A COMBINATORIAL PROBLEM; STABILITY AND ORDER FOR MODELS AND THEORIES IN INFINITARY LANGUAGES

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Some infinite combinatorial problems of Erdös and Makkai are solved, and we use them to investigate the connection between unstability and the existence of ordered sets; we also prove the existence of indiscernible sets under suitable conditions.

0. Introduction. In § 1 we deal with combinatorial problems raised by Erdös and Makkai in [5] (they appear later in Erdös and Hajnal [3], [18] Problem 71).

Let us define: $P_2(\lambda, \mu, \alpha)$ holds when for every set $A$ of cardinality $\mu$, and family $S$ of subsets of $A$ of cardinality $\lambda$, there are $a_k \in A$, $X_k \in S$ for $k < \alpha$, such that either $k, l < \alpha$ implies $a_k \in X_l$ or $k, l < \alpha$ implies $a_k \in X_l$ is a $\lambda$.

Erdös and Makkai proved in [5] that if $\lambda > \mu + 2$ then $P_2(\lambda, \mu, \omega)$ holds. Assuming G.C.H. for simplicity only, our theorems imply $P_2(\aleph_\beta, \beta^+, \alpha)$ holds for every $\beta$.

In § 2 we mainly generalize results on stability from Morley [9] and Shelah [12] to models, and theories of infinitary languages. We first deal with stable models. Let $M$ be a model, $L$ the first-order language associated with it, $\Delta$ a set of formulas of $L_{\geq \lambda}^+, \omega$ (for any $\lambda$) each with finite number of free variables. We shall assume $\Delta$ is closed under some simple operations. $M$ is $(\Delta, \lambda)$-stable, if for each $A \subset |M|$, $|A| \leq \lambda$, the elements of $M$ realize over $A$ no more than $\lambda$ different $\Delta$-types. Let $\lambda \in Od_\lambda(M)$ if there is $\varphi(\bar{x}, \bar{y}) \in \Delta$ and sequences $\bar{a}^k, k < \lambda$, of elements of $M$ such that for every $k, l < \lambda$, $M \not= \varphi[\bar{a}^k, \bar{a}^l]$ if and only if $k < l$.

By Theorem 2.1, if $M$ is not $(\Delta, \kappa)$-stable $\kappa^{(|\Delta|)} = \kappa = \sum_{n<\kappa} (\kappa^n + 2^{\kappa^n})$, then $\lambda \in Od_\lambda(M)$. Theorem 2.2 says that if $M$ is $(\Delta, \lambda)$-stable, $\lambda \in Od_\lambda(M)$, $|M| > \lambda$, $A \subset |M|$, $|A| \leq \lambda$, and the cofinality of $\lambda$ is $> |\Delta|$, then in $M$ there is an indiscernible set over $A$ of cardinality $> \lambda$. This generalizes Theorem 4.6 of Morley [9] for models of totally transcendental theories.

A theory $T$, $T \subset L_{\geq \lambda}^+, \omega$ for some $\lambda$, is $(\Delta, \mu)$-stable, if every model of $T$ is $(\Delta, \mu)$-stable. By Theorem 2.4, if $T$, $\Delta \subset L_{\geq \lambda}^+, \omega$ $|T| \leq \lambda$, and $\mu(\lambda) \in Od_\lambda(M)$ for some model $M$ of $T$, then for every $\kappa$, $T$ is not $(\Delta, \kappa)$-stable. This is a converse of Theorem 2.1. (Morley [9] proved a particular case of this theorem (3.9) that if $T$ is a first-order, counta-
ble, complete, totally transcendental theory, (i.e., $T$ is $(\Delta, \gamma^0)$-stable, where $\Delta$ is the set of all formulas of $L$), then $\mathfrak{S}_\alpha \in \mathcal{O}_\delta(M)$ for any model $M$ of $T$. (In fact he used a little stronger definition for $\mathfrak{S}_\alpha \in \mathcal{O}_\delta(M)$.)

By Theorem 2.5, if $T \subseteq L_{\lambda^+, \omega}$, and $\Delta$ is arbitrary, and for every $\kappa$, $T$ is not $(\Delta, \kappa)$-stable, then for some $\Delta_i \subseteq L_{\lambda^+, \omega}$, $| \Delta_i | \leq \lambda$, $T$ is $(\Delta_i, \kappa)$-unstable for every $\kappa$. By Shelah [16], we deduce that for every $\kappa > |T| + \lambda$, $T$ has $2^\kappa$ nonisomorphic models of cardinality $\kappa$.

