Pacific Journal of Mathematics

CANONICAL EXTENSIONS OF MEASURES AND THE EXTENSION OF REGULARITY OF CONDITIONAL PROBABILITIES

LOUIS HARVEY BLAKE

Vol. 41, No. 1 November 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 1, 1972

CANONICAL EXTENSIONS OF MEASURES AND THE EXTENSION OF REGULARITY OF CONDITIONAL PROBABILITIES*

LOUIS H. BLAKE

Let $(\Omega, \mathfrak{A}, P)$ be a probability space with $\mathfrak{B}$ a sub σ-field of $\mathfrak{A}$. Let $\mathfrak{A}' \equiv \sigma(\mathfrak{A}, H)$, the σ-field generated by $\mathfrak{A}$ and H, where H is a subset of Ω not in $\mathfrak{A}$. P_e will be called a simple extension of P to $\mathfrak{A}'$ if P_e is a probability measure on $\mathfrak{A}'$ which agrees with P on $\mathfrak{A}$.

The purpose of this paper is to use a particular type of simple extension called a canonical extension, denoted as P_c to examine under what conditions the regularity of the conditional probability $P^{\mathfrak{B}}$ will extend to the regularity of $P_c^{\mathfrak{B}}$. Also, if $\mathfrak{A}$ is countably generated and $P_c^{\mathfrak{B}}$ is regular, a characterization of $P_c^{\mathfrak{B}}$ in terms of $P^{\mathfrak{B}}$ will be given.

The terminology in the following definitions will be used throughout this paper.

DEFINITION. The conditional probability of a set $A \in \mathfrak{A}$ given the σ-field $\mathfrak{B}$ is a $\mathfrak{B}$-measurable function denoted by $P^{\mathfrak{B}}(\cdot, A)$ such that for every $B \in \mathfrak{B}$

$$\int_B P^{\mathfrak{B}}(\cdot, A) dP_{\mathfrak{B}} = P(AB).$$

DEFINITION. The conditional probability (given $\mathfrak{B}$) is the collection of functions

$$\{P^{\mathfrak{B}}(\cdot, A) \mid A \in \mathfrak{A}\} .$$

This collection is denoted by $P^{\mathfrak{B}}$.

DEFINITION. For $A \in \mathfrak{A}$, a version of $P^{\mathfrak{B}}(\cdot, A)$ is a selection from the equivalence class of $P^{\mathfrak{B}}(\cdot, A)$ which will be denoted by $p(\cdot, A \mid \mathfrak{B})$.

DEFINITION. A version of the conditional probability $P^{\mathfrak{B}}$ is a function $p(\cdot, \cdot \mid \mathfrak{B})$ on $X \times \mathfrak{A}$ such that for each $A \in \mathfrak{A}$ $p(\cdot, A \mid \mathfrak{B})$ is a version of $P^{\mathfrak{B}}(\cdot, A)$. Also $p(w, \cdot \mid \mathfrak{B})$ will denote a section of $p(\cdot, \cdot \mid \mathfrak{B})$ at $w \in X$.

DEFINITION. A conditional probability $P^{\mathfrak{B}}$ is called regular if there exists a version, $p(\cdot, \cdot \mid \mathfrak{B})$, such that $p(w, \cdot \mid \mathfrak{B})$ is a measure on $\mathfrak{A}$ $P_{\mathfrak{B}}$ a.e.

Before the main body of the paper is presented, it should be observed that the regularity of $P^{\mathfrak{B}}$ itself is not in general sufficient to insure the regularity of $P_c^{\mathfrak{B}}$; for example, see [2], p. 210.

Finally, the scope of this paper is limited to results on canonical extensions. A forthcoming paper will deal with the preservation of regularity for simple extensions.

The main results. Observe that the σ-field

$$\mathfrak{A}' = \{A_1H + A_2H^c \mid A_1, A_2 \in \mathfrak{A}\} ,$$

and make

DEFINITION 1. Let A' be any element of $\mathfrak{A}'$ with $A' = A_1H + A_2H^c$ for some A_1 and A_2 in $\mathfrak{A}$. A simple extension will be called a canonical extension, P_c, if there exists a number α between zero and one with $\beta = 1 - \alpha$ and $K \in \mathfrak{A}$ so that

$$\text{(1.1)} \qquad \text{(a)} \quad A'K^c \in \mathfrak{A}$$
$$\text{(b)} \quad P_c(A') = P(A'K^c) + \alpha P(A_1K) + \beta P(A_2K)$$

with P_c a well defined probability measure on $\mathfrak{A}'$.

