Pacific Journal of Mathematics

PEAK INTERPOLATION SETS FOR SOME ALGEBRAS OF ANALYTIC FUNCTIONS

ALEXANDER MUNRO DAVIE AND BERNT KARSTEN OKSENDAL

Vol. 41, No. 1 November 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 1, 1972

PEAK INTERPOLATION SETS FOR SOME ALGEBRAS OF ANALYTIC FUNCTIONS

A. M. DAVIE AND B. K. ØKSENDAL

For certain algebras of analytic functions on holomorphically convex sets in C^n metric sufficient conditions are given for a set (not necessarily compact) to be an interpolation set. The results extend the Rudin-Carleson theorem for the disc algebra.

Let K be a compact subset of C^n which is holomorphically convex, i.e. K is the intersection of a decreasing sequence of pseudoconvex domains (see [4], Ch. 2). We denote by $H(K)$ the uniform closure on K of the algebra of all functions analytic in a neighborhood of K, and by $A(K)$ the algebra of all continuous functions on K analytic on K^0 (the interior of K). If E is any subset of the boundary ∂K of K then we denote by H_E^∞ the algebra of all bounded continuous functions on $K^0 \cup E$ which are analytic on K^0. We show that if the boundary of K is well behaved at each point of E, and E satisfies a metric condition which says roughly that E has zero 2-dimensional measure in the directions of the complex tangent and zero one dimensional measure in the orthogonal direction, then E is a peak interpolation set (in an appropriate sense) for $H^\infty_{E\cup(\partial K\setminus\bar{E})}$. If E is compact then it is a peak interpolation set in the usual sense ([2], p. 59) for the uniform algebra $H(K)$. We show also that if E has zero one-dimensional measure then the conditions on ∂K can be relaxed.

We say that ∂K is strictly pseudoconvex in a neighborhood of a point $\zeta \in \partial K$ if there is an open neighborhood V of ζ such that $V \cap \partial K$ is a C^2-submanifold of V and the Levi form is positive definite at ζ. Then we can find an open neighborhood V of ζ and a C^2 strictly plurisubharmonic function ρ in V such that $K \cap V = \{z \in V: \rho(z) \leqq 0\}$ and grad $\rho \neq 0$ on $V \cap \partial K$. (See [3] Prop. IX. A4).

LEMMA 1. *Let K be a holomorphically convex compact set in C^n and let ζ be a point of ∂K in a neighborhood of which ∂K is strictly pseudoconvex. We can find positive numbers m_ζ and M_ζ and $G_\zeta \in H(K)$, such that*

(a) $\operatorname{Re} G_\zeta(z) \geqq m_\zeta |\zeta - z|^2$, $z \in K$
(b) $\operatorname{Re} G_\zeta(z) \leqq M_\zeta |\zeta - z|^2$, $z \in \partial K$
(c) $\operatorname{grad} (\operatorname{Re} G_\zeta)(\zeta) = -\operatorname{grad} \rho(\zeta)$.

Proof. Put

$$F(z) = \sum_{i=1}^{n} \frac{\partial \rho(\zeta)}{\partial \zeta_i}(z_i - \zeta_i) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_i \partial \zeta_j}(z_i - \zeta_i)(z_j - \zeta_j) .$$

Then the Taylor expansion of ρ about ζ is

$$\rho(z) = 2 \operatorname{Re} F(z) + \sum_{i,j=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_i \partial \bar{\zeta}_j}(z_i - \zeta_i)(z_j - \zeta_j) + o(|z - \zeta|^2) .$$

Since ρ is strictly plurisubharmonic at ζ it follows that, shrinking V if necessary, we can find $m > 0$ with $\operatorname{Re} F(z) \leqq -m|\zeta - z|^2$ for $z \in K \cap V$. Since $\rho = 0$ on $\partial K \cap V$ we also deduce that for some constant M

$$\operatorname{Re} F(z) \geqq -M|\zeta - z|^2;\ z \in \partial K \cap V .$$

Choose a pseudoconvex open neighborhood U of K so that $\operatorname{Re} F < 0$ on an open neighborhood W of $\partial V \cap U$ in U. Let $W_1 = W \cup (V \cap U)$ and $W_2 = W \cup (U \backslash \bar{V})$, so that $W_1 \cup W_2 = U$ and $W_1 \cap W_2 = W$. By solving a Cousin problem in U (see [4], Theorem 5.5.1) we can find analytic functions g_1 and g_2 on W_1 and W_2 respectively such that $g_2 - g_1 = F^{-2} \log F$ on W.

