INVERSE SEMIGROUPS OF PARTIAL TRANSFORMATIONS
AND $\theta$-CLASSES

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If $S$ is an inverse semigroup and $\theta$ is the relation on the lattice $A(S)$ of congruences on $S$ defined by saying that two congruences $\rho_1, \rho_2$ are $\theta$-equivalent if and only if they induce the same partition of the idempotents then $\theta$ is a congruence on $A(S)$ and each $\theta$-class is a complete modular sublattice of $A(S)$. If $X$ is a partially ordered set then $J_X$ denotes the inverse semigroup of one-to-one partial transformations of $X$ which are order isomorphisms of ideals of $X$ onto ideals of $X$, while if $X$ is a semilattice, $T_X$ denotes the inverse subsemigroup of $J_X$ consisting of those elements $\alpha$ whose domain $\Delta(\alpha)$ and range $\rho(\alpha)$ are principal ideals. It is shown that any inverse semigroup is isomorphic to an inverse subsemigroup of $J_X$ for some semilattice $X$.

For an inverse subsemigroup of $J_X$, $\theta(S) = A(S)/\theta$ is related to certain equivalence relations on $X$. The weakest of these is a convex congruence which is an equivalence relation on $X$, convex in the partial ordering and compatible with the operation in $S$. It is shown that there is a natural order preserving mapping $\alpha$ of $\theta(S)$ into the lattice $\Gamma(X)$ of convex congruences. If $X$ is a semilattice, the set of those convex congruences which are also semilattice congruences on $X$ is denoted by $\Gamma_2(X)$. If $S$ contains the idempotents of $T_X$, that is, if $S$ is full in $J_X$, then $\alpha$ is a semilattice homomorphism of $\theta(S)$ onto $\Gamma_2(X)$. If $S$ is full in $T_X$ then $\alpha$ is a lattice isomorphism of $\theta(S)$ onto $\Gamma_2(X)$. Conversely, there exists an order preserving mapping $\beta$ of $\Gamma_2(X)$ into $\theta(S)$. If $S$ is full in $J_X$, then $\beta$ is an order isomorphism into $\theta(S)$: if $S$ is full in $T_X$, then $\beta$ is a lattice isomorphism onto $\theta(S)$ and $\beta = \alpha^{-1}$.

We adopt the notation and terminology of (2). In particular, a semigroup $S$ is called an inverse semigroup if $a \in aSa$, for all $a \in S$, and the idempotents of $S$ commute. Then there is a unique element $x$ such that $a = axa$ and $a = xax$. We call $x$ the inverse of $a$ and write $x = a^{-1}$. For any inverse semigroup $S$, we denote by $E_S$ the subsemigroup of idempotents of $S$. If we define a partial ordering on $E_S$ by saying that $e \leq f$ if $ef = e$ then $S$ is a semilattice where, by a semilattice, we mean a partially ordered set in which any two elements have a greatest lower bound. For the basic results on inverse semigroups the reader is referred to (2). All semigroups considered in this paper will be inverse semigroups.
Denote by $\Lambda(S)$ the lattice of congruences on the inverse semigroup $S$; that is, the lattice of equivalence relations $\rho$ such that, for $a, b, c \in S$, $(a, b) \in \rho$ implies that $(ac, bc) \in \rho$ and $(ca, eb) \in \rho$. Define the relation $\theta$ (cf. 9) on $\Lambda(S)$ by

$$(\rho_1, \rho_2) \in \theta \text{ if and only if } \rho_1|E_s = \rho_2|E_s$$

where $\rho_i|E_s$ denotes the restriction of the congruence $\rho_i$ to $E_s$. Then

**Lemma 1.1.** ((9) Theorem 5.1). Let $S$ be an inverse semigroup and the relation $\theta$ be defined as above. Then

(i) $\theta$ is a congruence on $\Lambda(S)$;

(ii) each $\theta$-class is a complete modular sublattice of $\Lambda(S)$ (with a greatest and least element).

We shall denote the lattice of $\theta$-classes of an inverse semigroup $S$ by $\Theta(S)$.

Now each congruence on an inverse semigroup $S$ determines a normal partition of $E_s$; that is a partition $P = \{E_a: \alpha \in J\}$ such that

- $E(1)$ $\alpha, \beta \in J$ implies that there exists $\gamma \in J$ such that $E_aE_\beta \subseteq E_\gamma$;
- $E(2)$ $\alpha \in J$ and $\alpha \in S$ implies that there exists $\beta \in J$ such that $\alpha E_a\alpha^{-1} \subseteq F_\beta$.

Likewise we call an equivalence relation $\rho$ on $E_s$ a normal equivalence if its classes constitute a normal partition of $E_s$.

Conversely, if $P$ is a normal partition of $E_s$ then $P$ is induced by some congruence on $S$. Thus the lattice of normal partitions of $E_s$ is, clearly, just (isomorphic to) $\Theta(S)$.

The least and greatest congruence in the $\theta$-class corresponding to the normal partition $P$ can be characterized as follows:

**Lemma 1.2.** ((9) Theorem 4.2) Let $P = \{E_a: a \in J\}$ be a normal partition of the semilattice of idempotents of $S$. Let $\sigma = \{(a, b) \in S \times S: \text{there exists an } \alpha \in J \text{ with } \alpha \alpha^{-1}, \beta \beta^{-1} \in E_a \text{ and, for some } e \in E_a, ea = eb\}$ and $\rho = \{(a, b) \in S \times S: \alpha \in J \text{ implies that, for some } \beta \in J, a E_a\alpha^{-1}, b E_\beta \beta^{-1} \subseteq E_\beta\}$. Then $\sigma$ and $\rho$ are, respectively, the smallest and largest congruences on $S$ in the $\theta$-class corresponding to the normal partition $P$.

By a one-to-one partial transformation of a set $X$ we mean a one-to-one mapping $\alpha$ of a subset $Y$ of $X$ onto a subset $Y' = Y\alpha$ of $X$. We call $Y$ the domain of $\alpha$, $Y'$ the range of $\alpha$ and write $A(\alpha) = Y, V(\alpha) = Y'$. If we denote by $I_x$ the set of all one-to-one partial transformations of $X$ then, with respect to the natural multiplication of mappings, $I_x$ is an inverse semigroup called the symmetric inverse.
Let $X$ be a partially ordered set. By an *ideal* of $X$ we mean a subset $Y$ of $X$ such that $x \leq y \in Y$ implies that $x \in Y$. If $X$ is trivially ordered, that is, if no two distinct elements are comparable, then any subset of $X$ will be an ideal. We consider the empty set $\emptyset$ as being an ideal of $X$. By a principal ideal we mean an ideal of the form $\{x: x \leq y\}$ for some fixed element $y$. Then we call $\{x: x \leq y\}$ the (principal) ideal generated by $y$ and denote it by $< y >$. For an arbitrary subset $A$ of $X$ we write $< A > = \{x \in X: x \leq a$, for some $a \in A\}$.

If $X$ is a partially ordered set, let $J_X$ denote the set of all $\alpha \in I_X$ such that

1. $\mathcal{A}(\alpha)$ and $\mathcal{V}(\alpha)$ are ideals of $X$;
2. $\alpha$ is an order isomorphism of $\mathcal{A}(\alpha)$ onto $\mathcal{V}(\alpha)$; that is, a one-to-one mapping of $\mathcal{A}(\alpha)$ onto $\mathcal{V}(\alpha)$ such that, for $x, y \in \mathcal{A}(\alpha)$, $x \leq y$ if and only if $\alpha x \leq \alpha y$.

It is straightforward to verify that $J_X$ is an inverse subsemigroup of $I_X$. If $X$ is trivially ordered then, of course $J_X = I_X$.

By the following theorem, any inverse semigroup $S$ can be embedded in $I_S$.

**Theorem 1.3.** (2) Theorem 1.20) Let $S$ be an inverse semigroup and for each $a \in S$ define the element $\alpha_a$ of $I_S$ by

1. $\mathcal{A}(\alpha_a) = Sa^{-1}$;
2. for $x \in \mathcal{A}(\alpha_a)$, $x\alpha_a = xa$.

Then the mapping $\alpha: a \rightarrow \alpha_a$ is an isomorphism of $S$ into $I_S$.

Considering $S$ as a trivially ordered set we then have that $S$ can be embedded in $J_S$. However, on any inverse semigroup $S$ there exists a partial ordering, called the *natural partial ordering* which can be defined as follows: for any $a, b \in S$,

$$a \leq b \quad \text{if and only if} \quad a^{-1}b = a^{-1}a.$$ 

For several equivalent definitions of this partial ordering see §7.1 of (2). The natural partial ordering is compatible with the multiplication of $S$.