Marczewski and Los have shown, [4], that for any subset of X not in $\mathfrak{A}$, say H, there always exists a canonical extension P_c on $\mathfrak{A}'$. (It has been shown by the author in [1] that there exist many simple extensions which are not canonical.)

REMARK 2. One way of obtaining the set K of Definition 1 is by letting K_1 be an element of $\mathfrak{A}$ such that $(PK_1) = P_*(H)$ and K_2 be an element of $\mathfrak{A}$ such that $P(K_2) = P^*(H)$ with $K_1 \subset H \subset K_2$. Then, simply define $K = K_2 \backslash K_1$. (See [2], P. 71). Observe that there exists another $K' \in \mathfrak{A}$ which will extend P canonically to $\mathfrak{A}'$ as in Definition 1 if and only if $P(K \varDelta K') = 0$.

LEMMA 3. *Let $(X, \mathfrak{A}, P)$, $\mathfrak{B} \subset \mathfrak{A}$ and $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$ be given. Let $p(\cdot, \cdot \mid \mathfrak{B})$ be a version of $P^{\mathfrak{B}}$ which makes $P^{\mathfrak{B}}$ regular. Let P_c be a canonical extension of P to $\mathfrak{A}'$ with α, β and K as in Definition 1. Suppose for w, $P_{\mathfrak{B}}$ a.e., $p_c(w, \cdot \mid \mathfrak{B})$ is a canonical extension of $p(w, \cdot \mid \mathfrak{B})$ to $\mathfrak{A}'$ with the same α and β and K as P_c. Then, $P_c^{\mathfrak{B}}$ is regular.*

Proof. It will suffice to produce a version of $P_c^{\mathfrak{B}}$ which makes $P_c^{\mathfrak{B}}$ regular.

Let $A' \in \mathfrak{A}'$ with $A' = A_1H + A_2H^c$ for some A_1 and A_2 in $\mathfrak{A}$. For w, $P_{\mathfrak{B}}$ a.e.,

$$\text{(3.1)} \qquad \begin{aligned} P_c(w, A' \mid \mathfrak{B}) &= p(w, A'K^c \mid \mathfrak{B}) + \alpha p(w, A_1K \mid \mathfrak{B}) \\ &\quad + \beta p(w, A_2K \mid \mathfrak{B}) . \end{aligned}$$

Thus it is immediate from (3.1) that $p_c(\cdot, A' | \mathfrak{B})$ is a $\mathfrak{B}$-measurable function for all $A' \in \mathfrak{A}'$ and for w, $P_{\mathfrak{B}}$ a.e., $p_c(w, \cdot | \mathfrak{B})$ is a measure on $\mathfrak{A}'$. It is also clear that for $A' \in \mathfrak{A}'$ and $B \in \mathfrak{B}$

$$\int_B P_c(\cdot, A' | \mathfrak{B}) dP_c = P_c(A'B) . \tag{3.2}$$

For, integrating the right side of (3.1) with respect to P gives

$$P(A'K^cB) + \alpha P(A_1KB) + \beta P(A_2KB) = P_c(A'B) .$$

But $P_c = P$ on $\mathfrak{B}$ and so the integral of the right side of (3.1) is exactly the left side of (3.2) .

Hence, $p_c(\cdot, \cdot | \mathfrak{B})$ is the desired version.

THEOREM 4. *Let $(X, \mathfrak{A}, P)$, $\mathfrak{B}$, and $\mathfrak{A}'$ be as in Lemma 3. Suppose $P^{\mathfrak{B}}$ is regular and $p(\cdot, \cdot | \mathfrak{B})$ is a version such that*

(4.1) $p(w, \cdot | \mathfrak{B})$ *is a measure* $P_{\mathfrak{B}}$ *a.e.*

(4.2) $p(w, \cdot | \mathfrak{B}) \ll Q(P_{\mathfrak{B}}$ *a.e.) where Q is a probability measure on $\mathfrak{A}$.*

Let P_c be a canonical extension of P to $\mathfrak{A}'$ with respect to α, β and K as in (1.1). *Then, $P_c^{\mathfrak{B}}$ is regular.*

