

Pacific Journal of Mathematics

ABSOLUTE TOTAL-EFFECTIVE $(N, p_n)(c, 1)$ METHOD

H. P. DIKSHIT

ABSOLUTE TOTAL-EFFECTIVE $(N, p_n)(C, 1)$ METHOD

H. P. DIKSHIT

In the direction of the total-effectiveness of a $(N, p_n)(C, 1)$ method, results concerning the summability of a Lebesgue Fourier series and its conjugate series by such a method are known. Supporting the observation that generally bounded variation is the property associated with absolute summability in the same way in which continuity is associated with ordinary summability, the absolute total-effectiveness of a $(N, p_n)(C, 1)$ method is established in the present paper and the corresponding effectiveness of the (C) method is deduced as a particular case.

Throughout the present paper we use the definitions and notations of [7] without further explanation. The following additional notations for the conditions concerning $\{p_n\}$ are also used.

(1.1) $\{p_n\} \in RS$ means: $p_0 > 0$, $p_n \geq 0$ ($n \geq 1$), $\{R_n\} \in BV$ and $\{S_n\} \in B$;

(1.2) $\{p_n\} \in MS$ means: $p_n > 0$, $p_{n+1}/p_n \leq p_{n+2}/p_{n+1} \leq 1$ ($n \geq 0$) and $\{S_n\} \in B$;

(1.3) $\{p_n\} \in NS$ means: $p_0 > 0$, $p_n \geq 0$ ($n \geq 1$), $\{R_n\} \in B$, $\{S_n\} \in B$, $\{p_n\}$ and $\{\Delta p_n\}$ monotone.

As we shall see in section 5 of the present paper, $MS \subset NS$, but no interrelation is known between the sets of conditions RS and MS or NS .

Using a result due to Mears [15], Kwee [13] has proved that the following conditions:

$$(1.4) \quad p_n = o(|P_n|), n \rightarrow \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \left| \frac{P_n}{P_{n+\nu}} - \frac{P_{n-1}}{P_{n+\nu-1}} \right| < \infty,$$

for all $\nu \geq 1$, are necessary and sufficient for the absolute regularity of the (N, p_n) method. It may be observed that Lemma 1 and Lemma 2 of the present paper imply *a fortiori* that the (N, p_n) method is absolutely regular, under each of the conditions: $\{p_n\} \in RS$, $\{p_n\} \in NS$ and $\{p_n\} \in MS$.

Concerning the absolute Fourier-effectiveness, we have the following.

THEOREM A. *If $\{p_n\} \in RS$, then the (N, p_n) method is absolute Fourier-effective.*

THEOREM B. *The (C, δ) method is absolute Fourier-effective for every $\delta > 0$.*

As pointed out by the present author in [4], Theorem A is an apparently lighter but actually equivalent version of results of Pati [17]. Theorem B emerges from Bosanquet [1] and Bosanquet and Hyslop ([2], Th. K, with $\alpha = 0$) and is known to be the best possible in the sense that it breaks down if $\delta = 0$. It may also be mentioned that Theorem B is a special case of Theorem A.

$|F|$ -effective part of Theorem B may also be deduced as a special case of the corresponding effectiveness of the (N, p_n) method proved in [6] and [5], under the hypothesis: $\{p_n\} \in MS$ or more generally that $\{p_n\} \in NS$.

The following result emerges from ([10], Ths. 1, 2 and 3), when we observe that its $|F'|$ -effective part is deducible from the proof of Theorem 1 in [10], while its absolute Fourier-effective part, follows from the result of Theorem A and the absolute regularity of the $(C, 1)$ method.

THEOREM C. *If $\{p_n\} \in RS$, then the $(C, 1)(N, p_n)$ method is absolute total-effective.*

However, in the direction of the absolute total-effectiveness of the $(N, p_n)(C, 1)$ method¹, we have only succeeded in proving the following (cf. [9]).