Suppose that $y \in Sa^{-1}$ and that $x \leq y$. Then $y = sa^{-1}$, for some $s \in S$ and $x^{-1}y = x^{-1}x$. Hence $x = xx^{-1}x = xx^{-1}y = xx^{-1}as^{-1} \in Sa^{-1}$. Thus $\mathcal{A}(\alpha_a)$ is an ideal in the partially ordered set $S$. Moreover, for any $x \leq y$, with $x, y \in \mathcal{A}(\alpha_a)$, $x\alpha_a = xa \leq ya = y\alpha_a$, since the natural partial ordering is compatible with the multiplication. Conversely, if $x\alpha_a \leq y\alpha_a$, for $x, y \in \mathcal{A}(\alpha_a)$ then $xa \leq ya$ and $x\alpha a^{-1} \leq y\alpha a^{-1}$. Since $x, y \in \mathcal{A}(\alpha_a) = Sa^{-1}$, $x\alpha a^{-1} = x$ and $y\alpha a^{-1} = y$. Thus $x \leq y$ and $\alpha_a$ is an order isomorphism of $\mathcal{A}(\alpha_a)$ onto $\mathcal{V}(\alpha_a)$. Thus
Proposition 1.4. Let $S$ be an inverse semigroup. Then the embedding $a \rightarrow \alpha_s$ of $S$ into $I_s$, of Theorem 1.3, also embeds $S$ in $J_s$ where $S$ is considered as a partially ordered set with respect to the natural partial ordering.

Let $X$ be a partially ordered set and $S \subseteq J_x$ (we shall sometimes just write $S \subseteq J_x$ for "$S$ is an inverse subsemigroup of $J_x$"). We shall be interested in certain kinds of equivalence relations on $X$. Consider the following conditions on an equivalence $\rho$ on $X$:

(i) $x \leq y \leq z$, $(x, z) \in \rho$ implies that $(x, y) \in \rho$;

(ii) $(x, y) \in \rho$, $x, y \in \Delta(a)$, $a \in S$, implies that $(xa, ya) \in \rho$.

If $\rho$ satisfies these conditions then we shall call $\rho$ a convex congruence, or just a $c$-congruence on $X$.

If $X$ is actually a semilattice and we denote by $x \wedge y$ the greatest lower bound of any two elements $x$, $y$ of $X$, then we can also consider the conditions:

(iii) $(x, y) \in \rho$ implies that $(x, x \wedge y) \in \rho$;

(iv) $(x, y) \in \rho$, $z \in X$ implies that $(x \wedge z, y \wedge z) \in \rho$.

If $\rho$ satisfies conditions (i), (ii) and (iii) we shall call $\rho$ an $s'$-congruence, while if $\rho$ satisfies (ii) and (iv) then we shall call $\rho$ a semilattice congruence or just an $s$-congruence. Although these definitions depend on $S$, $S$ will generally be held fixed and so the terminology should not lead to any confusion. If $X$ is a semilattice and $\rho$ satisfies condition (iv), then clearly $\rho$ satisfies conditions (i) and (iii). Thus an $s$-congruence is an $s'$-congruence and an $s'$-congruence is a $c$-congruence.

If $X$ is totally ordered then the three types of congruence coincide.

By a complete sublattice $A$ of a lattice $B$ we mean a sublattice such that for any nonempty subset $C$ of $A$ the least upper bound (greatest lower bound) of $C$ in $A$ exists and is the least upper bound (greatest lower bound) of $C$ in $B$.

Proposition 1.5. Let $X$ be a partially ordered set and $S \subseteq J_x$. Then the set $\Gamma(X)$ of $c$-congruences on $X$, partially ordered by set inclusion (as subsets of $X \times X$) is a complete lattice.

If $X$ is a semilattice then the set $\Gamma_s(X)$ of $s'$-congruences on $X$ is a complete lattice (but not necessarily a sublattice of $\Gamma(X)$) and the set $\Gamma_s(X)$ of $s$-congruences is a complete sublattice of $\Gamma(X)$.

Proof. Let $\{\rho_i; i \in I\}$ be a family of $c$-congruences ($s'$-congruences, $s$-congruences). Then clearly $\bigcap_{i \in I} \rho_i$ is also a $c$-congruence ($s'$-congruence, $s$-congruence). Since $\Gamma(X)$ ($\Gamma_s(X)$, $\Gamma_s(X)$) has a largest element, the universal congruence $\rho = X \times X$, it follows from purely lattice theoretic considerations that $\Gamma(X)$ ($\Gamma_s(X)$, $\Gamma_s(X)$) is a complete
lattice.

Now let $C$ be a nonempty subset of $\Gamma_2(X)$. Clearly the greatest lower bound of $C$ in $\Gamma(X)$ and $\Gamma_2(X)$ is just $\bigcap_{\rho_i \in C} \rho_i$. Now define a relation $\eta$ on $X$ by

$$(x, y) \in \eta \iff \text{for some } x = x_{i_0}, x_i, \ldots, x_n = y \in X, \quad (x_{i-1}, x_i) \in \rho_i, i = 1, \ldots, n, \text{ for some } \rho_i \in C.$$  

Then, from (1) Chapter 2, Theorem 4, $\eta$ is an equivalence relation on $X$ such that, if $(x, y) \in \eta$ and $z \in X$ then $(x \land z, y \land z) \in \eta$. Hence, to show that $\eta \in \Gamma_2(X)$, it only remains to be shown that if $(x, y) \in \eta$ and $(x, y) \in A(a)$ then $(xa, ya) \in \eta$. Let $x = x_{i_0}, x_i, \ldots, x_n = y \in X$ and $\rho_i, \ldots, \rho_n \in C$ be such that $(x_{i-1}, x_i) \in \rho_i$, for $i = 1, \ldots, n$. Then $(x_0 \land x_{i-1}, x_0 \land x_i) \in \rho_i, i = 1, \ldots, n$ and, since $x_0 \land x_i \leq x_{i_0}, x_0 \land x_i \in A(a)$, for $i = 1, \ldots, n$. Therefore, $((x_0 \land x_{i-1})a, (x_0 \land x_i)a) \in \rho_i$, for $i = 1, \ldots, n$ and so $(xa, (x \land y)a) = ((x_0 \land x_{i_0})a, (x_0 \land x_{i_1})a) \in \eta$. Similarly, $(ya, (x \land y)a) \in \eta$. Hence $(xa, ya) \in \eta$ and $\eta \in \Gamma_2(X)$.

But $\eta$ is the least upper bound of $C$ in the lattice of equivalence relations on $X$ and hence is the least upper bound of $C$ in $\Gamma(X)$. Thus $\Gamma_2(X)$ is a complete sublattice of $\Gamma(X)$; in fact, we proved that $\Gamma_2(X)$ is a complete sublattice of the lattice of equivalence relations on $X$.

We now give an example to illustrate some of the points that have arisen.

**Example.** Let $X$ be the semilattice of Figure 1 and $S = E_{x,y}$.

![Figure 1](image.png)

Let $\rho_1$ be the equivalence relation on $X$ which partitions $X$ as $X = \{u\} \cup \{y\} \cup \{x, v\}$; let $\rho_2$ be the equivalence relation partitioning $X$ as $X = \{x, u\} \cup \{v\} \cup \{y\}$ and let $\rho_3$ be the equivalence relation partitioning $X$ as $X = \{x\} \cup \{y\} \cup \{u, v\}$.

Now $\rho_1$ is a c-congruence but not an s'-congruence since $(x, x \land v) = (x, y) \in \rho_1$. Also $\rho_2$ is an s'-congruence but not an s-congruence since $(x, u) \in \rho_2$ but $(x \land u, v \land v) = (y, v) \in \rho_2$. Similarly $\rho_3$ is an s'-congruence, but not an s-congruence. Finally, the least upper bound of $\rho_2$ and $\rho_3$ in $\Gamma(X)$ partitions $X$ as $X = \{x, u, v\} \cup \{y\}$ which is not an s'-congruence.

2. From normal equivalences to congruences. Throughout this
section, let $X$ be a partially ordered set and $S$ be an inverse subsemigroup of $J_X$. We now begin to relate the $\theta$-classes of $S$ and the congruences on $X$.

If $A$ is a subset of $S$ then we shall denote by $A\omega$ the set $\{s \in S: \alpha \leq s$, for some $\alpha \in A\}$.

Let $\tau$ be a normal equivalence on $E_S$ and $x \in X$. Let $V(x) = \{e \in E_S: x \in \Delta(e)\}$ and $V_\tau(x) = \bigcup_{e \in V(x)} e\tau\omega$.

Then we have

**Lemma 2.1.** $V(x) \subseteq V_\tau(x)$ implies that $V_\tau(x) \subseteq V_\tau(y)$.

**Proof.** Let $f, f_1 \in E_S$, $(f, f_1) \in \tau$ and $f_1 \in V(x)$. Then $f_1 \in V_\tau(y)$ and so $f_1 \subseteq f_{\alpha_1} (f_{\beta_1}, f_{\beta_3}) \in \tau$ and $f_3 \in V(y)$, for some $f_{\alpha_1}, f_{\beta_1}, f_{\beta_3} \in E_S$. Hence $f \geq f_{\alpha_1}$, $(f_{\alpha_1} f_{\beta_1} f_{\beta_3}) \in \tau$, $f_{\alpha_1} f_{\beta_1} = f_{\alpha_1}$, $(f_{\alpha_1}, f_{\beta_3}) \in \tau$ and $f_{\beta_3} \in V(y)$; that is, $f \geq f_{\alpha_1}$, $(f_{\alpha_1}, f_{\beta_3}) \in \tau$ and $f_{\beta_3} \in V(y)$. Hence $f \in V_\tau(y)$. Thus $\bigcup_{e \in V(x)} e\tau \subseteq V_\tau(y)$ and so $V_\tau(x) \subseteq V_\tau(y)$.

