

# Pacific Journal of Mathematics

**CANONICAL EXTENSIONS OF MEASURES AND THE  
EXTENSION OF REGULARITY OF CONDITIONAL  
PROBABILITIES**

LOUIS HARVEY BLAKE

## CANONICAL EXTENSIONS OF MEASURES AND THE EXTENSION OF REGULARITY OF CONDITIONAL PROBABILITIES\*

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Let  $(\Omega, \mathfrak{A}, P)$  be a probability space with  $\mathfrak{B}$  a sub  $\sigma$ -field of  $\mathfrak{A}$ . Let  $\mathfrak{A}' \equiv \sigma(\mathfrak{A}, H)$ , the  $\sigma$ -field generated by  $\mathfrak{A}$  and  $H$ , where  $H$  is a subset of  $\Omega$  not in  $\mathfrak{A}$ .  $P_e$  will be called a simple extension of  $P$  to  $\mathfrak{A}'$  if  $P_e$  is a probability measure on  $\mathfrak{A}'$  which agrees with  $P$  on  $\mathfrak{A}$ .

The purpose of this paper is to use a particular type of simple extension called a canonical extension, denoted as  $P_e$  to examine under what conditions the regularity of the conditional probability  $P^{\mathfrak{B}}$  will extend to the regularity of  $P_e^{\mathfrak{B}}$ . Also, if  $\mathfrak{A}$  is countably generated and  $P_e^{\mathfrak{B}}$  is regular, a characterization of  $P_e^{\mathfrak{B}}$  in terms of  $P^{\mathfrak{B}}$  will be given.

The terminology in the following definitions will be used throughout this paper.

**DEFINITION.** The conditional probability of a set  $A \in \mathfrak{A}$  given the  $\sigma$ -field  $\mathfrak{B}$  is a  $\mathfrak{B}$ -measurable function denoted by  $P^{\mathfrak{B}}(\cdot, A)$  such that for every  $B \in \mathfrak{B}$

$$\int_B P^{\mathfrak{B}}(\cdot, A) dP_{\mathfrak{B}} = P(AB).$$

**DEFINITION.** The conditional probability (given  $\mathfrak{B}$ ) is the collection of functions

$$\{P^{\mathfrak{B}}(\cdot, A) \mid A \in \mathfrak{A}\}.$$

This collection is denoted by  $P^{\mathfrak{B}}$ .

**DEFINITION.** For  $A \in \mathfrak{A}$ , a version of  $P^{\mathfrak{B}}(\cdot, A)$  is a selection from the equivalence class of  $P^{\mathfrak{B}}(\cdot, A)$  which will be denoted by  $p(\cdot, A \mid \mathfrak{B})$ .

**DEFINITION.** A version of the conditional probability  $P^{\mathfrak{B}}$  is a function  $p(\cdot, \cdot \mid \mathfrak{B})$  on  $X \times \mathfrak{A}$  such that for each  $A \in \mathfrak{A}$   $p(\cdot, A \mid \mathfrak{B})$  is a version of  $P^{\mathfrak{B}}(\cdot, A)$ . Also  $p(w, \cdot \mid \mathfrak{B})$  will denote a section of  $p(\cdot, \cdot \mid \mathfrak{B})$  at  $w \in X$ .

**DEFINITION.** A conditional probability  $P^{\mathfrak{B}}$  is called regular if there exists a version,  $p(\cdot, \cdot \mid \mathfrak{B})$ , such that  $p(w, \cdot \mid \mathfrak{B})$  is a measure on  $\mathfrak{A}$   $P_{\mathfrak{B}}$  a.e.

Before the main body of the paper is presented, it should be

observed that the regularity of  $P^{\mathfrak{B}}$  itself is not in general sufficient to insure the regularity of  $P_c^{\mathfrak{B}}$ ; for example, see [2], p. 210.

Finally, the scope of this paper is limited to results on canonical extensions. A forthcoming paper will deal with the preservation of regularity for simple extensions.

The main results. Observe that the  $\sigma$ -field

$$\mathfrak{A}' = \{A_1H + A_2H^c \mid A_1, A_2 \in \mathfrak{A}\},$$

and make

DEFINITION 1. Let  $A'$  be any element of  $\mathfrak{A}'$  with  $A' = A_1H + A_2H^c$  for some  $A_1$  and  $A_2$  in  $\mathfrak{A}$ . A simple extension will be called a canonical extension,  $P_c$ , if there exists a number  $\alpha$  between zero and one with  $\beta = 1 - \alpha$  and  $K \in \mathfrak{A}$  so that

$$(1.1) \quad \begin{aligned} (a) \quad & A'K^c \in \mathfrak{A} \\ (b) \quad & P_c(A') = P(A'K^c) + \alpha P(A_1K) + \beta P(A_2K) \end{aligned}$$

with  $P_c$  a well defined probability measure on  $\mathfrak{A}'$ .