THEOREM D. *If $\{p_n\} \in RS$, then the $(N, p_n)(C, 1)$ method is $|F_1|$ - and $|F'|$ -effective.*

2. **The main results.** That under the hypothesis: $\{p_n\} \in MS$, it is indeed possible to prove a more *powerful* effectiveness of the $(N, p_n)(C, 1)$ method than that obtained in Theorem D is demonstrated by our Theorem 2, where we succeed in establishing absolute total-effectiveness of such a method. The absolute total-effectiveness of the $(C, 1 + \delta)$ method ($\delta > 0$), which emerges from Bosanquet [1], Bosanquet and Hyslop ([2], Th. 1 for $\alpha = 0$ and Th. 5) and Hyslop [11], reduces to a special case of our Theorem 2.

We first prove the following which corresponds to Theorem D.

THEOREM 1. *If $\{p_n\} \in NS$, then the $(N, p_n)(C, 1)$ method is $|F_1|$ - and $|F'|$ -effective.*

¹ It is known [18] that the matrix $(C, 1)(N, p_n) \neq (N, p_n)(C, 1)$, unless (N, p_n) is a Cesàro matrix.

Using the result of Theorem 1, we prove:

THEOREM 2. *If $\{p_n\} \in MS$, then the $(N, p_n)(C, 1)$ method is absolute total-effective.*

In view of the result of Das ([3], Th. 5) that if $p_n > 0$, $p_{n+1}/p_n \leq p_{n+2}/p_{n+1} \leq 1$ ($n \geq 0$), then

$$|(N, p_n)(C, 1)| \sim |N, P_n| \sim |(C, 1)(N, p_n)|,$$

the absolute total-effectiveness of the (N, P_n) and the $(C, 1)(N, p_n)$ methods, under the hypotheses: $\{p_n\} \in MS$, follow from the result of Theorem 2.

3. Some preliminary results. We need the following lemmas for the proofs of our theorems.

LEMMA 1. *If $\{p_n\} \in RS$, then the (N, p_n) method is absolutely regular.*

Lemma 1 is included in Lemma 8 of [8].

LEMMA 2. *If $p_0 > 0$, $p_n \geq 0$ ($n \geq 1$), $\{p_n\}$ is monotone and $\{R_n\} \in B$, then the (N, p_n) method is absolutely regular.*

Proof. Since $\{R_n\} \in B$ implies that $p_n = o(P_n)$, $n \rightarrow \infty$, in order to prove Lemma 2, it is sufficient to show that for all $\nu \geq 1$

$$(3.1) \quad \Sigma^* = \sum_{n=0}^{\infty} \left| \frac{P_n}{P_{n+\nu}} - \frac{P_{n-1}}{P_{n+\nu-1}} \right| = \sum_{n=0}^{\infty} \frac{P_n}{P_{n+\nu-1}} \left| \frac{p_n}{P_n} - \frac{p_{n+\nu}}{P_{n+\nu}} \right| \leq K.$$

The case in which $\{p_n\}$ is monotonic nonincreasing, (3.1) follows directly from Corollary 1 due to Mears [15].

Since $\{P_n\}$ is positive monotonic nondecreasing, we have by suitable changes in orders of summations (cf. [9], proof of Lemma 10)

$$\begin{aligned} \Sigma^* &\leq \sum_{n=0}^{\infty} \frac{P_n}{P_{n+\nu-1}} \sum_{k=n}^{n+\nu-1} \left| \Delta \left(\frac{p_k}{P_k} \right) \right| \\ &= \sum_{k=0}^{\nu-1} \left| \Delta \left(\frac{p_k}{P_k} \right) \right| \sum_{n=0}^k \frac{P_n}{P_{n+\nu-1}} \\ &\quad + \sum_{k=\nu}^{\infty} \left| \Delta \left(\frac{p_k}{P_k} \right) \right| \sum_{n=k-\nu+1}^k \frac{P_n}{P_{n+\nu-1}} \\ &\leq \frac{1}{P_{\nu-1}} \sum_{k=0}^{\nu-1} (k+1)P_k \left| \frac{\Delta p_k}{P_k} + \frac{(p_{k+1})^2}{P_k P_{k+1}} \right| + \nu \sum_{k=\nu}^{\infty} \left| \frac{\Delta p_k}{P_k} + \frac{(p_{k+1})^2}{P_k P_{k+1}} \right| \\ &= \Sigma_1^* + \Sigma_2^*, \end{aligned}$$