**Theorem 2.2.** Let $X$ be a partially ordered set and $S \subseteq J_X$. Let $\tau$ be a normal equivalence on $E_S$. Define the relation $\rho = \rho_\tau$ on $X$ by $(x, y) \in \rho$ if and only if $V_\tau(x) = V_\tau(y)$.

Then $\rho$ is a $c$-congruence on $X$. Moreover, if $\sigma$ is another normal equivalence on $E_S$ and $\tau \subseteq \sigma$, then $\rho_\tau \subseteq \rho_\sigma$.

**Proof.** (i) Suppose that $x \leq y \leq z$ and $(x, z) \in \rho$. Then $V(z) \subseteq V(y) \subseteq V(x)$ and so $V_\tau(z) \subseteq V_\tau(y) \subseteq V_\tau(x) = V_\tau(z)$, by Lemma 2.1. Hence $V_\tau(x) = V_\tau(y)$ and so $(x, y) \in \rho$.

(ii) Suppose that $(x, y) \in \rho$, $a \in S$ and $x, y \in \Delta(a)$. Let $f \in V(xa)$. Then $xa \in \Delta(af a^{-1})$ and so $x \in \Delta(af a^{-1})$. Hence $a^{-1}f a \in V(x) \subseteq V_\tau(y)$. Therefore, for some $f_{\alpha_1}, f_{\beta_1} \in E_S$, we have $a^{-1}f a \subseteq f_{\alpha_1}$, $(f_{\alpha_1}, f_{\beta_1}) \in \tau$ and $f_{\beta_1} \in V(y)$. Hence $ya = yf_{\alpha_1} a \in \Delta(a^{-1}f a) = \Delta(a^{-1}f a)$, where $(a^{-1}f a, a^{-1}f a) \in \tau$, $a^{-1}f a \subseteq a^{-1}a^{-1}a \subseteq f$. Thus $f \in V_\tau(ya)$ and, by Lemma 2.1, $V_\tau(xa) \subseteq V_\tau(ya)$. Similarly we have the converse inclusion and so $V_\tau(xa) = V_\tau(ya)$ and $(xa, ya) \in \rho$. Hence $\rho$ is a $c$-congruence. Now $\tau \subseteq \sigma$ implies that $V_\tau(x) \subseteq V_\sigma(x)$, for all $x \in X$, and so $(x, y) \in \rho_\sigma$ implies that $V(x) \subseteq V_\sigma(y)$. Therefore $V_\tau(x) \subseteq V_\sigma(y)$, by Lemma 2.1, and similarly the converse inclusion holds. Thus $(x, y) \in \rho_\sigma$ and $\rho_\tau \subseteq \rho_\sigma$.

In general, of course, this mapping from normal equivalences to $c$-congruences is not one-to-one. However, in some circumstances, as we now show, it will be.

For any sets $A$ and $B$ let $A \setminus B = \{x: x \in A, x \notin B\}$. For $e \in E_S$, let $\delta(e) = \Delta(e) \setminus \bigcup_{f < e} \Delta(f) = \{x: x \in \Delta(e), x \notin \Delta(f)\}$ for any $f \in E_S$ such that $f < e$.

By an order isomorphism $\alpha$ of one partially ordered set $X$ into
another $Y$, we mean a one-to-one mapping $\alpha$ of $X$ into $Y$ such that, for $x, y \in X$, $x \leq y$ if and only if $x\alpha \leq y\alpha$.

**Proposition 2.3.** Let $X$ be a partially ordered set and $S \subseteq J_X$. Let the normal equivalence $\tau$ on $E_S$ induce the $\rho$-congruence $\rho = \rho_\tau$ on $X$ as in Theorem 2.2. Let $e, f \in E_S$, $x \in \delta(e)$ and $y \in \delta(f)$. Then

\[(2.1) \quad (x, y) \in \rho \text{ if and only if } (e, f) \in \tau.\]

Thus, if $X = \bigcup_{e \in E_S} \delta(e)$, then the definition of $\rho$ in Theorem 2.2 may be replaced by the statement (2.1).

Finally, if $\delta(e) \neq \emptyset$, for all $e \in E_S$, then the mapping $\tau \rightarrow \rho_\tau$ defines an order isomorphism of the lattice $\Theta(S)$ into $\Gamma(X)$.

**Proof.** Let $e, f \in E_S$, $x \in \delta(e)$, $y \in \delta(f)$. First suppose that $(e, f) \in \tau$. Then, for $g \in V(x)$ we have that, $g \geq e$, $(e, f) \in \tau$ and $f \in V(y)$. Thus $V(x) \subseteq V_z(x)$, $V_z(x) \subseteq V_z(y)$ and, by similarity, $V_z(x) = V_z(y)$; that is, $(x, y) \in \rho$.

Now suppose that $(x, y) \in \rho$. Then $V_z(x) = V_z(y)$. Hence $e \in V(x) \subseteq V_z(y)$. Thus, for some $e_1, e_2 \in E_S$, $e \geq e_1$, $(e, e_2) \in \tau$, $e_2 \geq f$. Similarly, for some $f_1, f_2 \in E_S$, $f \geq f_1$, $(f_1, f_2) \in \tau$ and $f_2 \geq e$. Then

\[e \geq e_2 f, (e_2 f, f) = (e, f, e_2 f) \in \tau\]

and

\[f \geq ef_1, (ef_1, e) = (ef_1, ef) \in \tau.\]

Hence

\[(e, f, ef) = (e \cdot \alpha, f, ef) \in \tau\]

and

\[(ef, e f) = (ef, ef) \in \tau.\]

Therefore $(e, f, ef) \in \tau$ and so $(e, f) \in \tau$.

The remainder of the theorem then follows easily.

A congruence $\rho$ on an inverse semigroup $S$ is called *idempotent separating* if no two distinct idempotents of $S$ lie in the same $\rho$-class. There exists a unique maximal idempotent separating congruence $\mu$ on $S$ which can be characterized as follows (Howie [4]):

\[(a, b) \in \mu \iff a^{-1}ea = b^{-1}eb \text{ for all } e \in E_S.\]

If $\mu$ is the identity congruence, then we shall call $S$ *fundamental*.

Although, for $S \subseteq J_X$ and $X$ a semilattice, we shall be considering
the general problem of defining a normal equivalence on $E_s$ from an $s'$-congruence on $X$ in the next section and although it appears essential in general to assume that $X$ is a semilattice and that the congruence on $X$ is an $s'$-congruence, we can, at least, establish the following theorem without these assumptions.

**Theorem 2.4.** Let $X$ be a partially ordered set and $S \subseteq J_x$. Define the relation $\nu$ on $X$ by:

$$(x, y) \in \nu \iff V(x) = V(y).$$

Then $\nu$ is $c$-congruence on $X$. Define the relation $\xi$ on $S$ by

$$(a, b) \in \xi \iff \begin{cases} x^*, x^* \cap J(a) \neq \emptyset; \quad \{x^*, x^* \cap J(b) \neq \emptyset\}; \\
(x, y) \in \nu \implies (xa, yb) \in \nu. \end{cases}$$

Then $\xi = \mu$, the maximum idempotent separating congruence on $S$.

**Proof.** Let $(x, z) \in \nu$ and $x \leq y \leq z$. Then $V(x) \supseteq V(y) \supseteq V(z) = V(x)$. Thus $V(x) = V(y)$ and $(x, y) \in \nu$.

Now let $(x, y) \in \nu$ and $x, y \in J(a)$. Let $e \in V(xa)$. Then $ae^{-1} \in V(x) = V(y)$. Thus $e \in V(ya)$ and $V(xa) \subseteq V(ya)$. Similarly $V(ya) \subseteq V(xa)$ and so $V(xa) = V(ya)$. Thus $(xa, ya) \in \nu$ and $\nu$ is a $c$-congruence.

It is straightforward to see that $\xi$ is an equivalence relation. To show that $\xi = \mu$, we first show that $\tau = \xi|_{E_s} = \tau$. Let $(e, f) \in \tau$ and $x \in J(e)$. Then $x^* \cap J(f) \neq \emptyset$ and so $y \in x^* \cap J(f)$, for some $y$. Then $f \in V(y) = V(x)$. Thus $x \in J(f)$ and $J(e) \subseteq J(f)$. Conversely, $J(f) \subseteq J(e)$ and so $J(e) = J(f)$ and $e = f$.