Marczewski and Los have shown, [4], that for any subset of  $X$  not in  $\mathfrak{A}$ , say  $H$ , there always exists a canonical extension  $P_c$  on  $\mathfrak{A}'$ . (It has been shown by the author in [1] that there exist many simple extensions which are not canonical.)

REMARK 2. One way of obtaining the set  $K$  of Definition 1 is by letting  $K_1$  be an element of  $\mathfrak{A}$  such that  $(PK_1) = P_*(H)$  and  $K_2$  be an element of  $\mathfrak{A}$  such that  $P(K_2) = P^*(H)$  with  $K_1 \subset H \subset K_2$ . Then, simply define  $K = K_2 \setminus K_1$ . (See [2], P. 71). Observe that there exists another  $K' \in \mathfrak{A}$  which will extend  $P$  canonically to  $\mathfrak{A}'$  as in Definition 1 if and only if  $P(K \Delta K') = 0$ .

LEMMA 3. Let  $(X, \mathfrak{A}, P)$ ,  $\mathfrak{B} \subset \mathfrak{A}$  and  $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$  be given. Let  $p(\cdot, \cdot | \mathfrak{B})$  be a version of  $P^{\mathfrak{B}}$  which makes  $P^{\mathfrak{B}}$  regular. Let  $P_c$  be a canonical extension of  $P$  to  $\mathfrak{A}'$  with  $\alpha, \beta$  and  $K$  as in Definition 1. Suppose for  $w$ ,  $P_{\mathfrak{B}}$  a.e.,  $p_c(w, \cdot | \mathfrak{B})$  is a canonical extension of  $p(w, \cdot | \mathfrak{B})$  to  $\mathfrak{A}'$  with the same  $\alpha$  and  $\beta$  and  $K$  as  $P_c$ . Then,  $P_c^{\mathfrak{B}}$  is regular.

*Proof.* It will suffice to produce a version of  $P_c^{\mathfrak{B}}$  which makes  $P_c^{\mathfrak{B}}$  regular.

Let  $A' \in \mathfrak{A}'$  with  $A' = A_1H + A_2H^c$  for some  $A_1$  and  $A_2$  in  $\mathfrak{A}$ . For  $w$ ,  $P_{\mathfrak{B}}$  a.e.,

$$(3.1) \quad \begin{aligned} P_c(w, A' | \mathfrak{B}) &= p(w, A'K^c | \mathfrak{B}) + \alpha p(w, A_1K | \mathfrak{B}) \\ &\quad + \beta p(w, A_2K | \mathfrak{B}). \end{aligned}$$

Thus it is immediate from (3.1) that  $p_c(\cdot, A'|\mathfrak{B})$  is a  $\mathfrak{B}$ -measurable function for all  $A' \in \mathfrak{A}'$  and for  $w, P_{\mathfrak{B}}$  a.e.,  $p_c(w, \cdot|\mathfrak{B})$  is a measure on  $\mathfrak{A}'$ . It is also clear that for  $A' \in \mathfrak{A}'$  and  $B \in \mathfrak{B}$

$$(3.2) \quad \int_B P_c(\cdot, A'|\mathfrak{B}) dP_c = P_c(A'B) .$$

For, integrating the right side of (3.1) with respect to  $P$  gives

$$P(A'K'B) + \alpha P(A_1KB) + \beta P(A_2KB) = P_c(A'B) .$$

But  $P_c = P$  on  $\mathfrak{B}$  and so the integral of the right side of (3.1) is exactly the left side of (3.2) .

Hence,  $p_c(\cdot, \cdot|\mathfrak{B})$  is the desired version.

