CANONICAL EXTENSIONS OF MEASURES AND THE EXTENSION OF REGULARITY OF CONDITIONAL PROBABILITIES

LOUIS HARVEY BLAKE
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Let \((\Omega, \mathcal{A}, P)\) be a probability space with \(\mathcal{B}\) a sub \(\sigma\)-field of \(\mathcal{A}\). Let \(\mathcal{A}' \equiv \sigma(\mathcal{A}, H)\), the \(\sigma\)-field generated by \(\mathcal{A}\) and \(H\), where \(H\) is a subset of \(\Omega\) not in \(\mathcal{A}\). \(P_e\) will be called a simple extension of \(P\) to \(\mathcal{A}'\) if \(P_e\) is a probability measure on \(\mathcal{A}'\) which agrees with \(P\) on \(\mathcal{A}\).

The purpose of this paper is to use a particular type of simple extension called a canonical extension, denoted as \(P_c\), to examine under what conditions the regularity of the conditional probability \(P^\mathcal{B}\) will extend to the regularity of \(P^\mathcal{A}_c\). Also, if \(\mathcal{A}\) is countably generated and \(P^\mathcal{B}\) is regular, a characterization of \(P^\mathcal{A}_c\) in terms of \(P^\mathcal{B}\) will be given.

The terminology in the following definitions will be used throughout this paper.

**Definition.** The conditional probability of a set \(A \in \mathcal{A}\) given the \(\sigma\)-field \(\mathcal{B}\) is a \(\mathcal{B}\)-measurable function denoted by \(P^\mathcal{B}(\cdot, A)\) such that for every \(B \in \mathcal{B}\)

\[
\int_B P^\mathcal{B}(\cdot, A) dP^\mathcal{B} = P(AB).
\]

**Definition.** The conditional probability (given \(\mathcal{B}\)) is the collection of functions

\[
\{P^\mathcal{B}(\cdot, A) | A \in \mathcal{A}\}.
\]

This collection is denoted by \(P^\mathcal{B}\).

**Definition.** For \(A \in \mathcal{A}\), a version of \(P^\mathcal{B}(\cdot, A)\) is a selection from the equivalence class of \(P^\mathcal{B}(\cdot, A)\) which will be denoted by \(p(\cdot, A | \mathcal{B})\).

**Definition.** A version of the conditional probability \(P^\mathcal{B}\) is a function \(p(\cdot, \cdot | \mathcal{B})\) on \(X \times \mathcal{A}\) such that for each \(A \in \mathcal{A}\) \(p(\cdot, A | \mathcal{B})\) is a version of \(P^\mathcal{B}(\cdot, A)\). Also \(p(w, \cdot | \mathcal{B})\) will denote a section of \(p(\cdot, \cdot | \mathcal{B})\) at \(w \in X\).

**Definition.** A conditional probability \(P^\mathcal{B}\) is called regular if there exists a version, \(p(\cdot, \cdot | \mathcal{B})\), such that \(p(w, \cdot | \mathcal{B})\) is a measure on \(\mathcal{A}\) \(P^\mathcal{B}\) a.e.

Before the main body of the paper is presented, it should be
observed that the regularity of $P_{\circ}$ itself is not in general sufficient to insure the regularity of $P_{\circ}$; for example, see [2], p. 210.

Finally, the scope of this paper is limited to results on canonical extensions. A forthcoming paper will deal with the preservation of regularity for simple extensions.

The main results. Observe that the σ-field
\[ \mathcal{A}' = \{ A_1 H + A_2 H^c | A_1, A_2 \in \mathcal{A} \}, \]
and make

**DEFINITION 1.** Let $A'$ be any element of $\mathcal{A}'$ with $A' = A_1 H + A_2 H^c$ for some $A_1$ and $A_2$ in $\mathcal{A}$. A simple extension will be called a canonical extension, $P_{\circ}$, if there exists a number $\alpha$ between zero and one with $\beta = 1 - \alpha$ and $K \in \mathcal{A}$ so that

\[ (1.1) \]
(a) $A' K^c \in \mathcal{A}$
(b) $P_{\circ}(A') = P(A' K^c) + \alpha P(A_1 K) + \beta P(A_2 K)$

with $P_{\circ}$ a well defined probability measure on $\mathcal{A}'$.