Let $(a, b) \in \xi$. Then, for any $x \in X$, $x^* \cap J(a) \neq \emptyset$ if and only if $x^* \cap J(b) \neq \emptyset$. But $J(a) = J(aa^{-1})$ and $J(b) = J(bb^{-1})$. Hence $x^* \cap J(aa^{-1}) \neq \emptyset$ if and only if $x^* \cap J(bb^{-1}) \neq \emptyset$. Moreover, for $(x, y) \in \nu$, $x \in J(aa^{-1})$, $y \in J(bb^{-1})$, $(xaa^{-1}, ybb^{-1}) = (x, y) \in \nu$. Hence $(a, b) \in \xi$ implies that $(aa^{-1}, bb^{-1}) \in \xi$ and so $aa^{-1} = bb^{-1}$ and $J(a) = J(b)$.

Now we show that $\xi$ is a congruence on $S$. Let $(a, b) \in \xi$ and $c \in S$. If $x \in J(ac)$ then $x \in J(a) = J(b)$ and $xa \in J(c)$. However, $(xa, xb) \in \nu$ and so $cc^{-1} \in V(xa) = V(xb)$. Thus $x \in J(bc)$ and $J(ac) \subseteq J(bc)$. By similarity, $J(ac) = J(bc)$ and condition (i) is satisfied by $ac$ and $bc$. If $x \in J(ac) = J(bc)$, then $(xa, xb) \in \nu$, since $(a, b) \in \xi$, and so $(xac, xcb) \in \nu$, since $\nu$ is a $c$-congruence. Thus $(ac, bc) \in \xi$.

Now $x \in J(ca)$ if and only if $x \in J(c)$ and $xc \in J(a) = J(b)$. Thus $J(ca) = J(cb)$ and condition (i) is satisfied by $ca$ and $cb$. Clearly $ca$ and $cb$ then satisfy condition (ii). Thus $(ca, cb) \in \xi$ and $\xi$ is a congruence.

Since $\xi|_{E_s} = \tau$ we have that $\xi \subseteq \mu$ and to complete the theorem we need only show that $\mu \subseteq \xi$. Suppose that $(a, b) \in \mu$. Then $aa^{-1} = \ldots$
$bb^{-1}, A(aa^{-1}) = A(bb^{-1})$ and condition (i) is satisfied. Now let $x \in A(a)$, $y \in A(b)$ and $(x, y) \in \nu$. Let $f \in V(xa)$. Then $xa \in A(f)$ and so $x \in A(afa^{-1})$. But, since $(a, b) \in \mu$, $afa^{-1} = bfb^{-1}$. Thus $x \in A(bfb^{-1})$. Now $V(x) = V(y)$ and so $y \in A(bfb^{-1})$. Hence $yb \in A(f)$ and $V(xa) \subseteq V(yb)$. By similarity, we have that $V(xa) = V(yb)$ and $(xa, yb) \in \nu$. Thus condition (ii) is also satisfied by $a$ and $b$ and so $(a, b) \in \xi$. Hence $\xi = \mu$.

If, in Theorem 2.4, $\nu$ is the identity relation on $X$, then clearly $(a, b) \in \xi$ if and only if $a = b$. Thus we have immediately:

**Corollary 2.5.** Let $X$ be a partially ordered set and $S \subseteq J_x$. If $\nu$ is the identity relation, then $S$ is fundamental.

Let $X$ be a partially ordered set and $x \in X$. Then we shall denote by $e_x$ the idempotent of $J_x$ with domain equal to the principal ideal $<x>$. Let $S \subseteq J_x$, then we say that $S$ is full in $J_x$ or (if $X$ is a semilattice and $S \subseteq T_x$) that $S$ is full in $T_x$ if $\{e_x : x \in X\} \subseteq E_S$, where $T_x$ is as defined in §3.

**Corollary 2.6.** Let $S$ be full inverse subsemigroup of $J_x$, then $S$ is fundamental.

**Proof.** If $S$ is full then $\nu$ must be the identity relation and then so must $\xi$.

Corollary 2.6 is a slight generalization of a theorem ([6] Theorem 2.6) of Munn’s and could be established directly along the same lines as Munn’s proof. Corollary 2.5 is a little stronger, however, as the following example shows:

**Example.** Let $X$ be the set of real numbers under their natural ordering. Let $S = \{\alpha \in J_x : A(\alpha) \text{ is not principal}\}$. Then $S$ is an inverse subsemigroup of $J_x$. Clearly $\nu$ is the identity relation and hence $S$ is fundamental. However, $S$ is not a full inverse subsemigroup of $J_x$.

3. $X$ a semilattice. Let $X$ be a semilattice, then we can define another subsemigroup of $I_x$ as follows. Let $T_x$ denote the set of $\alpha \in I_x$ such that

(i) $A(\alpha)$ and $F(\alpha)$ are principal ideals;

(ii) $\alpha$ is an order isomorphism of $A(\alpha)$ onto $A(\alpha)$.

It is straightforward to verify that $T_x$ is an inverse subsemigroup of $I_x$ and $J_x$. For a discussion of $T_x$ and its importance in connection with bisimple inverse semigroups see Munn [7].

**Proposition 3.1.** Let $X$ be a partially ordered set and let $\bar{X}$
denote the set of all ideals of $X$, partially ordered by set inclusion. Then $\bar{X}$ is a semilattice and there exists an embedding $\kappa: J_x \rightarrow T_{\bar{X}}$.

Proof. Clearly $\bar{X}$ is a semilattice. For $\alpha \in J_x$ define $\kappa_\alpha \in T_{\bar{X}}$ by:

(i) $\Delta(\kappa_\alpha) = \{I \in \bar{X}: I \subseteq \Delta(\alpha)\}$;
(ii) for $I \in \Delta(\kappa_\alpha)$, $I\kappa_\alpha = \{ax: x \in I\}$.

Then $\kappa: \alpha \rightarrow \kappa_\alpha$ is an isomorphism of $J_x$ into $T_{\bar{X}}$.

We now give several ways in which inverse semigroups might be considered as subsemigroups of $T_x$ for some semilattice $X$. First, from [7] Lemma 3.1,

**Proposition 3.2.** Let $S$ be an inverse semigroup and $E_S = E$. Define a mapping $\theta: S \rightarrow T_E$ by the rule that $a\theta = \theta_a$ where

(i) $\Delta(\theta_a) = Eaa^{-1}$;
(ii) for $e \in \Delta(\theta_a)$, $e\theta_a = a^{-1}ea$.

Then $\theta$ is a homomorphism of $S$ into $T_E$ inducing the maximum idempotent separating congruence on $S$ and hence is an isomorphism if $S$ is fundamental.

Combining either Theorem 1.3 (considering $S$ as a trivially ordered set) or Proposition 1.4 with Proposition 3.1 we have:

**Proposition 3.3** Let $S$ be an inverse semigroup then there exists a semilattice $X$ and an isomorphism $\kappa: S \rightarrow T_x$.

Presently we shall be considering inverse subsemigroups $S$ of $J_x$, where $X$ is a semilattice, such that $X = \bigcup_{e \in E_S} \delta(e)$ or such that $\delta(e) \neq \emptyset$, for all $e \in E_S$. In this connection, we have

**Proposition 3.4.** Let $S$ be an inverse semigroup then there exists a semilattice $X$ and an isomorphism $\kappa: S \rightarrow J_x$ such that

(i) $\delta(e\kappa) \neq \emptyset$ for all $e \in E_S$;
(ii) $X = \bigcup_{e \in E_S} \delta(e\kappa)$.

Proof. Let $\theta: S \rightarrow J_S$ be the embedding of Proposition 1.4. Let $X$ denote the set of all subsets of $S$ which are inversely well ordered with respect to the natural partial ordering of $S$, together with the empty set. Partially order $X$ by set inclusion. Then $X$ is clearly a semilattice. Define $\phi: J_S \rightarrow J_x$ as follows: for $\alpha \in J_S$,

(i) $\Delta(\alpha\phi) = \{A \in X: A \subseteq \Delta(\alpha)\}$;
(ii) for $A \in \Delta(\alpha\phi)$, $A(\alpha\phi) = \{a\alpha: a \in A\}$.

Then $\phi$ is an isomorphism and so $\kappa = \theta \circ \phi$ is an isomorphism of $S$ into $J_x$.

For $e \in E_S$, $e \in \Delta(e\theta)$ and so $\{e\} \in \Delta(e\kappa)$. Clearly $\{e\} \in \Delta(f\kappa)$, for $f \in$
If and only if $e \leq f$ in the natural partial order on $S$. Thus $\{e\} \in \delta(ek)$ and $\delta(ek) \neq \emptyset$ for all $e \in E_S$.

Let $A \in X$ have greatest element $a$, in the natural partial order on $S$. Then $a \in \delta((a^{-1}a)\kappa)$. Thus $X = \bigcup_{e \in E_S} \delta(ek)$.

Finally, we give a representation of slightly less general applicability which is interesting on account of the relationship that the set $X$ bears to the semigroup.

Before doing so, we need the following special case of Lemma 1.2 due to Munn [5]:

**Lemma 3.5.** Let $S$ be an inverse semigroup and let a relation $\sigma$ be defined on $S$ by the rule that $x \sigma y$ if and only if there is an idempotent $e$ in $S$ such that $ex = ey$ (or, equivalently, $xe = ye$). Then $\sigma$ is a congruence on $S$ and $S/\sigma$ is a group. Further, if $\tau$ is any congruence on $S$ with the property that $S/\tau$ is a group, then $\sigma \subseteq \tau$ and so $S/\tau$ is isomorphic with some quotient group of $S/\sigma$.