$$\text{Put } h(z) = \begin{cases} F(z) \exp(F(z)^2 g_1(z)), & z \in W_1 \\ \exp(F(z)^2 g_2(z)), & z \in W_2 . \end{cases}$$

The definitions agree on W so h is an analytic function on U, $h(z) = 0$ only when $z = \zeta$, and in a neighborhood of ζ, $h(z) = F(z) + 0(|z - \zeta|^3)$. Thus $\operatorname{Re} h \leqq 0$ near ζ, so there exists $\varepsilon > 0$ such that if $z \in K$ and $|h(z) - \varepsilon| \leqq \varepsilon$ then $z = \zeta$. Put

$$G(z) = -\frac{h(z)}{\varepsilon - h(z)},\ z \in K .$$

Then $G \in \bar{H}(K)$, $\operatorname{Re} G(z) > 0$ for $z \in K \backslash \{\zeta\}$. Finally, near ζ, $\operatorname{Re} G(z) = -\varepsilon^{-1} \operatorname{Re} F(z) + \varepsilon^{-2}(\operatorname{Im} F)^2 + 0(|z - \zeta|^3)$ from which it follows that $G_\zeta = 2\varepsilon G$ has the required properties.

If S is a real subspace of C^n and Y is any subset we denote by $d_S(Y)$ the diameter (in the Euclidean metric) of the (real) orthogonal projection of Y on S.

Let K be a compact holomorphically convex subset of C^n and suppose ∂K is strictly pseudoconvex in a neighborhood of a point $\zeta \in \partial D$. Then in a neighborhood of ζ we can write $\partial K = \{z\colon \rho(z) = 0\}$ where $\rho(z)$ is strictly plurisubharmonic in a neighborhood of ζ and $\operatorname{grad} \rho \neq 0$. The vector $i \operatorname{grad} \rho$ is orthogonal to $\operatorname{grad} \rho$ and so lies in the (real) tangent space to ∂K at ζ; let $T(\zeta)$ be the orthogonal complement to $i \operatorname{grad} \rho$ in this space. Then $T(\zeta)$ is the unique complex subspace of the real tangent space with complex dimension $n - 1$. Let $L(\zeta)$ be the real

line spanned by the vector i grad ρ.

If E is any subset of ∂K we denote by H_E^∞ the set of all bounded continuous functions on $K^0 \cup E$ which are analytic on K^0. We define $A(K) = H_{\partial K}^\infty$.

THEOREM 1. *Let F be a subset of ∂K such that ∂K is strictly pseudoconvex in a neighborhood of F. Suppose that for every $\varepsilon > 0$ the set F can be covered by a sequence $\{V_i\}$ of open sets with diameter $< \varepsilon$ such that if $\zeta_i \in F \cap V_i$ for each i then $\sum_i d_{L(\zeta_i)}(V_i) < \varepsilon$ and $\sum_i \{d_{T(\zeta_i)}(V_i)\}^2 < \varepsilon$. Let V be a neighborhood of F, let $\eta > 0$, and let g be a bounded continuous function on F with $\|g\| \leqq 1$.*

Then we can find $f \in H_{F \cup (K \setminus \bar{F})}^\infty$ with $f|F = g$, $\|f\| \leqq 1$, and $|f| < \eta$ on $K \setminus V$.

The proof will be split up into lemmas.

LEMMA 2. *Let F, V and η be as in the theorem. Then we can find $f \in H_{F \cup (K \setminus \bar{F})}^\infty$ with $f = 1$ on F, $\|f\| \leqq 2$ and $|f| < \eta$ on $K \setminus V$.*

Proof. For each $\zeta \in F$ we choose m_ζ, $M_\zeta > 0$, and a function $G_\zeta \in H(K)$ as in Lemma 1.

If W_ζ is a sufficiently small open neighborhood of ζ, then whenever $\zeta \in U \subseteqq W_\zeta$ and $z \in U \cap \partial K$ we have

$$\begin{aligned} |G_\zeta(z)| &\leqq \operatorname{Re} G_\zeta(z) + |\operatorname{Im} G_\zeta(z)| \\ &\leqq A_\zeta |z - \zeta|^2 + |\langle \operatorname{grad} (\operatorname{Im} G_\zeta)(\zeta), z - \zeta \rangle| \\ &\leqq 2A_\zeta(d_1^2 + d_2^2) + |\operatorname{grad} \rho(\zeta)| d_1 \\ &\leqq B_\zeta(d_1 + d_2^2) \end{aligned}$$

where $d_1 = d_{L(\zeta)}(U)$, $d_2 = d_{T(\zeta)}(U)$, A_ζ, B_ζ do not depend on z, and $\langle , \rangle$ denotes the real scalar product.