**NOTATIONS.** Let $\lambda$, $\kappa$, $\mu$, $\chi$ denote cardinals (infinite, if not clear otherwise). Let $\alpha, \beta, \gamma, i, j, k, l$ denote ordinals and $m, n$ denote natural numbers. We shall indentify cardinals with initial ordinals, and $\gamma_\alpha$ will be the $\alpha$th infinite cardinal ($\gamma_0$-the first). The first infinite ordinal is denoted by $\omega$. $\lambda^+$ is the first cardinal greater than $\lambda$. $|A|$ is the cardinality of the set $A$.

1. Combinatorial problems. Let $A$ denote a set, $S$ a family of subsets of $A$. Let $A (-) S$ be the family $\{A - B: B \in S\}$. $A^\alpha$ is the set of sequences of length $\alpha$ of $A$; and if $\bar{a} \in A^\alpha$, $l(\bar{a}) = \alpha$ and $\bar{a}_\beta$ is the $\beta$th element in the sequence. After Erdős and Makkai [5], $\bar{a}$ if strongly cut by $S$ if for every $\beta < \alpha$, there is $X_\beta \subseteq S$ such that $\bar{a}_\gamma \in X_\beta$ if and only if $\gamma < \beta$. Erdős and Makkai [5] proved that if $|S| > |A| \geq \mathfrak{K}_\omega$, then there is a sequence $\bar{a} \in A^\omega$ which is strongly cut by $S$ or by $A (-) S$. They asked several questions ([5] p. 159 and [3] problem 71 p. 45). We shall here answer some of their questions.

Let us define

**DEFINITION 1.1.** $P_1(\lambda, \mu, \alpha)$ holds, if $|S| = \lambda$, $|A| = \mu$ implies there are $\bar{a}, \bar{b} \in A^\alpha$, $\bar{X} \in S^\alpha$ such that: for every $\beta, \gamma < \alpha$,

$$\bar{a}_\beta \in \bar{X}_\gamma \iff \bar{b}_\beta \in \bar{X}_\gamma$$

if and only if $\gamma < \beta$.

**DEFINITION 1.2.** $P_2(\lambda, \mu, \alpha)$ holds, if $|S| = \lambda$, $|A| = \mu$ implies there are $\bar{a} \in A^\alpha$, $\bar{X} \in S^\alpha$ such that:

either $\beta, \gamma < \alpha$ implies $\bar{a}_\beta \in \bar{X}_\gamma \iff \beta < \gamma$

or

$$\beta, \gamma < \alpha \text{ implies } \bar{a}_\beta \in \bar{X}_\gamma \iff \gamma \leq \beta.$$  

**REMARK.** This means that $\bar{a}$ is strongly cut by $S$ or by $A (-) S$.

**DEFINITION 1.3.** $P_3(\lambda, \mu, \alpha)$ holds if $|S| = \lambda$, $|A| = \mu$ implies
there are $\bar{a} \in A^a$, $\bar{X} \in S^a$ such that for every $\beta, \gamma < \alpha$, $\bar{a}_{\beta} \subseteq \bar{X} \implies \beta < \gamma$.

**Remark.** This means $\bar{a}$ is strongly cut by $S$.

**Notation.** In each of $P_1, P_2, P_3$ we shall always implicitly assume $2^\mu > \lambda > \mu$. For otherwise, those relations are not interesting.

Clearly, the theorem of [5] is by our notation, that $P_2(\lambda^+, \lambda, \omega)$ holds. Let us now list the results proved here about those three properties.

**Theorem 1.1.** For every $\lambda$, $P_3(\lambda^+, \lambda, \omega)$ does not hold. (This solves negatively problem 1 in [5], which is the same as problem 71A, in [3] p. 45.) (In fact, we prove a stronger result.)

**Theorem 1.2.** If $\lambda > \sum_{\delta < \varepsilon} (\mu^\varepsilon + 2^{\varepsilon})$ then $P_1(\lambda, \mu, \varepsilon)$ holds.

**Theorem 1.3.** If $\lambda > \mu^{\varepsilon}$ then $P_2(\lambda, \mu, \varepsilon^+)$ holds. Moreover if $\varepsilon = \sum_{\delta < \varepsilon} 2^\varepsilon$, $\lambda > \mu^\varepsilon$ then $P_2(\lambda, \mu, \varepsilon)$ holds.