Then $\sigma$ is called the minimum group congruence on $S$.

**Proposition 3.6.** Let $S$ be an inverse semigroup, let $\sigma$ be the minimum group congruence on $S$, let $\mu$ be the maximum idempotent separating congruence on $S$ and let $\sigma \cap \mu = \iota$, the identity congruence on $S$. Let $X = E_S \cup S/\sigma \cup \{0\}$, where for $x, y \in X$, we have $x \leq y$ if and only if

- either (i) $x, y \in E_S$ and $x \leq y$ in the natural partial ordering of $E_S$;
- or (ii) $y \in E_S$ and $x \in S/\sigma$;
- or (iii) $x = 0$.

Then $X$ is a semilattice and there exists an embedding $\kappa: S \to T_x$, such that $\delta(ek) \neq \emptyset$ for all $e \in E_S$.

**Proof.** Let $\theta: a \to \theta_a$ be the Munn representation of $S$ of Proposition 3.2. Then, for $a \in S$, define $ak \in T_x$ as follows:

- (i) $\Delta(ak) = E_xa^{-1}a \cup S/\sigma \cup \{0\}$;
- (ii) $x(ak) = x\theta_a$ if $x \in E_S \cap \Delta(ak)$;
- (iii) $x(ak) = x(\sigma)$ if $x \in S/\sigma$;
- (iv) $x(ak) = x$ if $x = 0$.

Then it is clear that $\kappa$ is a homomorphism of $S$ into $T_x$ inducing the congruence $\sigma \cap \mu$, that is, the identity congruence. Thus $\kappa$ is an isomorphism.

We now turn to the problem of relating, for $S \subseteq J_x$ and $X$ a semilattice, $s'$-congruences on $X$ to normal equivalences or $\theta$-classes of $S$. For $\rho$ an $s'$-congruence on $X$ and $a \in S$ we shall denote by $U(a)$
the set \{x\rho: x\rho \cap A(a) \neq \emptyset\}. We suppress any indication of the dependence of \(U(a)\) on \(\rho\) since this will not lead to any confusion.

**Theorem 3.7.** Let \(X\) be a semilattice, \(S\) be an inverse subsemigroup of \(J_X\) and \(\rho\) be an \(s^r\)-congruence. For \(a \in S\), define \(a \in J_{X/\rho}\), as follows:

1. \(a \in J_{X/\rho}\)
2. for \(x \rho \in A(a)\), \((x \rho)\alpha_a = (x \rho)\alpha\) where \(x\) is any element in \(x \rho \cap A(a)\).

Then \(\alpha: a \rightarrow a\) is a homomorphism of \(S\) into \(J_{X/\rho}\). If \(\rho\) is an \(s\)-congruence then a partial ordering of \(X/\rho\) can be defined as follows:

\[ x \rho \leq y \rho \Rightarrow x_i \leq y_i \text{ for some } x_i \in x \rho, y_i \in y \rho. \]

With respect to this partial ordering \(X/\rho\) is a semilattice and \(S\alpha \subseteq J_{X/\rho}\).

**Proof.** Since \(\rho\) is a \(c\)-congruence, \(\alpha_a\) is clearly well defined and it is straightforward to show that \(\alpha_a \in J_{X/\rho}\), that is, that \(\alpha_a\) is one-to-one. Let \(a, b \in S\) and \(x \rho \in A(aab)\). Then there exists an \(x \in x \rho \cap A(ab)\). Hence \(x \in x \rho \cap A(a)\) and \(x, a \in A(b)\). Thus \(x \rho \in A(a)\) and \(x_a \in A(b)\). Thus \((x \rho)\alpha_a \in A(ab)\) and \(x \rho \in A(a)\). Conversely, let \(x \rho \in A(a)\). Then there exists an \(x \in x \rho \cap A(a)\) and an \(x \in (x \rho)\alpha_a \cap A(b)\). With \(x = x_1 \land x_1 a\), we have \(x = x_1 \rho = (x \rho)\alpha_a\) and \(x \in A(a)\). Since \(x, a \in A(ab)\), we have \(x \rho \in A(ab)\). Hence \(x \rho \in A(a)\) and \(x, a \in A(ab)\). Thus \(x \rho \in A(ab)\). Hence \(x \rho \in A(a)\). Thus \(A(a) = A(ab)\). Now let \(x \rho \in A(a)\). Then \(x \rho \alpha_a = (x \rho)\alpha_a\). and \(x \in x \rho \cap A(ab)\). Then

\[ (x \rho)\alpha_a = (x \rho)\alpha_a \rho \]

and

\[ (x \rho)\alpha_a \rho = (x \rho)\alpha_a = (x \rho)\rho. \]

Hence \(\alpha_a \rho = \alpha_{ab}\) and \(\alpha\) is a homomorphism.

If \(\rho\) is an \(s\)-congruence then \(X/\rho\) is clearly a semilattice and it only remains to be shown that \(S\alpha \subseteq J_{X/\rho}\).

So suppose that \(x \rho \leq y \rho\) and \(y \rho \in A(a)\). Then there exists \(x \in x \rho, y_1, y_2 \in y \rho\) such that \(x \leq y_1\) and \(y_2 \in A(a)\). Hence \(x \rho, x_1 \land y_2 = (x \rho)\alpha\) and so \((x, x \rho) \in \rho\) where \(x \rho \subseteq y \rho \leq y_2 \in A(a)\). Thus \(x \rho \in A(a)\) and \(x \rho \in A(a)\). Therefore \(A(a)\) is an ideal and it is routine to verify that \(\alpha_a\) is order preserving. Thus \(S\alpha \subseteq J_{X/\rho}\).

To see the difficulty that arises if \(\rho\) is just a \(c\)-congruence, consider the semilattice \(X\) of Figure 2.

Let \(S\) be the inverse subsemigroup of \(J_X\) consisting of the idem-
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Let \( \rho \) be the \( c \)-congruence on \( X \) determined by the partition \( X = \{x, v\} \cup \{u\} \cup \{w\} \). Then there is no natural homomorphism of \( S \) into \( J_{x/\rho} \).

From Theorem 3.7, we have

**Corollary 3.8.** Let \( X \) be a semilattice and \( S \) be an inverse subsemigroup of \( J_x \). Let \( \rho \) be an \( s \)-congruence on \( X \) and define the relation \( \tau = \tau_\rho \) on \( E_s \) as follows: for \( e, f \in E_s \),

\[
(e, f) \in \tau \iff U(e) = U(f).
\]

Then \( \tau \) is a normal equivalence on \( E_s \). If \( \rho \subseteq \rho' \) then \( \tau \subseteq \tau' \).

In certain circumstances we can give a more direct definition of the normal equivalence induced by an \( s \)-congruence.

**Lemma 3.9.** Let \( X \) be a semilattice and \( S \) be an inverse subsemigroup of \( J_x \). Let \( \rho \) be an \( s \)-congruence on \( X \) and let \( \rho \) induce the normal equivalence \( \tau \) on \( E_s \). If \( e_x, e_y \in E_s \) then

\[
(e_x, e_y) \in \tau \iff (x, y) \in \rho.
\]

In particular, if \( S \subseteq T_x \) then this defines \( \tau \).

**Proof.** Let \( (x, y) \in \rho \) and \( z \rho \cap \Delta(e_x) \neq \emptyset \). Without loss of generality, let \( z \in \Delta(e_x) \). Then \( z \leq x \), \( (z, z \wedge y) = (z \wedge x, z \wedge y) \in \rho \) and \( z \wedge y \in \Delta(e_y) \). Thus \( z \rho \cap \Delta(e_y) \neq \emptyset \) and \( U(e_x) \subseteq U(e_y) \). By similarity, we have the converse inclusion and so \( (e_x, e_y) \in \tau \).

Now suppose that \( (e_x, e_y) \in \tau \). Then \( x \in z \rho \cap \Delta(e_x) \) and so there exists an \( x_i \) such that \( (x, x_i) \in \rho \) and \( x_i \in \Delta(e_y) \), that is, \( x_i \leq y \). Similarly, there exists a \( y_i \) such that \( (y, y_i) \in \rho \) and \( y_i \in \Delta(e_x) \), that is, \( y_i \leq x \). Then \( (x \wedge x_i, x_i) = (x \wedge y, x_i \wedge y) \in \rho \) and \( (x \wedge y, y_i) = (x \wedge y, x \wedge y_i) \in \rho \). Hence \( (x_i, y_i) \in \rho \) and so \( (x, y) \in \rho \) as required.

We conclude this section with an instance where the mapping \( \rho \rightarrow \tau \) is one-to-one.