say. We now assume, that $\{p_n\}$ is monotonic nondecreasing and therefore

$$\begin{aligned}\Sigma_1^* &\leq \frac{\nu}{P_{\nu-1}} \sum_{k=0}^{\nu-1} (p_{k+1} - p_k) + \frac{1}{P_{\nu-1}} \sum_{k=0}^{\nu-1} p_{k+1} R_{k+1} \\ &\leq KR_\nu + K \leq K,\end{aligned}$$

by virtue of the hypothesis that $\{R_n\} \in B$, which implies that $P_n/P_{n-1} = O(1)$, $n \rightarrow \infty$. To prove that $\Sigma_2^* \leq K$, we observe that by an application of Abel's transformation,

$$\begin{aligned}\nu \sum_{k=\nu}^n \frac{1}{P_k} (p_{k+1} - p_k) + \nu \sum_{k=\nu}^n \frac{(R_{k+1})^2}{(k+1)^2} \\ \leq \nu \sum_{k=\nu}^{n-1} \frac{p_{k+1}}{P_k P_{k+1}} \sum_{\mu=\nu}^k (p_{\mu+1} - p_\mu) + \nu \frac{1}{P_n} \sum_{\mu=\nu}^n (p_{\mu+1} - p_\mu) + K \\ \leq K\nu \sum_{k=\nu}^{n-1} \frac{(R_{k+1})^2}{(k+1)^2} + K\nu \frac{R_{n+1}}{n} + K \\ \leq K,\end{aligned}$$

as $n \rightarrow \infty$, by virtue of the conditions: $\{p_n\}$ is monotonic nondecreasing and $\{R_n\} \in B$. Thus, $\Sigma_2^* \leq K$, and we complete the proof of (3.1) in the case in which $\{p_n\}$ is monotonic nondecreasing. This completes the proof of Lemma 2.

The condition $\{R_n\} \in B$ is automatically satisfied if $\{p_n\}$ is nonnegative and nonincreasing and thus, we observe that the present Lemma 2 extends the result of Corollary 1 of [15].

LEMMA 3. *If $\{p_n\} \in NS$, then uniformly in $0 < t \leq \pi$*

$$(3.2) \quad \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) \frac{\sin\left(n - k + \frac{1}{2}\right)t}{n - k + \frac{1}{2}} \right| \leq K.$$

Proof. The proof of (3.2) is similar to the proof of the following as given in [5].

$$\sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) \frac{\sin(n-k)t}{n-k} \right| \leq K.$$

LEMMA 4. *If $\theta(t) \in BV(0, \pi)$ and $\{p_n\} \in NS$, then $\{t_n(u)\} \in BV$, where $t_n(u)$ is the n th (N, p_n) mean of $\{u_n\}$ defined by*

$$u_n = \int_0^\pi \theta(t) \left\{ \sin(n+1)t / \sin\left(\frac{1}{2}t\right) \right\} dt.$$

Proof. Following the proof of a theorem in Pati ([16], p. 156), we observe that if $\theta(t) \in BV(0, \pi)$, then in order to prove that $\{t_n(u)\} \in BV$, it is sufficient to show that (3.2) holds uniformly in $0 < t \leq \pi$. Thus Lemma 4, follows from Lemma 3.

LEMMA 5. *If $\theta(t) \in BV(0, \pi)$, then $\{v_n\} \in BV$, where*

$$v_n = \frac{1}{n+1} \int_0^\pi \theta(t) \left\{ \sin\left(\frac{1}{2}(n+1)t\right) / \sin\left(\frac{1}{2}t\right) \right\}^2 dt .$$

Proof. Writing

$$v_n = \frac{1}{n+1} \sum_{k=0}^n \int_0^\pi \theta(t) \left\{ \sin\left(k + \frac{1}{2}\right)t / \sin\left(\frac{1}{2}t\right) \right\} dt ,$$

we observe that under the hypothesis: $\theta(t) \in BV(0, \pi)$, the result of Lemma 5 follows from the proof of $|F|$ -effective part of Theorem B, when we appeal to a well known inclusion relation for the absolute (C) method.