**Theorem 1.4.** If $P_1(\lambda, \mu, \varepsilon)$ and $\varepsilon \rightarrow (\kappa)^\varepsilon_1$ holds, then $P_2(\lambda, \mu, \varepsilon)$ holds.

**Remark.** (1) $\varepsilon \rightarrow (\kappa)^\varepsilon_1$ is defined in Erdős, Hajnal and Rado [4]. As the proof is straightforward, we leave it to the reader.

(2) We can combine theorems 1.2 and 1.4 to get results about $P_2(\lambda, \mu, \alpha)$. For example by Ramsey [11], $\aleph_1 \rightarrow (\aleph_0)^\varepsilon_1$, hence $P_2(\lambda, \mu, \omega)$ holds (which is the result of [5]). (Here, as usual, we implicitly assume $\lambda > \mu \geq \aleph_0$.)

(3) Theorems 1.2, 1.3, 1.4 give partial answer to a question which naturally arises from [5], and problem 2, [5], and 71B [3] are the most simple cases of it.

**Theorem 1.5.** $P_2(\lambda, \mu, \omega + 1)$ holds. Moreover, if $\lambda > \mu = \mu^\varepsilon$, $\varepsilon < \omega$, then $P_2(\lambda, \mu, \omega + \varepsilon)$ holds.

**Remark.** This answers problem 3 of [5] (in fact even stronger) and partially answer problem 2 of [5] (= 71B of [3]). The proof gives several more results of this kind.

To clarify our results let us assume G.C.H.

**Corollary 1.6.** (G.C.H.) For every regular cardinality $\mu$, and any cardinal $\chi < \mu$, $P_2(\mu^+, \mu, \chi)$ holds. Moreover, if $\mu$ is singular, $\chi$ is less than the cofinality of $\mu$, then $P_2(\mu^+, \mu, \chi)$ holds. If $\chi$ is
not greater than the cofinality of $\mu$, $P_1(\mu^+, \mu, \chi)$ holds.

**Proof.** Immediate from Theorems 1.2, 1.3, 1.4, and by [4], $(2^\chi)^+ \rightarrow (\lambda^+)^\chi$ holds.

The question naturally arises whether those are the best possible results. Prikry essentially proved this. See [18] Problem 72.

**Theorem 1.7.** Suppose $\lambda = \mu^+ > \sum_{\beta < \chi} \mu^\beta = \mu_0$ then $P_2(\lambda, \mu_0, \chi + 2)$ does not holds. ($\chi + 2$—this is an ordinal addition). Moreover $P_1(\lambda, \mu_0, \chi + 2)$ does not holds.

In [5], not $P_2(\kappa, \kappa_0, \omega + 2)$ was proved; and as the proof is similar and straightforward we leave it to the reader.

The most simple open problems are: (for simplicity only we assume G.C.H.)

**Problem 1.** If $\kappa$ is regular, does $P_1(\kappa_\alpha, \kappa_\alpha, \kappa_\beta)$ hold? Does $P_2(\kappa_{\alpha + 1}, \kappa_\alpha, \kappa_\beta)$ hold?

**Problem 2.** If $\kappa$ is singular, $\kappa_\beta$ is the cofinality of $\kappa_\alpha$, does $P_2(\kappa_{\alpha + 1}, \kappa_\alpha, \kappa_\beta)$ hold?

Maybe the answers are independent of $ZF + AC$.

Let us summarize the trivial facts about our properties.

**Lemma 1.8.** (A) If $\lambda_1 \geq \lambda$, $\mu_1 \leq \mu$, $\alpha_i \leq \alpha$ and $P_1(\lambda_1, \mu_1, \alpha_i)$ hold, then $P_1(\lambda, \mu, \alpha_i)$ holds. The same is true for $P_2$ and $P_3$.

(B) $P_3(\lambda, \mu, \alpha)$ implies $P_2(\lambda, \mu, \alpha)$; $P_2(\lambda, \mu, \alpha)$ implies $P_1(\lambda, \mu, \alpha)$, where $\alpha$ is a limit ordinal; and $P_2(\lambda, \mu, \alpha + 1)$ implies $P_1(\lambda, \mu, \alpha)$.

(C) If $\alpha < \omega$, $\lambda > \mu$ then $P_3(\lambda, \mu, \alpha)$ holds.

(D) If $\mathrm{cf}(\lambda) \leq \mu < \lambda$, $(\forall \chi < \lambda) \rightarrow P_2(\chi, \mu, \alpha)$ then not $P_2(\lambda, \mu, \alpha)$.