**Theorem 3.10.** Let \( X \) be a semilattice and \( S \) be a full inverse subsemigroup of \( J_x \). If \( \tau \) is a normal equivalence on \( E_s \) then \( \tau \) induces
an s-congruence on $X$. On the other hand, if $\rho$ is an s-congruence on
$X$, if $\rho$ induces the normal equivalence $\tau$ on $E_s$ and $\tau$, in turn, induces the s-congruence $\rho'$ on $X$, then $\rho = \rho'$. In particular, the mapping $\beta: \rho \rightarrow \tau$ defines an order isomorphism of $\varGamma_2(X)$ into $\Theta(S)$, and the mapping $\tau \rightarrow \rho$ into $\varGamma_2(X)$ is into $\varGamma_2(X)$. Thus, if $S$ is full in $T_X$ then, by Proposition 2.3, the mapping $\tau \rightarrow \rho$ defines an order isomorphism of $\Theta(S)$ onto $\varGamma_2(X)$.

**Proof.** Let the normal equivalence $\tau$ on $E_s$ induce the c-congruence $\rho$ on $X$. For any $x, y \in X$, we clearly have

$$
\Delta(e_x e_y) = \Delta(e_x) \cap \Delta(e_y)
$$

$$
= \{z: z \leq x\} \cap \{z: z \leq y\}
$$

$$
= \{z: z \leq x \land y\}
$$

$$
= \Delta(e_{x \land y}).
$$

Hence $e_x e_y = e_{x \land y}$. Also, from Proposition 2.3, we have that $(x, y) \in \rho$ if and only if $(e_x, e_y) \in \tau$. So now suppose that $(x, y) \in \rho$ and $z \in X$. Then $(e_z, e_y) \in \tau$ and so $(e_z, (e_{x \land z}, e_{y \land z})) = (e_z e_y, e_z e_z) \in \tau$. Hence $(x \land z, y \land z) \in \rho$ and $\rho$ is an s-congruence.

Now suppose that $\rho$ is an s-congruence, that $\rho$ induces the normal equivalence $\tau$ and $\tau$, in turn, induce $\rho'$. Let $(x, y) \in \rho$. Then, by Lemma 3.9, $(e_x, e_y) \in \tau$. Hence, for $e \in V(x)$, $e \geq e_x$, $(e_x, e_y) \in \tau$ and $e_y \in V(y)$. Thus $e \in V_\tau(x)$ and $V(x) \subseteq V_\tau(y)$. Similarly, $V(y) \subseteq V_\tau(x)$ and so $V_\tau(x) = V_\tau(y)$ and $(x, y) \in \rho'$. Thus $\rho \subseteq \rho'$.

Conversely, let $(x, y) \in \rho'$. Then $V_\tau(x) = V_\tau(y)$. Hence $e_x \in V_\tau(y)$ and $e_y \in V_\tau(x)$. Thus there exist $e_1, e_2, f_1, f_2 \in E_s$ such that

\begin{equation}
(3.1)\quad e_x \geq e_1, (e_1, e_2) \in \tau \quad \text{and} \quad e_2 \geq e_y
\end{equation}

and

\begin{equation}
(3.2)\quad e_y \geq f_1, (f_1, f_2) \in \tau \quad \text{and} \quad f_2 \geq e_x.
\end{equation}

Therefore

$$
e_x \geq e_1 e_y, (e_1 e_y, e_y) = (e_1 e_y, e_2 e_y) \in \tau,
$$

and

$$
e_y \geq f_1 e_x, (f_1 e_x, e_x) = (f_1 e_x, f_2 e_x) \in \tau.
$$

Hence

$$
(e_1 e_y, e_2 e_y) = (e_1 e_y, e_x e_y) \in \tau
$$

and
Thus \((e, f, e_x e_y) \in \tau\) and \((e_x, e_y) \in \tau\). Hence, by Lemma 3.9, \((x, y) \in \rho'\) and \(\rho' \subseteq \rho\). Thus \(\rho = \rho'\).

Let the \(s\)-congruences \(\rho\) and \(\rho'\) induce the normal equivalences \(\tau\) and \(\tau'\). If \(\rho \subseteq \rho'\) then \(\tau \subseteq \tau'\), by Corollary 3.8. Let \(\tau \subseteq \tau'\). Since, by the above \(\tau\) and \(\tau'\) induce, in turn, \(\rho\) and \(\rho'\) it follows from Theorem 2.2 that \(\rho \subseteq \rho'\). Hence \(\beta\) is an order isomorphism of \(\Gamma_s(X)\) into \(\Theta(S)\).

4. The case \(\delta(e) \neq \emptyset\). Throughout this section we assume that \(X\) is a semilattice, that \(S \subseteq J_x\) and that \(\delta(e) \neq \emptyset\) for all \(e \in E_s\). The representations of Propositions 3.2, 3.3, 3.4 and 3.6 all satisfy this condition. However, for the main result of this section we shall require further hypotheses.

**Lemma 4.1.** Let \(X\) be a semilattice, \(S \subseteq J_x\) and \(\delta(e) \neq \emptyset\), for all \(e \in E_s\). Let \(\tau\) be a normal equivalence on \(E_s\) and suppose that \(\tau\) induces an \(s\)-congruence \(\rho\) on \(X\). Let \(\rho\), in turn, induce the normal equivalence \(\tau'\) on \(E_s\). Then \(\tau' \subseteq \tau\).

**Proof.** Let \((e, f) \in \tau'\). Then \(U(e) = U(f)\). Let \(x \in \delta(e)\). Then \(x\rho \cap A(f) \neq \emptyset\) and so there exists a \(y \in x\rho\) such that \(y \in A(f)\) or \(f \in V(y)\). Thus \(f \in V(y) \subseteq V(x) = V_\tau(x)\) and so there exist \(f_1, f_2 \in E_s\) such that

\[(4.1) \quad f \geq f_1, (f_1, f_2) \in \tau \quad \text{and} \quad f_2 \geq e,
\]

since \(f_2 \in V(x)\) if and only if \(f_2 \geq e\). Similarly, there exist \(e_1, e_2 \in E_s\) such that

\[(4.2) \quad e \geq e_1, (e_1, e_2) \in \tau \quad \text{and} \quad e_2 \geq f.
\]

Now (4.1) and (4.2) are just the statements (3.1) and (3.2) with \(e\) and \(f\) replacing \(e_x\) and \(e_y\). Hence, as in Theorem 3.10, we can deduce that \((e, f) \in \tau\).

In the absence of the assumption that \(\delta(e) \neq \emptyset\), for all \(e \in E_s\), Lemma 4.1 need not hold.

**Example.** Let \(I = [0, 1]\), the interval of real numbers from 0 to 1 under the natural ordering. Let \(I'\) denote the half open interval \([0, 1)\). Let \(S\) be the subsemigroup \(\{e_i; i \in I\}\) of idempotents of \(J_{I'}\), where

\[
\Delta(e_i) = \begin{cases} 
\{r \in I; r \leq i\} & \text{if } i \neq 1, \\
\{r \in I; r < 1\} & \text{if } i = 1.
\end{cases}
\]

Let \(\tau\) be the normal equivalence on \(S = E_s\) determined by the parti-
tion $S = \{e; i < 1\} \cup \{e_i\}$ of $S$. Then $\tau$ induces the $s$-congruence $\rho = I' \times I'$ on $I'$ and $\rho$, in turn, induces the normal equivalence $\tau' = S \times S$ on $S$. Thus $\tau \subset \tau'$.

Even in the presence of the assumption that $\delta(e) \neq \emptyset$, for all $e \in E$, we may not have $\tau = \tau'$.

**Example.** Let $X$ be the semilattice of Figure 2.

Let $S$ be the subsemigroup of $J_x$ consisting of the idempotents $f, g, h$ where $A(f) = \{u, v, w, x\}, A(g) = \{v, w\}, A(h) = \{w\}$. If $\tau$ is the normal equivalence partitioning $S$ as $S = \{f, g\} \cup \{h\}$ then $\rho_\tau$ has classes $\{^+, v\}, \{w\}, \{x\}$ and $\rho_\tau$ is an $s$-congruence.

However, if $\rho_\tau$ induces the normal equivalence $\tau'$ then $\tau'$ is the identity equivalence and so $\tau' \subset \tau$.

**Theorem 4.2.** Let $X$ be a semilattice, $S$ be an inverse subsemigroup of $J_x$ and $\delta(e) \neq \emptyset$, for all $e \in S$. Let a normal equivalence $\tau$ on $E_S$ induce an $s'$-congruence $\rho$ on $X$. Let $\rho$, in turn, induce the normal equivalence $\tau'$ on $E_S$. If any of the following conditions hold then $\tau = \tau'$:

1. $X$ is totally ordered;
2. $\rho$ is an $s'$-congruence and $X = \bigcup_{e \in E_S} \delta(e)$; in particular, if $S$ is full in $T_x$;
3. $\rho$ is an $s$-congruence and $S \subseteq T_x$.

**Note.** If $X$ is totally ordered or, by Theorem 3.10, if $S$ is full in $T_x$, then every normal equivalence induces an $s$-congruence.

