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A CLASS OF OPERATORS ON HILBERT SPACE

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If T is an operator (bounded endomorphism) on the complex Hilbert space H , then $T \in \mathcal{R}$ if and only if $\|(T - zI)^{-1}\| = 1/d(z, W(T))$ for all $z \in \text{Cl } W(T)$, where $\text{Cl } W(T)$ is the closure of the numerical range of T and $d(z, W(T)) = \inf \{|z - u| : u \in W(T)\}$. The main results of this paper are: (1) $T \in \mathcal{R}$ if and only if the boundary of the numerical range of T is a subset of $\sigma(T)$, the spectrum of T ; and (2) \mathcal{R} is an arc-wise connected, closed nowhere dense subset of the set of all operators on H (norm topology) when $\dim H \geq 2$.

Introduction. If T is an operator (bounded endomorphism) on the complex Hilbert space H , then

$$1/d(z, \sigma(T)) \leq \|(T - zI)^{-1}\| \text{ and } \|(T - zI)^{-1}\| \leq 1/d(z, W(T)) ,$$

where the first inequality holds for all $z \notin \sigma(T)$, the spectrum of T ; and the second inequality holds for all $z \in \text{Cl } W(T)$, the closure of the numerical range of T [3]. Here $d(z, S)$ denotes the distance from z to the set S . In the current literature, much space has been devoted to the study of those operators T such that $\|(T - zI)^{-1}\|$ is equal to its smallest possible value for all $z \notin \sigma(T)$. In this paper, the properties of operators T with $\|(T - zI)^{-1}\|$ equal to its largest possible value, for all $z \in \text{Cl } W(T)$, are investigated. Let \mathcal{R} denote this set of operators.

We first characterize the operators in \mathcal{R} in terms of the boundary of the numerical range and spectrum of the operator. If S is a set of complex numbers, then let $\text{co } S$ denote the convex hull of S , let ∂S denote the boundary of S , and let $\text{Cl } S$ denote the closure of S . For $z \notin \sigma(T)$, let $R(T, z) = (T - zI)^{-1}$.

THEOREM 1. $T \in \mathcal{R}$ if and only if $\partial W(T) \subseteq \sigma(T)$.

Proof. First suppose that $\partial W(T) \subseteq \sigma(T)$. Let $z \in \text{Cl } W(T)$. Then $d(z, \sigma(T)) = d(z, W(T))$ so that

$$1/d(z, W(T)) = 1/d(z, \sigma(T)) \leq \|R(T, z)\| \leq 1/d(z, W(T)) .$$

Therefore $T \in \mathcal{R}$.

Suppose $T \in \mathcal{R}$ and let $z_0 \in \partial W(T)$. Then there exists a sequence $\{z_n\}$ approaching z_0 such that $|z_n - z_0| = d(z_n, W(T)) > 0$ for all $n = 1, 2, 3, \dots$. Then $\|R(T, z_n)\| = 1/d(z_n, W(T)) \rightarrow \infty$. Therefore $z_0 \in \sigma(T)$.

COROLLARY 1. *If $T \in \mathcal{R}$, then $\text{co } \sigma(T) = \text{Cl } W(T)$.*

COROLLARY 2. *If $T \in \mathcal{R}$ and $\sigma(T)$ is a finite set, then there exists a complex number α such that $T = \alpha I$.*

Corollary 1 follows immediately from Theorem 1. To prove Corollary 2 we do the following: Since $W(T)$ is a convex set, $\partial W(T) \subseteq \sigma(T)$ implies that the finite set $\sigma(T)$ must contain exactly one point, say α . Then $\sigma(T) = W(T) = \{\alpha\}$ so that $((T - \alpha I)x, x) = 0$ for all $x \in H$. Therefore $T = \alpha I$.

It follows from Corollary 2 that \mathcal{R} is just the set of all scalar multiples of the identity operator when $\dim H < \infty$.

To see that all operators satisfying $\text{co } \sigma(T) = \text{Cl } W(T)$ are not in \mathcal{R} , simply let N be a normal operator whose spectrum is a finite set with more than one point. Then by [3, problem 171], $\text{co } \sigma(N) = \text{Cl } W(N)$, and by Corollary 2, $N \notin \mathcal{R}$.

Let $B(H)$ denote the set of all operators on the complex Hilbert space H and give $B(H)$ the norm topology.

THEOREM 2. *\mathcal{R} is an arc-wise connected, closed, nowhere dense subset of $B(H)$ when $\dim H \geq 2$.*

Proof. If $T \in \mathcal{R}$, then $\alpha T \in \mathcal{R}$ for every complex number α . Therefore the ray through T in $B(H)$ is contained in \mathcal{R} , and thus \mathcal{R} is arc-wise connected.