Marczewski and Los have shown, [4], that for any subset of $X$ not in $\mathcal{A}$, say $H$, there always exists a canonical extension $P_{\circ}$ on $\mathcal{A}'$. (It has been shown by the author in [1] that there exist many simple extensions which are not canonical.)

**REMARK 2.** One way of obtaining the set $K$ of Definition 1 is by letting $K_1$ be an element of $\mathcal{A}$ such that $(PK_1) = P_{\circ}(H)$ and $K_2$ be an element of $\mathcal{A}$ such that $P(K_2) = P^*(H)$ with $K_1 \subset H \subset K_2$. Then, simply define $K = K_2 \setminus K_1$. (See [2], P. 71). Observe that there exists another $K' \in \mathcal{A}$ which will extend $P$ canonically to $\mathcal{A}'$ as in Definition 1 if and only if $P(K \Delta K') = 0$.

**LEMMA 3.** Let $(X, \mathcal{A}, P), \mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{W} = \sigma(\mathcal{A}, H)$ be given. Let $p(\cdot, \cdot | \mathcal{B})$ be a version of $P_{\circ}$ which makes $P_{\circ}$ regular. Let $P_{\circ}$ be a canonical extension of $P$ to $\mathcal{W}$ with $\alpha, \beta$ and $K$ as in Definition 1. Suppose for $w, P_{\circ}$ a.e., $p_{\circ}(w, \cdot | \mathcal{B})$ is a canonical extension of $p(w, \cdot | \mathcal{B})$ to $\mathcal{W}$ with the same $\alpha$ and $\beta$ and $K$ as $P_{\circ}$. Then, $P_{\circ}$ is regular.

**Proof.** It will suffice to produce a version of $P_{\circ}$ which makes $P_{\circ}$ regular.

Let $A' \in \mathcal{A}'$ with $A' = A_1 H + A_2 H^c$ for some $A_1$ and $A_2$ in $\mathcal{A}$. For $w, P_{\circ}$ a.e.,

\[ P_{\circ}(w, A' | \mathcal{B}) = p(w, A' K^c | \mathcal{B}) + \alpha p(w, A_1 K | \mathcal{B}) \]
\[ + \beta p(w, A_2 K | \mathcal{B}). \]
Thus it is immediate from (3.1) that \( p_c(\cdot, A'|\mathcal{B}) \) is a \( \mathcal{B} \)-measurable function for all \( A' \in \mathcal{A}' \) and for \( w, P_\alpha \) a.e., \( p_c(w, \cdot|\mathcal{B}) \) is a measure on \( \mathcal{A}' \). It is also clear that for \( A' \in \mathcal{A}' \) and \( B \in \mathcal{B} \)

\[
(3.2) \quad \int_B P_c(\cdot, A'|\mathcal{B})dP_c = P_c(A'B) .
\]

For, integrating the right side of (3.1) with respect to \( P \) gives

\[
P(A'K^cB) + \alpha P(A_1KB) + \beta P(A_2KB) = P_c(A'B) .
\]

But \( P_c = P \) on \( \mathcal{B} \) and so the integral of the right side of (3.1) is exactly the left side of (3.2).

Hence, \( p_c(\cdot, \cdot|\mathcal{B}) \) is the desired version.