**Proof.** We have from Lemma 4.1, that $\tau' \subseteq \tau$ in each case.

1. Let $(e, f) \in \tau$ and suppose that $x \rho \cap A(e) \neq \emptyset$. Without loss of generality let $x \in A(e)$. Since $X$ is totally ordered so also must $E_S$ be totally ordered. If $f \geq e$ then $A(f) \supseteq A(e)$ and $x \rho \cap A(f) \neq \emptyset$. So suppose that $f < e$ and that $y \in \delta(f)$. If $y \geq x$ then $x \in A(f)$ and again $x \rho \cap A(f) \neq \emptyset$. Suppose that $x > y$. Then $V(x) \subseteq V(y)$ and so $V_\tau(x) \subseteq V_\tau(y)$. Now let $g \in V(y)$. Then $g \geq f$, $(f, e) \in \tau$ and $e \in V(x)$. Hence $g \in V_\tau(x)$. Thus $V(y) \subseteq V_\tau(x)$, $V_\tau(y) = V_\tau(x)$ and $(x, y) \in \rho$. Thus we again have $x \rho \cap A(f) \neq \emptyset$. Thus $U(e) \subseteq U(f)$ and conversely, by similarity. Thus $(e, f) \in \tau'$ and so $\tau = \tau'$.

2. Let $(e, f) \in \tau$ and $x \rho \cap A(e) \neq \emptyset$. Let $x \in A(e)$ and $x \in \delta(k)$. Then $k \leq e$ and $(k, kf) = (ke, kf) \in \tau$. Let $y \in \delta(kf)$. Then, by Proposition 2.3, $(x, y) \in \rho$ and $y \in A(kf) \subseteq A(f)$. Thus $U(e) \subseteq U(f)$ and conversely, by similarity. Hence $(e, f) \in \tau'$ and $\tau = \tau'$.

3. Let $(e, f) \in \tau$. Let $A(e) = \langle x_e \rangle$ and $A(f) = \langle x_f \rangle$. By
Proposition 2.3, \((x_e, x_f) \in \rho\). Let \(x_\rho \cap \Delta(e) \neq \emptyset\) and suppose that \(x \in \Delta(e)\). Then \(x \leq x_e\) and \((x, x \land x_f) = (x \land x_e, x \land x_f) \in \rho\), since \(\rho\) is an \(s\)-congruence. Also \(x \land x_f \in \Delta(f)\) and so \(x_\rho \cap \Delta(f) \neq \emptyset\). Hence \(U(e) \subseteq U(f)\) and conversely. Thus \((e, f) \in \tau'\) and \(\tau = \tau'\).

5. Inducing congruences on \(S\). Let \(X\) be a semilattice, \(S \subseteq J_X\) and \(\rho\) be an \(s'\)-congruence on \(X\). We have seen that \(\rho\) induces a normal equivalence on \(E_s\) and in this section we show how to define two congruence relations on \(S\) in the corresponding \(\theta\)-class directly. In certain circumstances these will be the smallest and largest congruences in that \(\theta\)-classes.

**Proposition 5.1.** Let \(X\) be a semilattice, \(S\) be an inverse subsemigroup of \(J_X\) and let \(\rho\) be an \(s'\)-congruence on \(X\). Define the relation \(\xi = \xi_\rho\) on \(S\) by

\[
(a, b) \in \xi : (i) \quad U(a) = U(b); \\
(ii) \quad x \in \Delta(a), y \in \Delta(b) and (x, y) \in \rho \\
\text{implies that } (xa, yb) \in \rho.
\]

Then \(\xi\) is a congruence on \(S\), in fact, the congruence induced on \(S\) by the homomorphism \(\alpha\) of Theorem 3.7. If \(\rho\) is induced by some normal equivalence \(\sigma\) on \(E_s\), as in Theorem 2.2, if \(\tau = \xi|_{E_s}\) and \(\delta(e) \neq \emptyset\), for all \(e \in E_s\), then \(\xi = \mu\), the maximum congruence in the \(\theta\)-class containing \(\xi\).

**Proof.** Since \(\xi\) is just the congruence on \(S\) induced by the homomorphism \(\alpha\) of Theorem 3.7, the first part of the theorem requires no verification.

For the final assertion, since we must have \(\xi \subseteq \mu\), it suffices to show that \(\mu, \subseteq \xi\).

Let \((a, b) \in \mu\). Then \((aa^{-1}, bb^{-1}) \in \tau\), while \(\Delta(a) = \Delta(aa^{-1})\) and \(\Delta(b) = \Delta(bb^{-1})\). Hence, by the definition of \(\tau\), \(a\) and \(b\) satisfy condition (i).

Now let \((x, y) \in \rho, x \in \Delta(a)\) and \(y \in \Delta(b)\). We want \((xa, yb) \in \rho\). Since \(\rho\) is induced from \(\sigma\) we wish to show that \(V_\sigma(xa) = V_\sigma(yb)\).

Let \(e \in V(xa)\). Then \(xa \in \Delta(e)\) and \(x \in \Delta(axa^{-1})\). Hence \(aea^{-1} \in V(x) \subseteq V_\sigma(y)\) and so, for some \(f_1, f_2 \in E_s\), we have

\[
\text{aea^{-1} \geq f_1, (f_1, f_2) \in \sigma } \quad \text{and} \quad f_2 \in V(y)
\]

Hence \(yb = yf_2b \in \Delta(b^{-1}f_2b)\), where \((b^{-1}f_2b, b^{-1}f_2b) \in \sigma\), since \(\sigma\) is a normal equivalence. Also \((b^{-1}f_2b, a^{-1}f_2a) \in \tau\), by Lemma 1.2, since \((a, b) \in \mu\).

But, by Lemma 4.1, \(\tau \subseteq \sigma\). Hence \((a^{-1}f_2a, b^{-1}f_2b) \in \sigma\) and

\[
e \geq a^{-1}aea^{-1}a \geq a^{-1}f_2a, (a^{-1}f_2a, b^{-1}f_2b) \in \sigma \quad \text{and} \quad b^{-1}f_2b \in V(yb)
\]

Thus \(e \in V_\sigma(yb)\) and \(V_\sigma(xa) \subseteq V_\sigma(yb)\). By similarity, we have equality and so \((xa, yb) \in \rho\), as required. Hence \((a, b) \in \xi, \mu, \subseteq \xi\) and so \(\mu, = \xi\).
PROPOSITION 5.2. Let $X$ be a semilattice and $S$ be an inverse sub-
semigroup $J_x$. Let $\rho$ be an $s'$-congruence on $X$. Define the relation
$\eta$ on $S$ by

$$(a, b) \in \eta \iff$$
(i) $U(a) = U(b)$
(ii) If $x \rho \in (a) = (b)$ then there exists a $y \in x \rho$ such
that $y \in \Delta(a) \cap \Delta(b)$ and $za = zb$, for all $z \leq y, z \in X$.

Then $\eta$ is a congruence on $S$. If $\eta|_{E_S} = \tau$ and either of the following
two conditions holds then $\eta = \sigma$, the minimum congruence in the $\theta$-
class containing $\eta$:

1. $S \supseteq E_{J_x}$;
2. $\rho$ is an $s$-congruence and $S$ is full in $T_x$.

Proof. Let $(a, b) \in \eta$. We first show that $(a, b) \in \xi$, where $\xi$ is as
in Proposition 5.1. Then, for any $c \in S$, we shall have $(ac, bc) \in \xi$ and so, since $\xi$ is a congruence, we shall have $U(ac) = U(bc)$ and $U(ca) = U(ca) = U(cb)$.

Since the conditions (i) are identical, we need only verify that $a$ and $b$ satisfy condition (ii) in Proposition 5.1. Let $x \in \Delta(a), y \in \Delta(b)$ and $(x, y) \in \rho$. Then there exists a $y_i$ such that $(x, y_i) \in \rho$ and $za = zb$, for all $z \leq y_i$. Hence $y.a = y_i.b$, $(xa, y_i.a) \in \rho$, $(yb, y_i.b) \in \rho$ and so $(xa, yb) \in \rho$. Thus $(a, b) \in \xi$, $U(ac) = U(bc)$ and $U(ca) = U(cb)$.

Now let $x \rho \in U(ac) = U(bc)$. Then $x \rho \cap \Delta(a) \neq \emptyset$ and $x \rho \cap \Delta(b) \neq \emptyset$. Hence there is a $y_i \in x \rho$ such that $za = zb$ for all $z \leq y_i$. Let

$y \in x \rho \cap \Delta(ac), y \in x \rho \cap \Delta(bc)$ and $y = y_1 \cap y_2 \cap y_3$.

Then $y \in x \rho \cap \Delta(ac) \cap \Delta(bc)$ and for all $z \leq y, zac = zbc$. Thus

$(ac, bc) \in \eta$.

The proof that $(ca, cb) \in \eta$ is similar and so $\eta$ is a congruence.

To show that $\eta = \sigma$, we need, by Lemma 1.2, to show that, for
any $(a, b) \in \eta$,

1. $(aa^{-1}, bb^{-1}) \in \tau$;
2. there exists an $e \in E_s$ such that $(e, aa^{-1}) \in \tau$ and $ea = eb$.