**THEOREM 4.** *Let  $(X, \mathfrak{A}, P)$ ,  $\mathfrak{B}$ , and  $\mathfrak{A}'$  be as in Lemma 3. Suppose  $P^{\mathfrak{B}}$  is regular and  $p(\cdot, \cdot|\mathfrak{B})$  is a version such that*

$$(4.1) \quad p(w, \cdot|\mathfrak{B}) \text{ is a measure } P_{\mathfrak{B}} \text{ a.e.}$$

$$(4.2) \quad p(w, \cdot|\mathfrak{B}) \ll Q(P_{\mathfrak{B}} \text{ a.e.}) \text{ where } Q \text{ is a probability measure on } \mathfrak{A}.$$

*Let  $P_c$  be a canonical extension of  $P$  to  $\mathfrak{A}'$  with respect to  $\alpha, \beta$  and  $K$  as in (1.1). Then,  $P_c^{\mathfrak{B}}$  is regular.*

*Proof.* Suppose  $K' = K_2 \setminus K_1$ , where  $K_1 \subset H \subset K_2$ ,  $Q_*(H) = Q(K_1)$  and  $Q^*(H) = Q(K_2)$ . Consider any set  $A \subset K_2 \setminus H$  where  $A \in \mathfrak{A}$ .  $Q(A) = 0$ . By (4.2)  $p(w, A|\mathfrak{B}) = 0$  ( $P_{\mathfrak{B}}$  a.e.) and so therefore  $P(A) = 0$  also. Similarly, if  $B \subset H \setminus K_1$ , where  $B \in \mathfrak{A}$ , then  $Q(B) = 0$  and hence  $p(w, B|\mathfrak{B}) = 0$  and so  $P(B) = 0$  also. Thus  $p^*(w, H|\mathfrak{B}) = p(w, K_2|\mathfrak{B})$  ( $P^{\mathfrak{B}}$  a.e.) and  $p(w, K_1|\mathfrak{B}) = p_*(w, H|\mathfrak{B})$  ( $P_{\mathfrak{B}}$  a.e.). Also,  $P(K_1) = P_*(H)$  and  $P^*(H) = P(K_2)$ . According to Remark 2,  $p(w, \cdot|\mathfrak{B})$  can be extended canonically to  $\mathfrak{A}'$  with respect to  $\alpha, \beta$  and  $K'$  and by Lemma 3 the proof is complete.

The following result is a consequence of Theorem 4.

**THEOREM 5.** *Let  $(X, \mathfrak{A}, P)$ ,  $\mathfrak{B}$  and  $\mathfrak{A}'$  be as in Lemma 3. Suppose  $P^{\mathfrak{B}}$  is regular and  $p(\cdot, \cdot|\mathfrak{B})$  is a version such that*

$$(5.1) \quad p(w, \cdot|\mathfrak{B}) \text{ is a measure } P_{\mathfrak{B}} \text{ a.e.}$$

$$(5.2) \quad \text{there exists a sequence } \{w_n\}_{n=1}^{\infty} \text{ such that for every } \varepsilon > 0 \text{ and any } w(P_{\mathfrak{B}} \text{ a.e.) there is an } w_n \text{ with}$$

$$\sup_{A \in \mathfrak{A}'} |p(w, A|\mathfrak{B}) - p(w_n, A|\mathfrak{B})| < \varepsilon .$$

*Let  $P_c$  be a canonical extension of  $P$  to  $\mathfrak{A}'$  with  $\alpha, \beta$  and  $K$  as in (1.1). Then,  $P_c^{\mathfrak{B}}$  is regular.*

*Proof.* Let  $Q$  be a probability measure defined as

$$\sum_{n=1}^{\infty} \frac{1}{2^n} p(w_n, \cdot | \mathfrak{B}) .$$

Condition (5.2) insures that  $p(w, \cdot | \mathfrak{B}) \ll Q P_{\mathfrak{B}}$  a.e. and the result follows from Theorem 4.

The following proposition is presented for the sake of completeness.

Let  $(X, \mathfrak{A}, P)$  be a probability space with  $(X, \bar{\mathfrak{A}}, \bar{P})$  denoting the completion. Suppose  $H$  is in  $\bar{\mathfrak{A}}$  but not in  $\mathfrak{A}$ . Let  $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$ .

**PROPOSITION 6.** *Let  $(X, \mathfrak{A}, P)$ ,  $\mathfrak{B} \subset \mathfrak{A}$ , and  $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$  with  $H \in \bar{\mathfrak{A}} \setminus \mathfrak{A}$  be given. Let  $P_1$  denote the restriction of  $\bar{P}$  to  $\mathfrak{A}'$ . If  $P^{\mathfrak{B}}$  is regular then so is  $P_1^{\mathfrak{B}}$ .*

The proof can be viewed as an easy consequence of Lemma 3 and is therefore omitted.