LEMMA 6. *Let $t_n^1(s)$ denotes the n th $(N, p_n)(C, 1)$ mean of $\sum_{n=0}^\infty a_n$ and $p_n > 0, p_{n+1}/p_n \leq p_{n+2}/p_{n+1} \leq 1$, for all $n \geq 0$. Then $\{t_n^1(s)\} \in BV$, if and only if*

$$\sum_{n=1}^\infty \frac{1}{nP_n} \left| \sum_{k=0}^n p_{n-k} \frac{1}{k+1} \sum_{r=0}^k ra_r \right| \leq K .$$

Proof. We have by a change of order of summation

$$(3.3) \quad t_n^1(s) = \frac{1}{P_n} \sum_{k=0}^n \sum_{r=k}^n \frac{p_{n-r}}{r+1} s_k = \frac{1}{P_n} \sum_{r=0}^n p_{n-r} \sigma_r^1(s) ,$$

where $\sigma_r^1(s)$ is the r th $(C, 1)$ mean of $\sum_{n=0}^\infty a_n$. But, by a well known identity due to Kogbetliantz

$$(3.4) \quad k\{\sigma_k^1(s) - \sigma_{k-1}^1(s)\} = \frac{1}{k+1} \sum_{r=0}^k ra_r .$$

In view of (3.3) and (3.4), Lemma 6 follows directly from Theorem 6 of Das [3].

LEMMA 7. *If $\{p_n\}$ is nonnegative and nonincreasing, then for $0 \leq a \leq b \leq \infty, 0 \leq t \leq \pi$ and for any n and a ,*

$$\left| \sum_{k=a}^b p_k \exp i(n-k)t \right| \leq KP_\tau .$$

Lemma 7 is contained in [14].

LEMMA 8. If $p_0 > 0$, $p_n \geq 0$ ($n \geq 1$) and $\{R_n\} \in B$, then the condition: $\{S_n\} \in B$ implies that

$$P_n \sum_{k=n}^{\infty} \frac{1}{(k+1)P_k} \leq K, \quad n = 0, 1, 2, \dots$$

The proof of Lemma 8 is contained in [4]. A slight modification in the proof given in [4] shows that the result of Lemma 8 holds, even without the hypothesis: $\{R_n\} \in B$.

4. **Proof of Theorem 1. (I).** $|F_1|$ -effectiveness: Denoting by $\sigma_n^1(L(x))$, the n th $(C, 1)$ mean of $L(x)$, we have

$$\begin{aligned} \sigma_n^1(L(x)) &= \frac{1}{\pi(n+1)} \int_0^\pi \phi(t) \left\{ \sum_{k=0}^n \sin\left(k + \frac{1}{2}\right)t / \sin \frac{1}{2}t \right\} dt \\ &= \frac{1}{\pi(n+1)} \int_0^\pi \phi(t) \left\{ \sin \frac{1}{2}(n+1)t / \sin \frac{1}{2}t \right\}^2 dt. \end{aligned}$$

Writing $\theta(t) = \Phi_1(t) / \sin(1/2)t$, we have on integration by parts

$$\begin{aligned} \sigma_n^1(L(x)) &= \frac{\theta(\pi)}{\pi(n+1)} \left\{ \sin \frac{1}{2}(n+1)\pi \right\}^2 - \frac{1}{2\pi} \int_0^\pi \theta(t) \left\{ \sin(n+1)t / \sin \frac{1}{2}t \right\} dt \\ (4.1) \quad &+ \frac{1}{\pi(n+1)} \int_0^\pi \theta(t) \cos \frac{1}{2}t \left\{ \sin \frac{1}{2}(n+1)t / \sin \frac{1}{2}t \right\}^2 dt \\ &= w_n + Ku_n + Kv_n, \end{aligned}$$

say. In view of (3.3), in order to prove the $|F_1|$ -effectiveness of the $(N, p_n)(C, 1)$ method, it is enough to show that $\{t_n(u)\} \in BV$, $\{t_n(v)\} \in BV$ and $\{t_n(w)\} \in BV$, where $t_n(u)$, $t_n(v)$ and $t_n(w)$ are the n th (N, p_n) means of $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$, respectively.