**Theorem 4.** Let \((X, \mathcal{A}, P), \mathcal{B}, \) and \( \mathcal{A}' \) be as in Lemma 3. Suppose \( P_\alpha \) is regular and \( p(\cdot, \cdot|\mathcal{B}) \) is a version such that

\[
(4.1) \quad p(w, \cdot|\mathcal{B}) \text{ is a measure } P_\alpha \text{ a.e.}
\]

\[
(4.2) \quad p(w, \cdot|\mathcal{B}) \ll Q(P_\alpha \text{ a.e.) where } Q \text{ is a probability measure on } \mathcal{A}.
\]

Let \( P_c \) be a canonical extension of \( P \) to \( \mathcal{A}' \) with respect to \( \alpha, \beta \) and \( K \) as in (1.1). Then, \( P_c^\alpha \) is regular.

**Proof.** Suppose \( K' = K_2\setminus K_1 \), where \( K_1 \subset H \subset K_2 \), \( Q_\alpha(H) = Q(K_1) \) and \( Q^*(H) = Q(K_2) \). Consider any set \( A \subset K_2\setminus H \) where \( A \in \mathcal{A} \). \( Q(A) = 0 \).

By (4.2) \( p(w, A|\mathcal{B}) = 0 \) (\( P_\alpha \) a.e.) and so therefore \( P(A) = 0 \) also. Similarly, if \( B \subset H\setminus K_1 \), where \( B \in \mathcal{A} \), then \( Q(B) = 0 \) and hence \( p(w, B|\mathcal{B}) = 0 \) and so \( P(B) = 0 \) also. Thus \( p^*(w, H|\mathcal{B}) = p(w, K_2|\mathcal{B}) \) (\( P_\alpha \) a.e.) and \( p(w, K_1|\mathcal{B}) = p_\alpha(w, H|\mathcal{B})(P_\alpha \text{ a.e.)}. \) Also, \( P(K_1) = P_\alpha(H) \) and \( P^*(H) = P(K_2) \).

According to Remark 2, \( p(w, \cdot|\mathcal{B}) \) can be extended canonically to \( \mathcal{A}' \) with respect to \( \alpha, \beta \) and \( K' \) and by Lemma 3 the proof is complete.

The following result is a consequence of Theorem 4.

**Theorem 5.** Let \((X, \mathcal{A}, P), \mathcal{B} \) and \( \mathcal{A}' \) be as in Lemma 3. Suppose \( P_\alpha \) is regular and \( p(\cdot, \cdot|\mathcal{B}) \) is a version such that

\[
(5.1) \quad p(w, \cdot|\mathcal{B}) \text{ is a measure } P_\alpha \text{ a.e.}
\]

\[
(5.2) \quad \text{there exists a sequence } \{w_n\}_{n=1} \text{ such that for every } \varepsilon > 0 \text{ and any } w(P_\alpha \text{ a.e.) there is an } w_n \text{ with}
\]

\[
\sup_{A \in \mathcal{A}} |p(w, A|\mathcal{B}) - p(w_n, A|\mathcal{B})| < \varepsilon .
\]

Let \( P_c \) be a canonical extension of \( P \) to \( \mathcal{A}' \) with \( \alpha, \beta \) and \( K \) as in (1.1). Then, \( P_c^\alpha \) is regular.
Proof. Let $Q$ be a probability measure defined as
\[ \sum_{n=1}^{\infty} \frac{1}{2^n} p(w_n, \cdot | \mathcal{B}) . \]
Condition (5.2) insures that $p(w, \cdot | \mathcal{B}) \ll Q$ $P_\mathcal{B}$ a.e. and the result follows from Theorem 4.

The following proposition is presented for the sake of completeness. Let $(X, \mathcal{A}, P)$ be a probability space with $(X, \mathcal{A}, \overline{P})$ denoting the completion. Suppose $H$ is in $\mathcal{A}$ but not in $\mathcal{A}$. Let $\mathcal{W} = \sigma(\mathcal{A}, H)$.

PROPOSITION 6. Let $(X, \mathcal{A}, P)$, $\mathcal{B} \subset \mathcal{A}$, and $\mathcal{W} = \sigma(\mathcal{A}, H)$ with $H \in \mathcal{W} \setminus \mathcal{A}$ be given. Let $P_\mathcal{W}$ denote the restriction of $\overline{P}$ to $\mathcal{W}$. If $P_{\mathcal{B}}$ is regular then so is $P_\mathcal{W}$.