The remainder of this paper is devoted to the single

**THEOREM 7.** *Let  $(X, \mathfrak{A}, P)$  be a probability space with  $\mathfrak{A}$  generated by a countable field,  $\mathfrak{A}$ . Let  $\mathfrak{A}'$  be the field generated by  $\mathfrak{A}$  and  $H$  and  $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$ . Let  $P_e$  be a canonical extension of  $P$  to  $\mathfrak{A}'$  with respect to  $\alpha, \beta$  and  $K$  and suppose  $P_e^{\mathfrak{B}}$  is regular where  $\mathfrak{B} \subset \mathfrak{A}$ . Then, there exists a version  $p'(\cdot, \cdot | \mathfrak{B})$  of  $P_e^{\mathfrak{B}}$  such that  $P_{\mathfrak{B}}$  a.e.  $p'(w, \cdot | \mathfrak{B})$  is a probability measure which is a canonical extension of  $p'(w, \cdot | \mathfrak{B}) | \mathfrak{A}$  with respect to the same  $\alpha, \beta$  and  $K$  that are associated with  $P_e$ .*

The following lemmas are introduced before presenting the main body of the proof.

**LEMMA 8.** *Let  $(X, \mathfrak{A}, P)$  be a probability space with  $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$  and  $P_e$  an arbitrary simple extension of  $P$  to  $\mathfrak{A}'$ . Let  $K$  be the set associated with a canonical extension of  $P$  to  $\mathfrak{A}'$  as in Remark 2. Then, for each set  $A \in \mathfrak{A}$  there exist constants  $\alpha_A$  and  $\beta_A$  with  $0 \leq \alpha_A \leq 1$  and  $0 \leq \beta_A \leq 1$  and such that  $P_e(AHK) = \alpha_A P(AK)$  and  $P_e(AH^c K) = \beta_A P(AK)$ .*

*Proof.* For  $A \in \mathfrak{A}, AK \supset AHK$ . If  $P(AK) \neq 0$ , then  $\alpha_A = P_e(AHK)/P(AK)$ ; otherwise, let  $\alpha_A$  be arbitrary between zero and one.  $\beta_A$  is obtained similarly.

**LEMMA 9.** *Assume the hypothesis of Lemma 8. Let  $\mathfrak{A}$  be a field which generates  $\mathfrak{A}$  and  $\mathfrak{A}'$  the field generated by  $\mathfrak{A}$  and  $H$ . Let  $\alpha(\mathfrak{A}) \equiv \sup_{A \in \mathfrak{A}} \alpha_A$  and  $\beta(\mathfrak{A}) \equiv \sup_{A \in \mathfrak{A}} \beta_A$ . Then, a necessary and sufficient condition that  $P_e$  be a canonical extension of  $P$  to  $\mathfrak{A}'$  is that*

$\alpha(\mathcal{A}) = \alpha_x$  or  $\beta(\mathcal{A}) = \beta_x$  for some  $\mathcal{A}$  which generates  $\mathfrak{A}$ .

*Proof.* Necessity is obvious and only sufficiency is proved. Let  $\mathcal{A}$  be some field which generates  $\mathfrak{A}$  and  $\alpha(\mathcal{A}) = \alpha_x$ . (For simplicity, write  $\alpha(\mathcal{A}) = \alpha$ .) By hypothesis,

$$P_e(HK) = \alpha P(K) .$$

For  $A \in \mathcal{A}$  it follows by Lemma 8 that

$$(9.1) \quad P_e(AHK) = \alpha_A P(AK)$$

and

$$(9.2) \quad P_e(A^c HK) = \alpha_{A^c} P(A^c K) .$$

The following equalities also hold

$$(9.3) \quad \alpha P(K) = \alpha P(AK) + \alpha P(A^c K)$$

$$(9.4) \quad P_e(HK) = P_e(AHK) + P_e(A^c HK) .$$

By (9.1) – (9.4) it follows that

$$(9.5) \quad 0 = (\alpha - \alpha_A)P(AK) + (\alpha - \alpha_{A^c})P(A^c K) .$$

If  $P(AK) = 0$ , set  $\alpha_A = \alpha$  or if  $P(A^c K) = 0$ , set  $\alpha_{A^c} = \alpha$  (see Lemma 8). Otherwise, (9.5) forces  $\alpha - \alpha_A = \alpha - \alpha_{A^c} = 0$  and hence for any  $A \in \mathcal{A}$ ,  $P_e(AHK) = \alpha P(AK)$ .