For each positive integer n let

$$F_n = \{\zeta \in F\colon B_\zeta < n,\ \Delta(\zeta, 1/n) \subseteqq W_\zeta,\ m_\zeta d(\zeta, K \setminus V)^2 > 1/n,\ m_\zeta > 1/n\}\ .$$

Then $F = \bigcup_n F_n$. For each n we choose a sequence $\{V_i^{(n)}\}$ of open sets with diameter less than $1/n$ such that each point of F_n is contained in infinitely many $V_i^{(n)}$, and $\sum_i \{d_{L(\zeta_i^n)}(V_i^n) + (d_{T(\zeta_i^n)}(V_i^n))^2\} < \eta n^{-2} 2^{-n-2}$ for some choice of $\zeta_i^{(n)} \in V_i^{(n)} \cap F_n$. Renumber the collection of all $V_i^{(n)}$ as $V_1, V_2, \cdots$. For each j choose n_j so that $V_j = V_i^{(n_j)}$ for some i, and let $\zeta_j = \zeta_i^{(n_j)}$. Let $G_j = G_{\zeta_j}$. Writing $c_j = d_{L(\zeta_j)}(V_j) + \{d_{T(\zeta_j)}(V_j)\}^2$ we define

$$B_r(z) = \prod_{j=1}^{r} \frac{G_j(z)}{2n_j c_j + G_j(z)}\ ,\ z \in K,\ r = 1, 2, \cdots .$$

Then $B_r \in H(K)$ and $|B_r| \leqq 1$ on K. We claim that $\{B_r\}$ converges pointwise on $F \cup (K \backslash \bar{F})$ to a limit B which is continuous on $F \cup (K \backslash \bar{F})$, analytic on K^0, zero at each point of F, with $\|B\| \leqq 1$ and $|1 - B| < \eta$ on $K \backslash V$.

If $z \in K \backslash V$ then $\operatorname{Re} G_j(z) \geqq m_{\zeta_j} |z - \zeta_j|^2 > 1/n_j$, so

$$\sum_{j=1}^{\infty} \left| 1 - \frac{G_j(z)}{2n_j c_j + G_j(z)} \right| = \sum_{j=1}^{\infty} \frac{2n_j c_j}{|2n_j c_j + G_j(z)|}$$
$$\leqq \sum_{j=1}^{\infty} 2n_j^2 c_j < \eta/2 ,$$

which proves that $B_r(z)$ converges to a limit $B(z)$ with $|1 - B(z)| < \eta$.

If $z \in K \backslash \bar{F}$ then

$$\sum_{j=j_0}^{\infty} \frac{2n_j c_j}{|2n_j c_j + G_j(z)|} \leqq \sum_{j=j_0}^{\infty} \frac{2n_j^2 c_j}{|z - \zeta_j|^2} \leqq d(z, F)^{-2} \sum_{j=j_0}^{\infty} 2n_j^2 c_j .$$

The series on the right converges, so B_r converges uniformly to a limit B on sets at positive distance from F, so B is continuous on $K \backslash \bar{F}$ and analytic on K^0.

Finally let $z \in F$. Then $z \in V_j$ for infinitely many j. For each such j we have $V_j \subseteqq W_{\zeta_j}$ and for all $w \in W_{\zeta_j}$,

$$\left| \frac{G_j(w)}{2n_j c_j + G_j(w)} \right| \leqq \frac{n_j c_j}{2n_j c_j} = \frac{1}{2} .$$

It follows that $B_r(z) \to 0$ and $\lim |B_r|$ is continuous at z. Thus B has the asserted properties, and $f = 1 - B$ satisfies the requirements of the theorem.

LEMMA 3. *Let X be a compact subset of K, W a neighborhood of X, and h a continuous function on K with support in X such that $\|h\| \leqq 1$. Let $\eta > 0$.*

Then there exists $f \in H^{\infty}_{F \cup (K \backslash \bar{F})}$ such that $|f - h| < \eta$ on F, $\|f\| \leqq 3$, and $|f| < \eta$ on $K \backslash W$.