The first requirement is satisfied since $\eta$ is a congruence and $\eta|_{E_S} = \tau$.

Now suppose that $S \supseteq E_{J_x}$. Let $U(a) = U(b) = \{x_i \rho: i \in I\}$. For
each $i \in I$, let $y_i \in x_i \rho$ be such that $za = zb$, for all $z \leq y_i$. Let $e$ be
the idempotent $S$ with domain $\bigcup_{i \in I} <y_i>$. Then clearly, by the de-
definition of $e$, $U(aa^{-1}) = U(a) \subseteq U(e)$. On the other hand, we clearly
have $e \leq aa^{-1}$ and so $U(e) \subseteq U(aa^{-1})$. Thus $U(e) = U(aa^{-1})$ and $(e, aa^{-1}) \in \tau$. Also $ea = eb$ and so $(a, b) \in \sigma$. Thus $\eta = \sigma$.

Finally suppose that $\rho$ is an $s'$-congruence and that $S \subseteq T_x$. Let

$aa^{-1} = e_x$ and $bb^{-1} = e_y$. Since $(e_x, e_y) \in \tau$, by Lemma 3.9, $(x, y) \in \rho$ and so there exists a $z$ such that $(x, z) \in \rho$ and $z_a = z_b$ for all $z \leq z$.

Then, again by Lemma 3.9, $(e_x, e_y) \in \tau$ while clearly $e_x a = e_y b$. Thus
(α, b) ∈ σ, and η = σ.*

**Corollary 5.3.** Let S be a full inverse subsemigroup of \( T_x \). Let \( \tau \) be a normal equivalence on \( E_S \) and let \( \tau \) induce the s-congruence \( \rho \) on \( X \). Then the congruences \( \xi \) and \( \eta \) of Propositions 5.1 and 5.2 are respectively \( \mu \), the maximum congruence, and \( \sigma_\rho \), the minimum congruence in the \( \theta \)-class determined by \( \tau \).

**Proof.** That \( \tau \) induces an s-congruence \( \rho \) and that \( \rho \), in turn induces \( \tau \) follows from Proposition 3.10. The result then follows from Propositions 5.1 and 5.2.

**6. \( \Theta(S) \) and \( \Gamma_2(X) \).** By a lattice (semilattice) homomorphism \( \alpha \) of a lattice (semilattice) \( A \) into a lattice (semilattice) \( B \) we mean a mapping \( \alpha \) of \( A \) into \( B \) such that \( (x \land y)\alpha = x\alpha \land y\alpha \) and \( (x \lor y)\alpha = x\alpha \lor y\alpha((x \land y)\alpha = x\alpha \land y\alpha) \) for all \( x, y \in A \). A lattice (semilattice) isomorphism is then a one-to-one lattice (semilattice) homomorphism.

In the next two theorems we essentially summarize some of the previous results.

**Theorem 6.1.** Let \( X \) be a semilattice. If \( X \) is a full inverse subsemigroup of \( J_x \), then the mapping \( \alpha: \tau \mapsto \rho_\tau \), of Theorem 2.2, from \( \Theta(S) \) into \( \Gamma(X) \) is a semilattice homomorphism onto \( \Gamma_2(X) \).

If \( S \) is a full inverse subsemigroup of \( T_x \) then \( \alpha \) is a lattice isomorphism of \( \Theta(S) \) onto \( \Gamma_2(X) \).

If \( X \) is totally ordered and \( \delta(e) \neq \emptyset \), for all \( e \in E_S \), then \( \alpha \) is an order isomorphism of \( \Theta(S) \) into \( \Gamma_2(X) \).

**Proof.** That \( \alpha \) maps \( \Theta(S) \) onto \( \Gamma_2(X) \), when \( S \) is full in \( J_x \), follows from Theorem 3.10. Let \( \tau_1 \) and \( \tau_2 \) be normal equivalences, let \( \tau_3 = \tau_1 \cap \tau_2 \) and \( \rho_i = (\tau_i)\alpha \), \( i = 1, 2, 3 \). Then from Theorem 2.2, \( \rho_3 \subseteq \rho_1 \cap \rho_2 \). Let \( (x, y) \in \rho_1 \cap \rho_2 \). Then by Proposition 2.3, \( (e_2, e_3) \in \tau_1 \cap \tau_2 = \tau_3 \). Hence, again by Proposition 2.3, \( (x, y) \in \rho_3 \). Thus \( \rho_3 = \rho_1 \cap \rho_2 \) and \( \alpha \) is a semilattice homomorphism.

If \( S \) is full in \( T_x \), then by Proposition 3.10, \( \alpha \) is a one-to-one semilattice homomorphism of \( \Theta(S) \) onto \( \Gamma_2(X) \) and hence is a lattice isomorphism.

If \( X \) is totally ordered, then every c-congruence is an s-congruence and so, by Proposition 2.3, \( \alpha \) is an \( o \)-isomorphism of \( \Theta(S) \) into \( \Gamma_2(X) \).

**Theorem 6.2.** Let \( X \) be a semilattice and \( S \) be an inverse subsemigroup of \( J_x \). Let \( \beta \) denote the mapping \( \rho \rightarrow \tau_\rho \) of Corollary 3.8.

If \( S \) is full in \( J_x \) then \( \beta \) is an \( o \)-isomorphism of \( \Gamma_2(X) \) into \( \Theta(S) \).

If \( S \) is full in \( T_x \) then \( \beta = \alpha^{-1} \), where \( \alpha \) is defined as in Theorem
6.1.

If \( X \) is totally ordered and \( \delta(e) \neq \emptyset \), for all \( e \in E_S \), then \( \beta \) is an order preserving mapping of \( \Gamma_2(X) \) onto \( \Theta(S) \).

**Proof.** If \( S \) is full in \( J_X \) then, from Theorem 3.10, \( \beta \) is an order isomorphism of \( \Gamma_2(X) \) into \( \Theta(S) \).

If \( S \) is full in \( T_X \) then, from Theorem 3.10, \( \beta \alpha = \iota_{\Gamma_2(X)} \) and, from Theorem 4.2, \( \alpha \beta = \iota_{\Theta(S)} \).

Hence \( \beta = \alpha^{-1} \).

Finally, if \( X \) is totally ordered and \( \delta(e) \neq \emptyset \), for all \( e \in E_S \), then \( \beta \) is order preserving, by Corollary 3.8, and \( \beta \) maps \( \Gamma_2(S) \) onto \( \Theta(S) \) by Theorem 4.2.

If \( S \) is a full inverse subsemigroup of \( J_X \), it is natural to ask to what extent the properties of \( S \) are determined by those of \( S \cap T_X \). We shall denote by \( ST_2(X) \) the lattice of \( s \)-congruences under \( S \) to distinguish it from the lattice of \( s \)-congruences \( T_2(X) \) under some other semigroup \( T \).

**Proposition 6.3.** Let \( X \) be a semilattice and \( S \) be a full inverse subsemigroup \( J_X \). Let \( T = S \cap T_X \). Then \( ST_2(X) = T_2(X) \).

**Proof.** Clearly \( ST_2(X) \subseteq T_2(X) \). Let \( \rho \in T_2(X) \), \( (x, y) \in \rho \), \( x, y \in A(a) \), for some \( x, y \in X \), \( a \in S \). Let \( e_a \) denote the idempotent of \( T \) with domain \( <x> \). Since \( \rho \in T_2(X) \), we have \( (x, x \wedge y) \in \rho \) and \( x, x \wedge y \in A(a) \). Also \( x, x \wedge y \in A(e_a) \). Hence \( x, x \wedge y \in A(e_a) \) and \( e_a x \in T \). Hence \( (x e_a, (x \wedge y)e_a) \in \rho \), that is, \( (x a, (x \wedge y)a) \in \rho \). Similarly \( (y a, (x \wedge y)a) \in \rho \). Thus \( \rho \in ST_2(X) \) and we have the result.

**Corollary 6.4.** Under the hypothesis of Proposition 6.3, there exists a semilattice homomorphism of \( \Theta(S) \) onto \( \Theta(T) \).

**Proof.** The result follows from Theorem 6.1 and Proposition 6.3.

**Remark.** Let \( S \) be an inverse semigroup and \( \mu \) be the maximum idempotent separating congruence on \( S \). Since \( \Theta(S) = \Theta(S/\mu) \) and since, by Proposition 3.2, \( S/\mu \) is isomorphic to a full inverse subsemigroup of \( T_{E_S} \) one might question the need to study other kinds of inverse subsemigroups of \( J_X \) apart from those that are full subsemigroups of \( T_X \). (If \( S \) is a full inverse subsemigroup of \( T_X \) then it is not difficult to see that the representation of \( S \) as a semigroup of partial transformations of \( X \) is isomorphic in a natural way to the representation of \( S \) given by Proposition 3.2. on \( E_S \).) However, this assumes a prior knowledge of the semigroup sufficient to identify the representation of \( S \) on \( E_S \). If the semigroup is known as a semi-
group of partial transformations, it may be quite difficult to identify the representation on $E_s$ while it might be relatively simple to work with the semigroup of partial transformations as given.

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