Since x is a $|F_1|$ -regular point, $\{t^{-1}\Phi_1(t)\} \in BV(0, \pi)$ and that $\{t_n(u)\} \in BV$, follows from Lemma 4. Similarly, $\{t_n(v)\} \in BV$, by virtue of Lemmas 2 and 5. Finally, we write (cf. [9])

$$w_n - w_{n-1} = \alpha_n + \beta_n,$$

where

$$\begin{aligned} \alpha_n &= (-1)^n A / \left(n + \frac{1}{2}\right) = A \sin \left\{ \left(n + \frac{1}{2}\right)\pi \right\} / \left(n + \frac{1}{2}\right); \\ \beta_n &= \begin{cases} -A / \left\{ 2 \left(n + \frac{1}{2}\right)(n+1) \right\} & (n \text{ even}); \\ -A / 2n \left(n + \frac{1}{2}\right) & (n \text{ odd}); \end{cases} \end{aligned}$$

and $A = \Phi_1(\pi) / \pi$.

Thus, in order to show that $\{t_n(w)\} \in BV$, it is enough to show

that $\{t_n(\alpha)\} \in BV$ and $\{t_n(\beta)\} \in BV$, where $t_n(\alpha)$ and $t_n(\beta)$ are the n th (N, p_n) means of $\sum_{n=0}^{\infty} \alpha_n$ and $\sum_{n=0}^{\infty} \beta_n$, respectively.

Now, we have

$$t_n(\alpha) - t_{n-1}(\alpha) = \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) \alpha_{n-k}$$

and therefore $\{t_n(\alpha)\} \in BV$ by virtue of (3.2).

That $\{t_n(\beta)\} \in BV$, follows from the absolute convergence of $\sum_{n=0}^{\infty} \beta_n$, when we appeal to the result of Lemma 2.

This completes the proof of $|F_1|$ -effective part of Theorem 1.

(II). $|F'|$ -effectiveness: Denoting by $\sigma_n^i(L'(x))$ the n th $(C, 1)$ mean of $L'(x)$, we have

$$\begin{aligned} \sigma_n^i(L'(x)) &= -\frac{1}{\pi(n+1)} \int_0^\pi \psi(t) \frac{d}{dt} \left\{ \sin \frac{1}{2}(n+1)t / \sin \frac{1}{2}t \right\}^2 dt \\ (4.2) \quad &= -\frac{1}{2\pi} \int_0^\pi \left\{ \psi(t) / \sin \frac{1}{2}t \right\} \left\{ \sin(n+1)t / \sin \frac{1}{2}t \right\} dt \\ &\quad + \frac{1}{\pi(n+1)} \int_0^\pi \left\{ \psi(t) / \tan \frac{1}{2}t \right\} \left\{ \sin \frac{1}{2}(n+1)t / \sin \frac{1}{2}t \right\}^2 dt. \end{aligned}$$

Comparing (4.2) with (4.1), and observing that $\{\psi(t)/t\} \in BV(0, \pi)$, since x is $|F'|$ -regular, it follows from Lemmas 2, 4 and 5 that the $(N, p_n)(C, 1)$ method is $|F'|$ -effective.

5. Proof of Theorem 2.(I). $|F_1|$ - and $|F'|$ -effectiveness: We observe that if $p_n > 0$ and $p_{n+1}/p_n \leq p_{n+2}/p_{n+1} \leq 1$, for all $n \geq 0$, then $(n+1)p_n \leq P_n$, i.e. $\{R_n\} \in B$ and $\{\Delta p_n\}$ is monotonic nonincreasing, for (see [12])

$$\Delta p_n - \Delta p_{n+1} = p_n - 2p_{n+1} + p_{n+2} \geq (\sqrt{p_n} - \sqrt{p_{n+2}})^2 \geq 0$$

and therefore $\{p_n\} \in NS$. Thus $|F_1|$ - and $|F'|$ -effective parts of Theorem 2 are included in Theorem 1.