The proof can be viewed as an easy consequence of Lemma 3 and is therefore omitted.

The remainder of this paper is devoted to the single

THEOREM 7. Let $(X, \mathcal{A}, P)$ be a probability space with $\mathcal{A}$ generated by a countable field, $\mathcal{A}$. Let $\mathcal{A}'$ be the field generated by $\mathcal{A}$ and $H$ and $\mathcal{W} = \sigma(\mathcal{A}')$. Let $P_\mathcal{W}$ be a canonical extension of $P$ to $\mathcal{W}$ with respect to $\alpha$, $\beta$ and $K$ and suppose $P_{\mathcal{B}}$ is regular where $\mathcal{B} \subset \mathcal{A}$. Then, there exists a version $p'(\cdot, \cdot | \mathcal{B})$ of $P_{\mathcal{B}}$ such that $P_{\mathcal{B}}$ a.e. $p'(w, \cdot | \mathcal{B})$ is a probability measure which is a canonical extension of $p'(w, \cdot | \mathcal{B}) | \mathcal{A}$ with respect to the same $\alpha$, $\beta$ and $K$ that are associated with $P_\mathcal{W}$.

The following lemmas are introduced before presenting the main body of the proof.

LEMMA 8. Let $(X, \mathcal{A}, P)$ be a probability space with $\mathcal{W} = \sigma(\mathcal{A}, H)$ and $P_\mathcal{W}$ an arbitrary simple extension of $P$ to $\mathcal{W}$. Let $K$ be the set associated with a canonical extension of $P$ to $\mathcal{W}$ as in Remark 2. Then, for each set $A \in \mathcal{A}$ there exist constants $\alpha_A$ and $\beta_A$ with $0 \leq \alpha_A \leq 1$ and $0 \leq \beta_A \leq 1$ and such that $P_\mathcal{W}(AHK) = \alpha_A P(AK)$ and $P_\mathcal{W}(AH^cK) = \beta_A P(\bar{AK})$.

Proof. For $A \in \mathcal{A}$, $AK \supset AHK$. If $P(\bar{AK}) \neq 0$, then $\alpha_A = P_\mathcal{W}(AHK)/P(\bar{AK})$; otherwise, let $\alpha_A$ be arbitrary between zero and one. $\beta_A$ is obtained similarly.

LEMMA 9. Assume the hypothesis of Lemma 8. Let $\mathcal{A}$ be a field which generates $\mathcal{A}$ and $\mathcal{A}'$ the field generated by $\mathcal{A}$ and $H$. Let $\alpha(\mathcal{A}) = \sup_{A \in \mathcal{A}} \alpha_A$ and $\beta(\mathcal{A}) = \sup_{A \in \mathcal{A}} \beta_A$. Then, a necessary and sufficient condition that $P_\mathcal{W}$ be a canonical extension of $P$ to $\mathcal{W}$ is that
\(\alpha(\mathcal{A}) = \alpha_x\) or \(\beta(\mathcal{A}) = \beta_x\) for some \(\mathcal{A}\) which generates \(\mathcal{A}\).

**Proof.** Necessity is obvious and only sufficiency is proved. Let \(\mathcal{A}\) be some field which generates \(\mathcal{A}\) and \(\alpha(\mathcal{A}) = \alpha_x\). (For simplicity, write \(\alpha(\mathcal{A}) = \alpha\).) By hypothesis,

\[ P_e(HK) = \alpha P(K). \]

For \(A \in \mathcal{A}\) it follows by Lemma 8 that

\[ P_e(AHK) = \alpha A P(AK). \]

and

\[ P_e(A^eHK) = \alpha A^e P(A^eK). \]

The following equalities also hold

\[ \alpha P(K) = \alpha P(AK) + \alpha P(A^eK) \]

\[ P_e(HK) = P_e(AHK) + P_e(A^eHK). \]

By (9.1) - (9.4) it follows that

\[ 0 = (\alpha - \alpha^A) P(AK) + (\alpha - \alpha^A) P(A^eK). \]

If \(P(AK) = 0\), set \(\alpha^A = \alpha\) or if \(P(A^eK) = 0\), set \(\alpha^A = \alpha\) (see Lemma 8). Otherwise, (9.5) forces \(\alpha - \alpha^A = \alpha - \alpha^A = 0\) and hence for any \(A \in \mathcal{A}\), \(P_e(AHK) = \alpha P(AK)\).