**Proof.** Immediate. We use (D) for (B).

Let us now prove the theorems.

**Definition 1.4.** $\text{Ded}(\mu)$ is the first cardinal $\lambda$ such that there is no ordered set of cardinality $\lambda$ with a dense subset of cardinality $\mu$.

**Remark.** Clearly $\mu^+ < \text{Ded}(\mu) \leq (2^\mu)^+$. By Mitchell [8] it is consistent with $ZF + AC$ that $\text{Ded}(\kappa) < (2^\kappa)^+$.

**Theorem 1.9.** If $\mu < \lambda < \text{Ded}(\mu)$ then $P_3(\lambda, \mu, \omega)$ does not hold.

**Remark.** Clearly Theorem 1.1 is an immediate conclusion of this theorem.
Proof. Let a tree mean a pair of a set and a well ordering of the set, which is not necessarily a total ordering. A branch of a tree is a maximal ordered subset. It can be easily shown that there is a tree \( \langle A, \prec \rangle \) (\( A \)—the set, \( \prec \)—the ordering) such that \(|A| = \mu\) and the tree has \( \geq \lambda \) branches. Let \( S_i \) be the family of the branches of the tree and \( S = A (-) S_i \). Clearly \(|S| \geq \lambda\), \(|A| = \mu\) and \( S \) is a family of subsets of \( A \). So it suffices to show that there is no \( \bar{a} \in A^\omega \) which is strongly cut by \( S \).

So suppose \( \bar{a} \in A^\omega \) is strongly cut by \( S \). By using Ramsey theorem ([11]) we know there is an infinite subsequence of \( \bar{a}, \bar{b} \), such that exactly one of the following conditions is fulfilled
(1) for every \( n < m < \omega \), \( \bar{b}_n < \bar{b}_m \) (in the tree)
(2) for every \( n < m < \omega \), \( \bar{b}_m = \bar{b}_n \)
(3) for every \( n < m < \omega \), \( \bar{b}_n > \bar{b}_m \)
(4) for every \( n < m < \omega \), \( \bar{b}_n, \bar{b}_m \) are incomparable, i.e., \( \bar{b}_n \neq \bar{b}_m \), not \( \bar{b}_n > \bar{b}_m \), and not \( \bar{b}_n < \bar{b}_m \).

Now clearly also \( \bar{b} \) is strongly cut by \( S \). Hence (2) cannot be fulfilled. As \( \prec \) is a well ordering (3) cannot be fulfilled. Now as \( \bar{b} \) is strongly cut by \( S \), there is a branch of \( \langle A, \prec \rangle \) which contains two of the \( \bar{b}_n \)'s and so they are comparable, in contradiction to (4).

So (1) is fulfilled. As \( \bar{b} \) is strongly cut by \( S \), there is \( \bar{a} \in A^\omega \) such that, for every \( \alpha, \beta < \chi \), \( \bar{a} \in X_{\beta} \iff \bar{b} \in X_{\beta} \), hence \( \bar{b} \in A - X \), a contradiction.

Theorem 1.2. If \( \lambda > \sum_{\omega \leq \chi} (\mu^\omega + 2^\omega) \) then \( P1(\lambda, \mu, \chi) \) holds.

Proof. Let \( S \) be a family of subsets of \( A \), \(|S| = \lambda\), \(|A| = \mu\).

We should prove there are \( \bar{a}, \bar{b} \in A^\omega \) and \( \bar{X} \in S^\omega \) such that, for every \( \alpha, \beta \in \chi \), \( \bar{a} \in \bar{X}_\beta \iff \bar{b} \in \bar{X}_\beta \) if and only if \( \beta < \alpha \).

Let us define, for every \( T \subset S \), an equivalence relation \( E_T \) on \( A \): \( \bar{a}E_T \bar{b} \) holds if and only if for every \( X \in T \), \( a \in X \iff b \in X \). Clearly \( E_T \) is an equivalence relation, and the number of equivalence classes is \( \leq 2^{\| T \|} \).

Let us also define that \( T \subset S \) fixes \( X \in S \) if for every \( a, b \in A \), \( aE_T b \) implies \( a \in X \iff b \in X \). Clearly the number of \( X \in S \) which are fixed by \( T \) cannot be more than the number of subsets of the set of the \( E_T \)-equivalence classes. Hence \(|\{X \in S : X \text{ is fixed by } T\}| \leq 2^{\| T \|} \).