Proof. Suppose $K' = K_2 \backslash K_1$, where $K_1 \subset H \subset K_2$, $Q_*(H) = Q(K_1)$ and $Q^*(H) = Q(K_2)$. Consider any set $A \subset K_2 \backslash H$ where $A \in \mathfrak{A}$. $Q(A) = 0$. By (4.2) $p(w, A | \mathfrak{B}) = 0$ ($P_{\mathfrak{B}}$ a.e.) and so therefore $P(A) = 0$ also. Similarly, if $B \subset H \backslash K_1$, where $B \in \mathfrak{A}$, then $Q(B) = 0$ and hence $p(w, B | \mathfrak{B}) = 0$ and so $P(B) = 0$ also. Thus $p^*(w, H | \mathfrak{B}) = p(w, K_2 | \mathfrak{B})$ ($P^{\mathfrak{B}}$ a.e.) and $p(w, K_1 | \mathfrak{B}) = p_*(w, H | \mathfrak{B})$ ($P_{\mathfrak{B}}$ a.e.). Also, $P(K_1) = P_*(H)$ and $P^*(H) = P(K_2)$. According to Remark 2, $p(w, \cdot | \mathfrak{B})$ can be extended canonically to $\mathfrak{A}'$ with respect to α, β and K' and by Lemma 3 the proof is complete.

The following result is a consequence of Theorem 4.

THEOREM 5. *Let $(X, \mathfrak{A}, P)$, $\mathfrak{B}$ and $\mathfrak{A}'$ be as in Lemma 3. Suppose $P^{\mathfrak{B}}$ is regular and $p(\cdot, \cdot | \mathfrak{B})$ is a version such that*

(5.1) $p(w, \cdot | \mathfrak{B})$ *is a measue* $P_{\mathfrak{B}}$ *a.e.*

(5.2) *there exists a sequence $\{w_n\}_{n=1}$ such that for every $\varepsilon > 0$ and any $w(P_{\mathfrak{B}}$ a.e.) there is an w_n with*

$$\sup_{A \in \mathfrak{A}} | p(w, A | \mathfrak{B}) - p(w_n, A | \mathfrak{B}) | < \varepsilon .$$

Let P_c be a canonical extension of P to $\mathfrak{A}'$ with α, β and K as in (1.1). *Then, $P_c^{\mathfrak{B}}$ is regular.*

Proof. Let Q be a probability measure defined as

$$\sum_{n=1}^{\infty} \frac{1}{2^n} p(w_n, \cdot|\mathfrak{B}) .$$

Condition (5.2) insures that $p(w, \cdot|\mathfrak{B}) \ll Q \; P_{\mathfrak{B}}$ a.e. and the result follows from Theorem 4.

The following proposition is presented for the sake of completeness.

Let $(X, \mathfrak{A}, P)$ be a probability space with $(X, \bar{\mathfrak{A}}, \bar{P})$ denoting the completion. Suppose H is in $\bar{\mathfrak{A}}$ but not in $\mathfrak{A}$. Let $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$.

PROPOSITION 6. *Let $(X, \mathfrak{A}, P)$, $\mathfrak{B} \subset \mathfrak{A}$, and $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$ with $H \in \bar{\mathfrak{A}} \backslash \mathfrak{A}$ be given. Let P_1 denote the restriction of $\bar{P}$ to $\mathfrak{A}'$. If $P^{\mathfrak{B}}$ is regular then so is $P_1^{\mathfrak{B}}$.*

The proof can be viewed as an easy consequence of Lemma 3 and is therefore omitted.

The remainder of this paper is devoted to the single

THEOREM 7. *Let $(X, \mathfrak{A}, P)$ be a probability space with $\mathfrak{A}$ generated by a countable field, $\mathscr{A}$. Let $\mathscr{A}'$ be the field generated by $\mathscr{A}$ and H and $\mathfrak{A}' = \sigma(\mathscr{A}')$. Let P_c be a canonical extension of P to $\mathfrak{A}'$ with respect to α, β and K and suppose $P_c^{\mathfrak{B}}$ is regular where $\mathfrak{B} \subset \mathfrak{A}$. Then, there exists a version $p'(\cdot, \cdot|\mathfrak{B})$ of $P_c^{\mathfrak{B}}$ such that $P_{\mathfrak{B}}$ a.e. $p'(w, \cdot|\mathfrak{B})$ is a probability measure which is a canonical extension of $p'(w, \cdot|\mathfrak{B})|\mathfrak{A}$ with respect to the same α, β and K that are associated with P_c.*

The following lemmas are introduced before presenting the main body of the proof.