Next, the fact that \(P_e(A^cHK) = \beta P(AK), \beta = 1 - \alpha\), is immediate from the following chain of equalities:

\[ P(A) = P_e(AH + AH^c) = P_e((AH + AH^c)K) + P_e(AHK) + P_e(AH^cK) = P(AK) + \alpha P(AK) + P_e(AH^cK). \]

Hence, where \(\mathcal{A}' = \{A_iH + A_iH^c | A_i \in \mathcal{A} i = 1, 2\}\), \(A' \in \mathcal{A}'\) can be written as \(A' = A_1H + A_2H^c\) and it follows that

\[ P_e(A') = P(A'K) + \alpha P(A_1K) + \beta P(A_2K). \]

Finally, let

\[ \phi_\alpha = \{A \in \mathcal{A} | P_e(AHK) = \alpha P(AK)\} \]

\[ \phi_\beta = \{A \in \mathcal{A} | P_e(A^cHK) = \beta P(AK)\}. \]

Both \(\phi_\alpha\) and \(\phi_\beta\) are monotone classes containing \(\mathcal{A}\); hence, the proof is complete by the monotone class theorem (see [3], p. 60).

Theorem 7 can now be proved.

**Proof.** For \(w \in X, P_\#\) a.e., and \(A \in \mathcal{A}\), write
where \( 0 \leq \alpha_{w,A} \leq 1 \) as in Lemma 8 and \( p(w, \cdot | \mathcal{B}) \) will be written for \( p'(w, \cdot | \mathcal{B}) \). For fixed \( A \in \mathcal{F} \), \( \alpha_{w,A} \) is a \( \mathcal{B} \)-measurable function where

\[
\begin{align*}
\alpha_{w,A} &= p'(w, AHK|\mathcal{B})/p(w, AK|\mathcal{B}) \quad \text{for } p(w, AK|\mathcal{B}) \neq 0 \\
\alpha_{w,A} &= \alpha \quad \text{if } p(w, AK|\mathcal{B}) = 0.
\end{align*}
\]

(In (7.1) \( \alpha \) is associated with \( P_\varepsilon \) and by Lemma 9, \( \alpha = \sup_{A \in \mathcal{F}} \alpha_A \)).

For \( A \in \mathcal{F} \) let

\[
U_A = \{ w | \alpha_{w,A} > \alpha \}.
\]

Observe that \( U_A \) is contained in the complement of the set of \( w \)'s where \( p(w, AK|\mathcal{B}) = 0 \). Also, \( U_A \in \mathcal{B} \) (see (7.1)). Hence, since \( P_\varepsilon \) is a canonical extension, it follows that

\[
\alpha P(AU_AK) = P_\varepsilon(AU_AHK) = \int_{U_A} p'(w, AHK|\mathcal{B})dP_\varepsilon.
\]

Also,

\[
\int_{U_A} p'(w, AHK|\mathcal{B})dP_\varepsilon = \int_{U_A} \alpha_{w,A} p(w, AK|\mathcal{B})dP = \alpha P(AU_AK).
\]

Hence, the defining properties of \( U_A \) together with (7.3) and (7.4) say that \( P(U_A) = 0 \).

If \( L_A = \{ w | \alpha_{w,A} < \alpha \} \), then an argument similar to the preceding one shows \( P(L_A) = 0 \).