Let us now define by induction the families \( S_\kappa \), for \( 0 \leq \kappa < \chi \) such that:
(1) \( S_\kappa \subset S \), \(|S_\kappa| \leq \mu^\kappa \)
(2) \( \kappa_\alpha < \kappa_\beta \) implies \( S_{\kappa_\alpha} \subset S_{\kappa_\beta} \)
(3) if \( B, C \subset A \), \(|B| \leq \kappa, |C| \leq \kappa \), and there is \( X \in S \) such that \( B \subset X, C \cap X = 0 \), then there is \( Y \in S_\kappa \) such that \( B \subset Y, C \cap Y = 0 \).

Clearly we can define the \( S_\kappa \). We shall now prove that
(*) there is \( Y \in S \) such that for any \( T, T \subseteq S_\kappa, 0 \leq \kappa < \chi, |T| \leq \kappa \), \( Y \) is not fixed by \( T \).

Suppose (*) does not hold and we shall get a contradiction. So

\[
S = \bigcup_{0 \leq \kappa < \chi} \bigcup_{T \subseteq S_\kappa} \{X \setminus \{X \in S, X \text{ is fixed by } T\}.
\]

We have proved that \(|\{X \in S, X \text{ is fixed by } T\}| \leq 2^{(|T|)}\), and by its contraction \(|S_\kappa| \leq \mu^\kappa\). Hence

\[
\lambda = |S| \leq \sum_{0 \leq \kappa < \chi} \sum_{T \subseteq S_\kappa} 2^{(|T|)}
\]

\[
\leq \sum_{0 \leq \kappa < \chi} |S_\kappa|^\kappa \times 2^{2\kappa} = \sum_{0 \leq \kappa < \chi} (|S_\kappa|^\kappa + 2^{2\kappa})
\]

\[
\leq \sum_{0 \leq \kappa < \chi} (\mu^\kappa + 2^{2\kappa}) < \lambda
\]
a contradiction. So (*) holds.

Now we shall define by induction \( a_k, b_k, X_k \) for \( k < \chi \) such that:

\( A \) \( a_k \in A, b_k \in A, \) and \( X_k \in S_{|\kappa|+1} \)

\( B \) if \( l \leq k \) then \( a_l \in X_k, a_l \in Y, b_l \in X_k, \) and \( b_l \in Y \)

\( C \) if \( l < k \) then \( a_l \in X_k \) if and only if \( b_l \in X_k \).

Suppose \( a_l, b_l \), and \( X_l \) has been defined for every \( l < k \). Let \( 1+|k| = \kappa, \) and \( T = \{X_l; l < k\} \). Clearly \( T \subseteq S_\kappa, |T| \leq \kappa \). Hence, by the definition of \( Y \), it is not fixed by \( T \). So there are \( a_k, b_k \in A \) such that: \( a_k \in Y, b_k \in Y \) and \( a_k E_\kappa b_k \), i.e., for every \( l < k \), \( a_k \in X_k \) if and only if \( b_k \in X_k \). Clearly \( \{a_l; l \leq k\} \subseteq Y, \{a_l; l \leq k\} \cap Y = 0, |\{a_l; l \leq k\}| \leq \kappa, |\{b_l; l \leq k\}| \leq \kappa, \) hence by the definition of \( S_\kappa \) there is \( X_k \in S_\kappa \) such that

\[
\{a_l; l \leq k\} \subseteq X_k, \{b_l; l \leq k\} \cap X_k = 0.
\]

Clearly \( \langle a_k; k < |\chi| \rangle, \langle b_k; k < |\chi| \rangle, \) and \( \langle X_k; k < |\chi| \rangle \) are the required sequences, and so Theorem 1.2 is proved.

**Theorem 1.3.** If \( \chi^\alpha = \sum_{\alpha \leq \kappa} 2^\alpha, \lambda > \mu^\lambda \), then \( P2(\lambda, \mu, \chi) \) holds.

**Proof.** As the proof is very similar to the proof of Theorem 2, we shall only sketch it.

Suppose \( S \) is a family of subsets of \( A, |S| = \lambda, |A| = \mu \). It is easy to find \( S_\kappa \subseteq S, |S_\kappa| \leq \mu^0 \) such that:

\( 1 \) if \( B \subseteq A, |B| \leq 2^\kappa, 0 \leq \kappa < \chi, \) and \( T \subseteq S_\kappa, |T| \leq \kappa \) and \( Y \subseteq S \) then there is \( X \subseteq S_\kappa \) such that: \( A \cap B = Y \cap X \)

\( B \) if \( C \) is an \( E_\kappa \)-equivalence class then \( C \subseteq X \Rightarrow C \subseteq Y \) and \( C \cap X = 0 \Rightarrow C \cap Y = 0 \).