LEMMA 8. *Let $(X, \mathfrak{A}, P)$ be a probability space with $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$ and P_e an arbitrary simple extension of P to $\mathfrak{A}'$. Let K be the set associated with a canonical extension of P to $\mathfrak{A}'$ as in Remark 2. Then, for each set $A \in \mathfrak{A}$ there exist constants α_A and β_A with $0 \leqq \alpha_A \leqq 1$ and $0 \leqq \beta_A \leqq 1$ and such that $P_e(AHK) = \alpha_A P(AK)$ and $P_e(AH^cK) = \beta_A P(AK)$.*

Proof. For $A \in \mathfrak{A}, AK \supset AHK$. If $P(AK) \neq 0$, then $\alpha_A = P_e(AHK)/P(AK)$; otherwise, let α_A be arbitrary between zero and one. β_A is obtained similarly.

LEMMA 9. *Assume the hypothesis of Lemma 8. Let $\mathscr{A}$ be a field which generates $\mathfrak{A}$ and $\mathscr{A}'$ the field generated by $\mathscr{A}$ and H. Let $\alpha(\mathscr{A}) \equiv \sup_{A \in \mathscr{A}} \alpha_A$ and $\beta(\mathscr{A}) \equiv \sup_{A \in \mathscr{A}} \beta_A$. Then, a necessary and sufficient condition that P_e be a canonical extension of P to $\mathfrak{A}'$ is that*

$\alpha(\mathscr{A}) = \alpha_X$ or $\beta(\mathscr{A}) = \beta_X$ for some $\mathscr{A}$ which generates $\mathfrak{A}$.

Proof. Necessity is obvious and only sufficiency is proved. Let $\mathscr{A}$ be some field which generates $\mathfrak{A}$ and $\alpha(\mathscr{A}) = \alpha_X$. (For simplicity, write $\alpha(\mathscr{A}) = \alpha$.) By hypothesis,

$$P_e(HK) = \alpha P(K) .$$

For $A \in \mathscr{A}$ it follows by Lemma 8 that

$$P_e(AHK) = \alpha_A P(AK) \tag{9.1}$$

and

$$P_e(A^c HK) = \alpha_{A^c} P(A^c K) . \tag{9.2}$$

The following equalities also hold

$$\alpha P(K) = \alpha P(AK) + \alpha P(A^c K) \tag{9.3}$$

$$P_e(HK) = P_e(AHK) + P_e(A^c HK) . \tag{9.4}$$

By (9.1) – (9.4) it follows that

$$0 = (\alpha - \alpha_A)P(AK) + (\alpha - \alpha_{A^c})P(A^c K) . \tag{9.5}$$

If $P(AK) = 0$, set $\alpha_A = \alpha$ or if $P(A^c K) = 0$, set $\alpha_{A^c} = \alpha$ (see Lemma 8). Otherwise, (9.5) forces $\alpha - \alpha_A = \alpha - \alpha_{A^c} = 0$ and hence for any $A \in \mathscr{A}$, $P_e(AHK) = \alpha P(AK)$.

Next, the fact that $P_e(AH^c K) = \beta P(AK)$, $\beta = 1 - \alpha$, is immediate from the following chain of equalities:

$$\begin{aligned} P(A) &= P_e(AH + AH^c) = P_e((AH + AH^c)K^c) + P_e(AHK) \\ &\quad + P_e AH^c K) = P(AK^c) + \alpha P(AK) + P_e(AH^c K) . \end{aligned}$$

Hence, where $\mathscr{A}' = \{A_1 H + A_2 H^c \mid A_i \in \mathscr{A}\ i = 1, 2\}$, A' in $\mathscr{A}'$ can be written as $A' = A_1 H + A_2 H^c$ and it follows that

$$P_e(A') = P(A'K^c) + \alpha P(A_1 K) + \beta P(A_2 K) .$$

Finally, let

$$\begin{aligned} \phi_\alpha &= \{A \in \mathfrak{A} \mid P_e(AHK) = \alpha P(AK)\} \\ \phi_\beta &= \{A \in \mathfrak{A} \mid P_e(AH^c K) = \beta P(AK)\} . \end{aligned}$$

Both ϕ_α and ϕ_β are monotone classes containing $\mathscr{A}$; hence, the proof is complete by the monotone class theorem (see [3], p. 60).