(II). $|\tilde{F}_1|$ -effectiveness: Since

$$nB_n(x) = -\frac{2}{\pi} \int_0^\pi \psi(t) \left(\frac{d}{dt} \cos nt \right) dt$$

and $\int_0^\pi t^{-1} |\psi(t)| dt \leq K$, by virtue of the hypothesis that x is a $|\tilde{F}_1|$ -regular point, it follows from Lemma 6, that in order to prove the $|\tilde{F}_1|$ -effective part of Theorem 2, it is sufficient to show that uniformly in $0 < t \leq \pi$

$$(5.1) \quad \Sigma = t \sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{k=0}^n p_{n-k} \frac{1}{k+1} \frac{d}{dt} \left\{ \sum_{r=0}^k \cos rt \right\} \right| \leq K.$$

Now we write²

$$\begin{aligned} \Sigma &\leq t \sum_{n \leq \tau} \frac{1}{nP_n} \left| \sum_{k=0}^n p_{n-k} \frac{1}{k+1} \sum_{r=1}^k r \sin rt \right| \\ &\quad + t \sum_{n > \tau} \frac{1}{nP_n} \left| \sum_{k=0}^n p_{n-k} \frac{1}{k+1} \frac{d}{dt} \left\{ \frac{\sin \left(k + \frac{1}{2} \right) t}{\sin \frac{1}{2} t} \right\} \right| \\ &= \Sigma_1 + \Sigma_2, \end{aligned}$$

say. But, we have

$$(5.2) \quad \Sigma_1 \leq Kt \sum_{n \leq \tau} \frac{1}{P_n} \sum_{k=0}^n p_{n-k} \leq K.$$

By virtue of Lemmas 7 and 8, we have

$$\begin{aligned} \Sigma_2 &\leq K \sum_{n > \tau} \frac{1}{nP_n} \left| \sum_{k=0}^n p_{n-k} \cos \left(k + \frac{1}{2} \right) t \frac{\left(k + \frac{1}{2} \right)}{k+1} \right| \\ (5.3) \quad &\quad + Kt^{-1} \sum_{n > \tau} \frac{1}{nP_n} \left| \sum_{k=0}^n p_{n-k} \frac{1}{k+1} \sin \left(k + \frac{1}{2} \right) t \right| \\ &\leq KP_{\tau} \sum_{n > \tau} \frac{1}{nP_n} + Kt^{-1} \sum_{n > \tau} \frac{1}{nP_n} \left| \sum_{k=0}^{\lfloor n/2 \rfloor - 1} p_{n-k} \frac{1}{k+1} \sin \left(k + \frac{1}{2} \right) t \right| \\ &\quad + Kt^{-1} \sum_{n > \tau} \frac{1}{nP_n} \left| \sum_{k=\lfloor n/2 \rfloor}^n p_{n-k} \frac{1}{k+1} \sin \left(k + \frac{1}{2} \right) t \right| \\ &\leq K + \Sigma_{21} + \Sigma_{22}, \end{aligned}$$

say. Since $p_{n+1} \leq p_n$, for all $n \geq 0$, we have by Abel's Lemma

$$\begin{aligned} \Sigma_{21} &\leq Kt^{-1} \sum_{n > \tau} \frac{p_{\lfloor n/2 \rfloor}}{nP_n} \max_{0 \leq \nu < \lfloor n/2 \rfloor} \left| \sum_{k=0}^{\nu} \frac{\sin \left(k + \frac{1}{2} \right) t}{k+1} \right| \\ (5.4) \quad &\leq Kt^{-1} \sum_{n > \tau} \frac{1}{n^2} R_{\lfloor n/2 \rfloor} \leq K, \end{aligned}$$

since

$$\left| \sum_{k=0}^{\nu} \left\{ \sin \left(k + \frac{1}{2} \right) t \right\} / (k+1) \right| \leq K$$

and $\{R_n\} \in B$, automatically.

Again by Abel's Lemma and Lemma 7, we have

² Throughout $[x]$ denotes the greatest integer not greater than x and $\tau = [2\pi/t]$.

$$\begin{aligned}
 \Sigma_{22} &\leq Kt^{-1} \sum_{n>\tau} \frac{1}{n^2 P_n} \max_{[n/2] \leq \nu \leq n} \left| \sum_{k=[n/2]}^{\nu} p_{n-k} \sin \left(k + \frac{1}{2} \right) t \right| \\
 (5.5) \quad &\leq Kt^{-1} P_{\tau} \sum_{n>\tau} \frac{1}{n^2 P_n} \leq K,
 \end{aligned}$$

by virtue of Lemma 8.