Next, the fact that  $P_e(AH^c K) = \beta P(AK)$ ,  $\beta = 1 - \alpha$ , is immediate from the following chain of equalities:

$$\begin{aligned} P(A) &= P_e(AH + AH^c) = P_e((AH + AH^c)K^c) + P_e(AHK) \\ &\quad + P_e(AH^c K) = P(AK^c) + \alpha P(AK) + P_e(AH^c K) . \end{aligned}$$

Hence, where  $\mathcal{A}' = \{A_1 H + A_2 H^c \mid A_i \in \mathcal{A} \ i = 1, 2\}$ ,  $A'$  in  $\mathcal{A}'$  can be written as  $A' = A_1 H + A_2 H^c$  and it follows that

$$P_e(A') = P(A'K^c) + \alpha P(A_1 K) + \beta P(A_2 K) .$$

Finally, let

$$\begin{aligned} \phi_\alpha &= \{A \in \mathfrak{A} \mid P_e(AHK) = \alpha P(AK)\} \\ \phi_\beta &= \{A \in \mathfrak{A} \mid P_e(AH^c K) = \beta P(AK)\} . \end{aligned}$$

Both  $\phi_\alpha$  and  $\phi_\beta$  are monotone classes containing  $\mathcal{A}$ ; hence, the proof is complete by the monotone class theorem (see [3], p. 60).

Theorem 7 can now be proved.

*Proof.* For  $w \in X$ ,  $P_{\mathfrak{B}}$  a.e., and  $A \in \mathcal{A}$ , write

$$p'(w, AHK|\mathfrak{B}) = \alpha_{w,A}p(w, AK|\mathfrak{B})$$

where  $0 \leq \alpha_{w,A} \leq 1$  as in Lemma 8 and  $p(w, \cdot|\mathfrak{B})$  will be written for  $p'(w, \cdot|\mathfrak{B})|_{\mathfrak{B}}$ . For fixed  $A \in \mathcal{A}$ ,  $\alpha_{w,A}$  is a  $\mathfrak{B}$ -measurable function where

$$(7.1) \quad \begin{aligned} \alpha_{w,A} &= p'(w, AHK|\mathfrak{B})/p(w, AK|\mathfrak{B}) \text{ for } p(w, AK|\mathfrak{B}) \neq 0 \\ \alpha_{w,A} &= \alpha \text{ if } p(w, AK|\mathfrak{B}) = 0. \end{aligned}$$

(In (7.1)  $\alpha$  is associated with  $P_c$  and by Lemma 9,  $\alpha = \sup_{A \in \mathcal{A}} \alpha_A$ ).

For  $A \in \mathcal{A}$  let

$$(7.2) \quad U_A \equiv \{w | \alpha_{w,A} > \alpha\}.$$

Observe that  $U_A$  is contained in the complement of the set of  $w$ 's where  $p(w, AK|\mathfrak{B}) = 0$ .

Also,  $U_A \in \mathfrak{B}$  (see (7.1)). Hence, since  $P_c$  is a canonical extension, it follows that

$$(7.3) \quad \alpha P(AU_A K) = P_c(AU_A HK) = \int_{U_A} p'(w, AHK|\mathfrak{B}) dP_c.$$

Also,

$$(7.4) \quad \begin{aligned} \int_{U_A} p'(w, AHK|\mathfrak{B}) dP_c &= \int_{U_A} \alpha_{w,A} p(w, AK|\mathfrak{B}) dP \geq \\ &\int_{U_A} \alpha p(w, AK|\mathfrak{B}) dP = \alpha P(AU_A K). \end{aligned}$$

Hence, the defining properties of  $U_A$  together with (7.3) and (7.4) say that  $P(U_A) = 0$ .

If  $L_A \equiv \{w | \alpha_{w,A} < \alpha\}$ , then an argument similar to the preceding one shows  $P(L_A) = 0$ .

Hence, for each set  $A \in \mathcal{A}$ , there exists a  $P_{\mathfrak{B}}$  null set on the complement of which  $\alpha_{w,A} = \alpha$ . But where  $\mathcal{A}$  is countable, it follows that there exists a  $P_{\mathfrak{B}}$  null set,  $N$ , on the complement of which  $\alpha_{w,A} = \alpha$  for all  $A \in \mathcal{A}$ . Thus,

$$(7.5) \quad p'(w, AHK|\mathfrak{B}) = \alpha p(w, AK|\mathfrak{B})$$

for all  $w \in N^c$  and  $A \in \mathcal{A}$ .

Finally, if  $\alpha_w \equiv \sup_{A \in \mathcal{A}} \alpha_{w,A}$ , then it is immediate from (7.5) that  $P_{\mathfrak{B}}$  a.e.  $\alpha_w = \alpha = \alpha_X$  and by Lemma 9 the theorem is proved.

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