Hence, for each set \( A \in \mathcal{F} \), there exists a \( P_\varepsilon \) null set on the complement of which \( \alpha_{w,A} = \alpha \). But where \( \mathcal{F} \) is countable, it follows that there exists a \( P_\varepsilon \) null set, \( N \), on the complement of which \( \alpha_{w,A} = \alpha \) for all \( A \in \mathcal{F} \). Thus,

\[
p'(w, AHK|\mathcal{B}) = \alpha p(w, AK|\mathcal{B})
\]

for all \( w \in N^c \) and \( A \in \mathcal{F} \).

Finally, if \( \alpha_w = \sup_{A \in \mathcal{F}} \alpha_{w,A} \), then it is immediate from (7.5) that \( P_\varepsilon \) a.e. \( \alpha_w = \alpha = \alpha_\varepsilon \) and by Lemma 9 the theorem is proved.

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<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anatole Beck and Peter Warren, Weak orthogonality</td>
<td>1</td>
</tr>
<tr>
<td>Jonnie Bee Bednar and Howard E. Lacey, Concerning Banach spaces whose duals are abstract L-spaces</td>
<td>13</td>
</tr>
<tr>
<td>Louis Harvey Blake, Canonical extensions of measures and the extension of regularity of conditional probabilities</td>
<td>25</td>
</tr>
<tr>
<td>R. A. Brooks, Conditional expectations associated with stochastic processes</td>
<td>33</td>
</tr>
<tr>
<td>Theodore Allen Burton and Ronald Calvin Grimmer, On the asymptotic behavior of solutions of $x'' + a(t)f(x) = e(t)$</td>
<td>43</td>
</tr>
<tr>
<td>Stephen LaVern Campbell, Operator-valued inner functions analytic on the closed disc</td>
<td>57</td>
</tr>
<tr>
<td>Yuen-Kwok Chan, A constructive study of measure theory</td>
<td>63</td>
</tr>
<tr>
<td>Alexander Munro Davie and Bernt Karsten Oksendal, Peak interpolation sets for some algebras of analytic functions</td>
<td>81</td>
</tr>
<tr>
<td>H. P. Dikshit, Absolute total-effective $(N, p_n)(c, 1)$ method</td>
<td>89</td>
</tr>
<tr>
<td>James Daniel Halpern, On a question of Tarski and a maximal theorem of Kurepa</td>
<td>111</td>
</tr>
<tr>
<td>Gerald L. Itzkowitz, A characterization of a class of uniform spaces that admit an invariant integral</td>
<td>123</td>
</tr>
<tr>
<td>Mo Tak Kiang, Semigroups with diminishing orbital diameters</td>
<td>143</td>
</tr>
<tr>
<td>Glenn Richard Luecke, A class of operators on Hilbert space</td>
<td>153</td>
</tr>
<tr>
<td>R. James Milgram, Group representations and the Adams spectral sequence</td>
<td>157</td>
</tr>
<tr>
<td>G. S. Monk, On the endomorphism ring of an abelian $p$-group, and of a large subgroup</td>
<td>183</td>
</tr>
<tr>
<td>Yasutoshi Nomura, Homology of a group extension</td>
<td>195</td>
</tr>
<tr>
<td>R. Michael Range, Approximation to bounded holomorphic functions on strictly pseudoconvex domains</td>
<td>203</td>
</tr>
<tr>
<td>Norman R. Reilly, Inverse semigroups of partial transformations and $\theta$-classes</td>
<td>215</td>
</tr>
<tr>
<td>Chris Rorres, Strong concentration of the spectra of self-adjoint operators</td>
<td>237</td>
</tr>
<tr>
<td>Saharon Shelah, A combinatorial problem; stability and order for models and theories in infinitary languages</td>
<td>247</td>
</tr>
<tr>
<td>George Gustave Weill, Vector space decompositions and the abstract imitation problem</td>
<td>263</td>
</tr>
<tr>
<td>Arthur Thomas White, On the genus of the composition of two graphs</td>
<td>275</td>
</tr>
</tbody>
</table>