Combining (5.3)–(5.5), we prove that $\Sigma_2 \leq K$. This result combined with (5.2), leads to (5.1) and we thus, complete the proof of $|\tilde{F}_1|$ -effective part of Theorem 2.

(III). $|F^*$ -effectiveness: We have by integration by parts

$$\begin{aligned}
 nB_n(x) &= \frac{2}{\pi} \int_0^{\pi} \psi(t) n \sin nt \, dt \\
 &= \frac{2}{\pi} \psi(\pi)(1 - \cos n\pi) - \frac{2}{\pi} \int_0^{\pi} (1 - \cos nt) d\psi(t).
 \end{aligned}$$

Since x is a $|F^*$ -regular point, we have $\int_0^{\pi} |d\psi(t)| \leq K$ and it follows from Lemma 6 that in order to prove $|F^*$ -effectiveness of the $(N, p_n)(C, 1)$ method, it is sufficient to show that uniformly in $0 < t \leq \pi$

$$(5.6) \quad \sum_{n=1}^{\infty} \frac{1}{n P_n} \left| \sum_{k=1}^n p_{n-k} \frac{1}{k+1} \sum_{r=1}^k r \Delta \{1 - \cos(r-1)t\} \right| \leq K.$$

But, we have

$$\sum_{r=0}^k r \Delta \{\cos(r-1)t\} = \frac{1}{2}(1 - \cos kt) + 2 \sin \frac{t}{2} \sum_{r=1}^k \left(r - \frac{1}{2}\right) \sin \left(r - \frac{1}{2}\right) t.$$

Thus, to prove (5.6), it is enough to show that uniformly in $0 < t \leq \pi$

$$(5.7) \quad \left| \sin \frac{t}{2} \left| \sum_{n=1}^{\infty} \frac{1}{n P_n} \left| \sum_{k=1}^n \frac{p_{n-k}}{k+1} \sum_{r=1}^k \left(r - \frac{1}{2}\right) \sin \left(r - \frac{1}{2}\right) t \right| \right| \right| \leq K$$

and

$$(5.8) \quad \Sigma' = \sum_{n=1}^{\infty} \frac{1}{n P_n} \sum_{k=1}^n \frac{p_{n-k}}{k+1} \leq K.$$

The proof of (5.7), runs exactly parallel to the proof of (5.1). To prove (5.8), we observe that by a change of order of summations

$$\Sigma' = \sum_{k=1}^{\infty} \frac{1}{k+1} \sum_{n=k}^{\infty} \frac{p_{n-k}}{n P_n} \leq K \sum_{k=1}^{\infty} \frac{1}{k+1} \frac{p_0}{P_{k-1}} \leq K,$$

by virtue of Lemma 8 and the condition that $0 < p_{n+1} \leq p_n$.

We thus complete the proof of $|F^*|$ -effective part of Theorem 2.

(IV). *Absolute Fourier-effectiveness*: of the $(N, p_n)(C, 1)$ method follows from the absolute regularity of the (N, p_n) method and the corresponding effectiveness of the $(C, 1)$ method, which is included in the result of Theorem B.

Combing (I)–(IV), we complete the proof of Theorem 2.

The author should like to thank a referee for his kind suggestions which have helped to improve the presentation of this paper.