Theorem 7 can now be proved.

Proof. For $w \in X$, $P_{\mathfrak{B}}$ a.e., and $A \in \mathscr{A}$, write

$$p'(w, AHK|\mathfrak{B}) = \alpha_{w,A}p(w, AK|\mathfrak{B})$$

where $0 \leqq \alpha_{w,A} \leqq 1$ as in Lemma 8 and $p(w, \cdot|\mathfrak{B})$ will be written for $p'(w, \cdot|\mathfrak{B})|_{\mathfrak{A}}$. For fixed $A \in \mathscr{A}$, $\alpha_{w,A}$ is a $\mathfrak{B}$-measurable function where

(7.1) $$\alpha_{w,A} = p'(w, AHK|\mathfrak{B})/p(w, AK|\mathfrak{B}) \text{ for } p(w, AK|\mathfrak{B}) \neq 0$$
$$\alpha_{w,A} = \alpha \text{ if } p(w, AK|\mathfrak{B}) = 0 \, .$$

(In (7.1) α is associated with P_c and by Lemma 9, $\alpha = \sup_{A \in \mathscr{A}} \alpha_A$).

For $A \in \mathscr{A}$ let

(7.2) $$U_A \equiv \{w|\alpha_{w,A} > \alpha\} \, .$$

Observe that U_A is contained in the complement of the set of w's where $p(w, AK|\mathfrak{B}) = 0$.
Also, $U_A \in \mathfrak{B}$ (see (7.1)). Hence, since P_c is a canonical extension, it follows that

(7.3) $$\alpha P(AU_AK) = P_c(AU_AHK) = \int_{U_A} p'(w, AHK|\mathfrak{B})dP_c \, .$$

Also,

(7.4) $$\int_{U_A} p'(w, AHK|\mathfrak{B})dP_c = \int_{U_A} \alpha_{w,A}p(w, AK|\mathfrak{B})dP \geqq \int_{U_A} \alpha p(w, AK|\mathfrak{B})dP = \alpha P(AU_AK) \, .$$

Hence, the defining properties of U_A together with (7.3) and (7.4) say that $P(U_A) = 0$.

If $L_A \equiv \{w|\alpha_{w,A} < \alpha\}$, then an argument similar to the preceding one shows $P(L_A) = 0$.

Hence, for each set $A \in \mathscr{A}$, there exists a $P_{\mathfrak{B}}$ null set on the complement of which $\alpha_{w,A} = \alpha$. But where $\mathscr{A}$ is countable, it follows that there exists a $P_{\mathfrak{B}}$ null set, N, on the complement of which $\alpha_{w,A} = \alpha$ for all $A \in \mathscr{A}$. Thus,

(7.5) $$p'(w, AHK|\mathfrak{B}) = \alpha p(w, AK|\mathfrak{B})$$

for all $w \in N^c$ and $A \in \mathscr{A}$.

Finally, if $\alpha_w \equiv \sup_{A \in \mathscr{A}} \alpha_{w,A}$, then it is immediate from (7.5) that $P_{\mathfrak{B}}$ a.e. $\alpha_w = \alpha = \alpha_X$ and by Lemma 9 the theorem is proved.

References

1. L. H. Blake, *The Preservation of Regularity of Conditional Probabilities*, Doctoral Dissertation, University of Maryland, 1969.

2. P. R. Halmos, *Measure Theory*, Eleventh Printing, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1966.

3. M. Loeve, *Probability Theory*, Third Edition, D. Van Nostrand Company, Inc., Pri-

nceton, New Jersey, 1963.
4. E. Marczewski and J. Łoś, '*Extensions of measure,*' Fund. Math., **36**, pp. 267-276.

Received November 30, 1970. This paper is based in part on the author's doctoral dissertation completed at the University of Maryland under the direction of Professor R. Syski.

WORCESTER POLYTECHNIC INSTITUTE

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 41, No. 1 November, 1972