To see that \mathcal{R} is closed, we let $\{T_n\}$ be a sequence of operators in \mathcal{R} approaching $T \in B(H)$ in norm. Then $W(T_n) \rightarrow W(T)$ in the Hausdorff metric [3, p. 176]. Let $z \notin \text{Cl } W(T)$. Then there exists a positive integer N such that for all $n \geq N, z \notin W(T_n)$. Then

$$\|R(T_n, z)\| = 1/d(z, W(T_n)) \longrightarrow 1/d(z, W(T)) .$$

Now choose $M \geq N$ so that for all $n \geq M, \|(T - T_n)R(T_n, z)\| < 1$. Then [2, p. 52]

$$R(T, z) = (I - (T - T_n)R(T_n, z))^{-1}R(T_n, z) .$$

Since $(T - T_n)R(T_n, z) \rightarrow 0$ as $n \rightarrow \infty$,

$$\|R(T, z)\| = \lim_{n \rightarrow \infty} \|R(T_n, z)\| = 1/d(z, W(T)) .$$

Therefore $T \in \mathcal{R}$ and hence \mathcal{R} is closed.

If \mathcal{C} is the set of all $T \in B(H)$ such that $\text{co } \sigma(T) = \text{Cl } W(T)$, then

by Corollary 1 $\mathcal{R} \subseteq \mathcal{C}$. By [4] \mathcal{C} is a nowhere dense subset of $B(H)$ when $\dim H \geq 2$. Therefore \mathcal{R} is also a nowhere dense subset of $B(H)$ when $\dim H \geq 2$.

The set of all operators on H ($\dim H = \infty$) that satisfy property (G_1) locally is a larger set than the set of all operators on H that satisfy property (G_1) [see 5]. This situation does not occur for \mathcal{R} . To see this, suppose T is an operator such that

$$\|R(T, z)\| = 1/d(z, W(T))$$

for all $z \in U - (\text{Cl } W(T))$ where U is an open set containing $\text{Cl } W(T)$. We now show that $T \in \mathcal{R}$, i.e., the above relationship holds for all $z \notin \text{Cl } W(T)$. Let $z_0 \in \partial W(T)$. Then there exists a sequence $\{z_n\} \subseteq U - (\text{Cl } W(T))$ such that $z_n \rightarrow z_0$ and $|z_n - z_0| = d(z_n, W(T))$. Then

$$\|R(T, z_n)\| = 1/d(z_n, W(T)) \longrightarrow \infty .$$

Thus $z_0 \in \sigma(T)$. Therefore, by Theorem 1, $T \in \mathcal{R}$.

We now give a method to construct nontrivial examples of operators in \mathcal{R} .

THEOREM 3. *If A is an operator on H , then $A \oplus N \in \mathcal{R}$ on $H \oplus K$ whenever N is a normal operator on K with $\sigma(N) \supseteq \partial W(A)$.*

Proof. Let N be as above and let $T = A \oplus N$. Since $W(A) \subseteq W(N)$,

$$W(T) = \text{co}(W(A) \cup W(N)) = W(N) .$$

For $z \in \sigma(T) = \sigma(A) \cup \sigma(N)$,

$$\|R(N, z)\| = 1/d(z, \sigma(N)) = 1/d(z, W(T)) .$$

Thus, since $\|R(A, z)\| \leq 1/d(z, W(A)) \leq 1/d(z, W(T))$,

$$\begin{aligned} \|R(T, z)\| &= \text{Max} \{ \|R(A, z)\|, \|R(N, z)\| \} \\ &= \text{Max} \{ \|R(A, z)\|, 1/d(z, W(T)) \} \\ &= 1/d(z, W(T)) . \end{aligned}$$

Therefore $T \in \mathcal{R}$.

There are a number of nice properties that not all operators in \mathcal{R} enjoy. Let $R_{sp}(T)$ denote the spectral radius of T .

THEOREM 4. *There exists $T \in \mathcal{R}$ such that*

- (i) $T^2 \notin \mathcal{R}$,
- (ii) $R_{sp}(T) < \|T\|$, and
- (iii) $T^{-1} \notin \mathcal{R}$.

Proof. Let $A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$, and let N be a normal operator with $\sigma(N) = W(A)$. Let $T = A \oplus N$. Then by Theorem 3, $T \in \mathcal{R}$. By [1] $W(A)$ is the closed disc of radius $1/2$ about $z = 1$, and $W(A^2)$ is the closed disc of radius 1 about $z = 1$. Therefore

$$0 \in W(A^2) \subseteq W(T^2) \text{ and } 0 \notin \text{co}(\sigma(T)^2) = \text{co} \sigma(T^2).$$

Therefore, $\text{co} \sigma(T^2) \neq \text{Cl } W(T^2)$ and so $T^2 \notin \mathcal{R}$. A computation yields $\|T\| = (3/2 + \sqrt{5/2})^{1/2}$. Thus $\|T\| > 3/2 = R_{sp}(T)$. If T^{-1} were in \mathcal{R} , then

$$\|T\| = \|R(T^{-1}, 0)\| = 1/d(0, W(T^{-1})).$$

But $1/d(0, W(T^{-1})) = 2 > \|T\|$. Therefore $T^{-1} \notin \mathcal{R}$.

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