REFERENCES

1. L. S. Bosanquet, *The absolute Cesàro summability of a Fourier series*, Proc. London Math. Soc., **41** (1936), 517–528.
2. L. S. Bosanquet and J. M. Hyslop, *On the absolute summability of the allied series of a Fourier series*, Math. Z., **42** (1937), 489–512.
3. G. Das, *Tauberian theorems for absolute Nörlund summability*, Proc. London Math. Soc., (3), **19** (1969), 357–384.
4. H. P. Dikshit, *Absolute summability of a Fourier series by Nörlund means*, Math. Z., **102** (1967), 166–170.
5. ———, *Absolute (N, p_n) summability of a Fourier series*, Rendiconti di Mat., **1** (1968), 319–330.
6. ———, *On a class of Nörlund means and Fourier series*, Pacific J. Math., **33** (1970), 559–569.
7. ———, *Absolute total-effective triangular matrix method*, Math. Annalen, **186** (1970), 101–113.
8. ———, *Absolute total-effective (N, p_n) means*, Proc. Camb. Phil. Soc., **69** (1971), 107–122.
9. ———, *Absolute summability of a Fourier series and its derived series by a product method*, J. Australian Math. Soc. (to appear).
10. H. P. Dikshit and B. L. Misra, *On the absolute summability of a Fourier series and its conjugate series*, Math. Japonicae, **14** (1969), 23–30.
11. J. M. Hyslop, *On the absolute summability of the successively derived series of Fourier series and its allied series*, Proc. London Math. Soc., **46** (1940), 55–80.
12. K. Känno, *On the absolute Nörlund summability of the factored Fourier series*, Tôhoku Math. J., **21** (1969), 434–447.
13. B. Kwee, *Absolute regularity of the Nörlund means*, J. Australian Math. Soc., **5** (1965), 1–7.
14. L. McFadden, *Absolute Nörlund summability*, Duke Math. J., **9** (1942), 168–207.
15. F. M. Mears, *Absolute regularity and the Nörlund means*, Annals Math., **38** (1937), 594–601.
16. T. Pati, *On the absolute Nörlund summability of a Fourier series*, J. London Math. Soc., **34** (1959), 153–160; *Addendum*: Ibid, **37** (1962), 256.
17. ———, *On the absolute summability of a Fourier series by Nörlund means*, Math. Z., **88** (1965), 244–249.
18. L. L. Silverman, *Products of Nörlund transformation*, Bull. Amer. Math. Soc., **43** (1937), 95–101.

Received March 3, 1971 and in revised form September 2, 1971.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

Pacific Journal of Mathematics

Vol. 41, No. 1

November, 1972

Anatole Beck and Peter Warren, <i>Weak orthogonality</i>	1
Jonnie Bee Bednar and Howard E. Lacey, <i>Concerning Banach spaces whose duals are abstract L-spaces</i>	13
Louis Harvey Blake, <i>Canonical extensions of measures and the extension of regularity of conditional probabilities</i>	25
R. A. Brooks, <i>Conditional expectations associated with stochastic processes</i>	33
Theodore Allen Burton and Ronald Calvin Grimmer, <i>On the asymptotic behavior of solutions of $x'' + a(t)f(x) = e(t)$</i>	43
Stephen LaVern Campbell, <i>Operator-valued inner functions analytic on the closed disc</i>	57
Yuen-Kwok Chan, <i>A constructive study of measure theory</i>	63
Alexander Munro Davie and Bernt Karsten Oksendal, <i>Peak interpolation sets for some algebras of analytic functions</i>	81
H. P. Dikshit, <i>Absolute total-effective $(N, p_n)(c, 1)$ method</i>	89
Robert E. Edwards, Edwin Hewitt and Kenneth Allen Ross, <i>Lacunarity for compact groups. II</i>	99
James Daniel Halpern, <i>On a question of Tarski and a maximal theorem of Kurepa</i>	111
Gerald L. Itzkowitz, <i>A characterization of a class of uniform spaces that admit an invariant integral</i>	123
Mo Tak Kiang, <i>Semigroups with diminishing orbital diameters</i>	143
Glenn Richard Luecke, <i>A class of operators on Hilbert space</i>	153
R. James Milgram, <i>Group representations and the Adams spectral sequence</i>	157
G. S. Monk, <i>On the endomorphism ring of an abelian p-group, and of a large subgroup</i>	183
Yasutoshi Nomura, <i>Homology of a group extension</i>	195
R. Michael Range, <i>Approximation to bounded holomorphic functions on strictly pseudoconvex domains</i>	203
Norman R. Reilly, <i>Inverse semigroups of partial transformations and θ-classes</i>	215
Chris Rorres, <i>Strong concentration of the spectra of self-adjoint operators</i>	237
Saharon Shelah, <i>A combinatorial problem; stability and order for models and theories in infinitary languages</i>	247
George Gustave Weill, <i>Vector space decompositions and the abstract imitation problem</i>	263
Arthur Thomas White, <i>On the genus of the composition of two graphs</i>	275