Proof. Choose $0 < \delta < d(X, K \backslash W)$ so small that $|h(x) - h(y)| < \eta/8$ whenever $x, y \in K$, $|x - y| < \delta$. We can easily find an integer $N > 0$, compact sets $X_1 \cdots X_r$ contained in X, and open sets $W_1 \cdots W_r$, with diameters $< \delta$, with $W_i \supseteqq X_i$, $W_i \subseteqq W$, such that

(a) if $x \in X$ and N_x is the number of integers i in $\{1, \cdots, r\}$ for which $x \in X_i$, $|N_x - N| < \eta N/8$

(b) if $x \in C^n$ the number of integers i for which $x \in W_i \backslash X_i$ is less than ηN.

Let $F_i = F \cap X_i$. For $i = 1, 2 \cdots r$ we can find by Lemma 2 functions $f_i \in H^{\infty}_{F_i \cup (K \backslash \bar{F}_i)}$ with $f_i = 1$ on F_i, $\|f_i\| \leqq 2$ and $|f_i| < \eta/3r$ on $K \backslash W_i$.

Choose $x_i \in X_i$ for each i and put $f(z) = 1/N \sum_{i=1}^{r} f_i(z)h(x_i)$, $z \in F \cup (K \backslash \bar{F})$. Clearly $f \in H^{\infty}_{F \cup (K \backslash \bar{F})}$ and $\|f\| \leqq 3$ by (a). If $z \in K \backslash W$ then $|f_i(z)| < \eta/r$ for each i so $|f(z)| < \eta$.

Finally let $z \in F$. Then

$$f(z) = \frac{1}{N}\Big(\sum_{z \in X_i} + \sum_{z \in W_i \backslash X_i} + \sum_{z \notin W_i}\Big) f_i(z)h(x_i)$$
$$= f_1(z) + f_2(z) + f_3(z), \text{ say.}$$

We have

$$|f_1(z) - h(z)| \leqq \left|\frac{1}{N}\sum_{z \in X_i} f_i(z)(h(z) - h(x_i))\right|$$
$$+ \left|1 - \frac{N_z}{N}\right| < \frac{\eta N_z}{8N} + \left|1 - \frac{N_z}{N}\right| < \eta/3\,,$$

by (a), since $|z - x_i| < \delta$. Moreover, $|f_2(z)| < \sum_{i=1}^{r} \eta/3$, by (b) and $|f_3(z)| < \sum_{i=1}^{r} \eta/3r = \eta/3$, so that we have $|f(z) - h(z)| < \eta$ as required.

LEMMA 4. *With F as in the theorem, if W is any open neighborhood of F and h a bounded continuous function on W with $\|h\| \leqq 1$, we can find $G \in H_{F \cup (K \backslash \bar{F})}$ with $|G - h| < \eta$ on F, $\|G\| \leqq 7$, and $|G| < \eta$ outside W.*

Proof. Choose a sequence $\{W_n\}$ of relatively compact open subsets of W with $W = \bigcup_{n=1}^{\infty} W_n$, such that $\bar{W}_m \cap \bar{W}_n = \varnothing$ if $|m - n| > 1$. We can write $h = \sum_{n=1}^{\infty} h_n$ on W where $h_n \in C(K)$ has support in W_n and $\|h_n\| \leqq 1$. By Lemma 3 for each n we can find $f_n \in H^{\infty}_{F \cup (K \backslash \bar{F})}$ with $|f_n - h_n| < 2^{-n}\eta$ on F, $|f_n| < 2^{-n}\eta$ on $K \backslash W$, and $\|f_n\| \leqq 3$. Then $G = \sum_{n=1}^{\infty} f_n$ has the required properties.

Proof of Theorem 1. By Lemma 4 and using the fact that g can be approximated uniformly by functions continuous in a neighborhood of F, we can construct by induction on n a sequence $\{G_n\}_{n=0}^{\infty}$ in $H^{\infty}_{F \cup (K \backslash \bar{F})}$ such that, writing $f_n = G_0 + \cdots + G_n$ we have:

$$(1) \qquad |G_0 - g| < \lambda/7 \text{ on } F\,,$$

$$(1)_n \qquad |G_n + f_{n-1} - (1 + \lambda + \cdots + \lambda^n)g| < \frac{\lambda^{n+1}}{7}$$

on F, $n > 1$, where $\lambda = 9/10$

$$(2)_n \qquad \|G_n\| \leqq 7\|f_{n-1} - (1 + \lambda + \cdots + \lambda^n)g\|_F < 8\lambda^n$$

$$(3)_n \qquad \|f_n\| \leqq 1 + \lambda + \cdots + 9\lambda^n\,.$$

(To get $(3)_n$ observe that by $(1)_{n-1}$ we have $|f_{n-1}| < 1 + \lambda + \cdots + \lambda^{n-1} + \lambda^n/7$ on F, and hence on a neighborhood of F; if we make $|G_n| < \lambda^{n-1}/10$ outside this neighborhood then $(3)_n$ follows from $(2)_n$ and $(3)_{n-1}$).

