). Otherwise, (9.5) forces $\alpha - \alpha_A = \alpha - \alpha_{A^c} = 0$ and hence for any $A \in \mathscr{A}$, $P_e(AHK) = \alpha P(AK)$.

Next, the fact that $P_e(AH^c K) = \beta P(AK)$, $\beta = 1 - \alpha$, is immediate from the following chain of equalities:

$$\begin{aligned} P(A) = P_e(AH + AH^c) &= P_e((AH + AH^c)K^c) + P_e(AHK) \\ &+ P_e AH^c K) = P(AK^c) + \alpha P(AK) + P_e(AH^c K) . \end{aligned}$$

Hence, where $\mathscr{A}' = \{A_1 H + A_2 H^c \mid A_i \in \mathscr{A}\ i = 1, 2\}$, $A'$ in $\mathscr{A}'$ can be written as $A' = A_1 H + A_2 H^c$ and it follows that

$$P_e(A') = P(A' K^c) + \alpha P(A_1 K) + \beta P(A_2 K) .$$

Finally, let

$$\begin{aligned} \phi_\alpha &= \{A \in \mathfrak{A} \mid P_e(AHK) = \alpha P(AK)\} \\ \phi_\beta &= \{A \in \mathfrak{A} \mid P_e(AH^c K) = \beta P(AK)\} . \end{aligned}$$

Both $\phi_\alpha$ and $\phi_\beta$ are monotone classes containing $\mathscr{A}$; hence, the proof is complete by the monotone class theorem (see [3], p. 60).

Theorem 7 can now be proved.

*Proof.* For $w \in X$, $P_{\mathfrak{B}}$ a.e., and $A \in \mathscr{A}$, write

$$p'(w, AHK|\mathfrak{B}) = \alpha_{w,A}p(w, AK|\mathfrak{B})$$

where $0 \leqq \alpha_{w,A} \leqq 1$ as in Lemma 8 and $p(w, \cdot|\mathfrak{B})$ will be written for $p'(w, \cdot|\mathfrak{B})|_{\mathfrak{A}}$. For fixed $A \in \mathscr{A}$, $\alpha_{w,A}$ is a $\mathfrak{B}$-measurable function where

(7.1) $$\alpha_{w,A} = p'(w, AHK|\mathfrak{B})/p(w, AK|\mathfrak{B}) \text{ for } p(w, AK|\mathfrak{B}) \neq 0$$
$$\alpha_{w,A} = \alpha \text{ if } p(w, AK|\mathfrak{B}) = 0 .$$

(In (7.1) $\alpha$ is associated with $P_c$ and by Lemma 9, $\alpha = \sup_{A \in \mathscr{A}} \alpha_A$).

For $A \in \mathscr{A}$ let

(7.2) $$U_A \equiv \{w|\alpha_{w,A} > \alpha\} .$$

Observe that $U_A$ is contained in the complement of the set of $w$'s where $p(w, AK|\mathfrak{B}) = 0$ .
Also, $U_A \in \mathfrak{B}$ (see (7.1)). Hence, since $P_c$ is a canonical extension, it follows that

(7.3) $$\alpha P(AU_AK) = P_c(AU_AHK) = \int_{U_A} p'(w, AHK|\mathfrak{B})dP_c .$$

Also,

(7.4) $$\int_{U_A} p'(w, AHK|\mathfrak{B})dP_c = \int_{U_A} \alpha_{w,A}p(w, AK|\mathfrak{B})dP \geqq \int_{U_A} \alpha p(w, AK|\mathfrak{B})dP = \alpha P(AU_AK) .$$

Hence, the defining properties of $U_A$ together with (7.3) and (7.4) say that $P(U_A) = 0$.

If $L_A \equiv \{w|\alpha_{w,A} < \alpha\}$, then an argument similar to the preceding one shows $P(L_A) = 0$.

Hence, for each set $A \in \mathscr{A}$, there exists a $P_{\mathfrak{B}}$ null set on the complement of which $\alpha_{w,A} = \alpha$. But where $\mathscr{A}$ is countable, it follows that there exists a $P_{\mathfrak{B}}$ null set, $N$, on the complement of which $\alpha_{w,A} = \alpha$ for all $A \in \mathscr{A}$. Thus,

(7.5) $$p'(w, AHK|\mathfrak{B}) = \alpha p(w, AK|\mathfrak{B})$$

for all $w \in N^c$ and $A \in \mathscr{A}$.

Finally, if $\alpha_w \equiv \sup_{A \in \mathscr{A}} \alpha_{w,A}$, then it is immediate from (7.5) that $P_{\mathfrak{B}}$ a.e. $\alpha_w = \alpha = \alpha_X$ and by Lemma 9 the theorem is proved.

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