$$(4)_n \qquad |G_n| < 2^{-n} \text{ on } K\backslash V .$$

Then $(2)_n$ shows that $f_n \to G$ say uniformly on K, so $G \in H^\infty_{F\cup(K\backslash\bar{F})}$; by $(1)_n$ $G = 10g$ on F and by $(3)_n$ $\|G\| \leqq 10$. Finally by $(4)_n |G| < \eta$ on $K\backslash V$. Then $f = (1/10)G$ is the required function.

REMARK. The metric condition on F in Theorem 1 is clearly satisfied if F has zero one-dimensional Hausdorff measure; however it is also satisfied by sets which are thicker in the direction of the complex tangent space, e.g. any smooth arc in ∂K whose tangent at each point lies in the complex tangent space.

If F is compact then of course it is a peak interpolation set, so Theorem 1 extends the Rudin-Carleson theorem. The extension to non-closed sets in the case of the disc has been obtained independently by Detraz [1], and subsequently generalized to other domains in the plane by A. Stray (private communication).

If we assume that F has zero one-dimensional Hausdorff measure then we can make do with a weaker pseudoconvexity hypothesis at the points of F. We say that ∂K is point pseudoconvex at ζ if there exists a neighborhood N of ζ and a real C^2 strictly plurisubharmonic function ρ in N such that $\rho(\zeta) = 0$ and $\rho(z) \leqq 0$ in $N \cap K$.

THEOREM 2. *Let K be holomorphically convex, and let F be a subset of ∂K with zero one-dimensional Hausdorff outer measure such that ∂K is point pseudoconvex at each point of F. Let V be a neighborhood of F in K, let $\eta > 0$, and let g be a bounded continuous function on F with $\|g\| \leqq 1$.*

Then we can find $f \in H^\infty_{F\cup(K\backslash\bar{F})}$ with $f|F = g$, $\|f\| \leqq 1$ and $|f| < \eta$ on $K\backslash V$.

Proof. We show that the conclusion of Lemma 2 holds; the rest of the proof is just as before. As in the proof of Lemma 2 for each $\zeta \in F$ we can find positive constants m_ζ and M_ζ, a neighborhood W_ζ of ζ, and $G_\zeta \in H(K)$ such that

(a) $m_\zeta |\zeta - z|^2 \leqq \operatorname{Re} G_\zeta(z), \qquad z \in K$

(b) $|G_\zeta(z)| \leqq M_\zeta |\zeta - z|, \qquad z \in K$.

Then whenever $\zeta \in U \subset W_\zeta$ and $z \in U$ we have $|G_\zeta(z)| \leqq M_\zeta \operatorname{diam}(U)$. We define F_n as before and cover F_n by balls $\Delta_i^{(n)}$ such that $\sum_i \operatorname{diam}(\Delta_i^n) < \varepsilon n^{-2} 2^{-n-2}$. The rest of the proof goes just as before, with c_j replaced by $\operatorname{diam}(\Delta_j)$.

COROLLARY. *Let F be a compact subset of ∂K with zero 1-dimensional Hausdorff measure and assume ∂K is point pseudoconvex at each point of F. Then F is a peak interpolation set for $A(K)$.*

Finally we remark that the functions obtained in Theorem 1 and 2 are actually pointwise limits on K^0 of bounded sequences in $H(K)$; this follows from the construction. If F is compact the peak-interpolating functions constructed are in $\bar{H}(K)$; in this case the proof simplifies somewhat since it is only necessary to take finite products in Lemma 2 and the theorem follows from Lemma 2 by general theorems on peak interpolation sets.

REFERENCES

1. J. Detraz, *Algebres de fonctions analytiques dans le disque*, Ann. Sci. École Norm. Sup., **3** (1970), 313-352.
2. T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.
3. R. C. Guning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N. J, 1965.
4. L. Hormander, *An Introduction to Complex Analysis in Several Variables*, Van Nostrand, Princeton, 1967.

Received May 10, 1971. The work of the first author was partially supported by NSF Grant GP-19067 and the work of the second author was partially supported by NAVF, Norway and NSF Grant GP-11475.

UNIVERSITY OF CALIFORNIA, LOS ANGELES

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 41, No. 1 November, 1972