\( 2 \) if \( X^\kappa, \kappa < \beta < \chi, l < \chi^\kappa, Y^\kappa, \kappa < \beta < \chi, l < \chi^\kappa, \) and \( Z_l, l < \chi^\kappa \) are sets from \( S_\kappa \), and there is \( X \subseteq S \) such that: for every \( l < \chi^\kappa \)
\[ X \cap \bigcap_{k \in \alpha} X_k \cap \bigcap_{k \in \beta} (A - Y^k_t) = Z_t \cap \bigcap_{k \in \alpha} X_k \cap \bigcap_{k \in \beta} (A - Y^k_t) \]

then there is \( X \in S_t \), which satisfies this condition.

Now we can repeat a construction similar to that which appears in the proof of Theorem 1.

As Theorem 1.4 is trivial, it remains to prove only

**Theorem 1.5.**

(A) If \( \lambda > \mu \) then \( P_2(\lambda, \mu, \omega + 1) \) holds.

(B) If \( \lambda > \mu = \sum_{\alpha^p < \chi} \mu^p, \alpha \leq \chi \) and \( P_2(\lambda, \mu, \alpha) \) holds then \( P_2(\lambda, \mu, \alpha + 1) \) holds. Hence for every \( n \), if in addition \( \alpha < \chi \), \( P_2(\lambda, \mu, \alpha + n) \) holds. (By 1.8D we can assume \( \text{cf}(\lambda) > \mu \).

(C) If \( \lambda > \mu^{\aleph_0} \), then \( P_2(\lambda, \mu, \omega + n) \).

**Remark.**

(1) Clearly (A) cannot be improved by [5] \( P_2(\aleph_1, \aleph_0, \omega + 2) \) does not hold.

(2) Part of the proof is a generalization of a proof of A. Máté which appeared in [5].

**Proof.**

As the proof of (B) is obvious from the proof of A, we shall prove A only. (C follow from B).

So let \( S \) be a family of subsets of \( A, |S| = \lambda, |A| = \mu \).

First, there is \( a^0 \in A \) such that \( S_1 = \{X: X \in S, a^0 \in X\} \) is of cardinality > \( \mu \). Otherwise

\[
\lambda = |S| = \left| \bigcup_{a \in A} \{X: X \in S, a \in X\} \cup \{0\} \right| \\
\leq \sum_{a \in A} |\{X: X \in S, a \in X\}| + 1 = \mu \cdot \mu + 1 = \mu < \lambda
\]

a contradiction. Similarly there is \( a^1 \in A \) such that \( S_2 = \{X: X \in S, a^1 \in X\} \) is of cardinality > \( \mu \). Now at first we assume

(*) there is \( A^1 \subset A \), and \( S^1 \subset \{Y \cap A^1: Y \in S_2\} \) such that \( |S^1| > \mu \); and for every \( X \in S^1 \),

\[
|\{Y \cap X: Y \in S^1\}| \leq \mu.
\]

Then it can be easily seen that if \( X_1, \ldots, X_n \in S^1 \), \( X = X_1 \cup \cdots \cup X_n \), then

\[
|\{Y \cap X: Y \in S^1\}| \leq \mu.
\]

So we can easily find \( S^2 \subset S^1, |S^2| \leq \mu \) such that: if \( X_1, \ldots, X_n \in S^2 \), \( X \in S^2 \) and \( X \subset X_1 \cup \cdots \cup X_n \) then \( X \in S^2 \); and if \( a_0, \ldots, a_n \in A, X \in S^2 \), then there is \( Y \in S^2 \) such that \( \{a_0, \ldots, a_n\} \cap X = \{a_0, \ldots, a_n\} \cap Y \).

Now let \( Y^0 \in S^1 \), \( Y^0 \in S^2 \). (\( Y^0 \) exists as \( |S^1| > \mu \geq |S^2| \)). Now we shall define by induction on \( n, a_n, X_n \) such that: \( a_n \in Y^0, X_n \in S^2 \), and
Suppose $a_n, X_n$ has been defined for every $n < m < \omega$. As $Y^0 \in S^2$, $Y^0 \not\in X_0 \cup \cdots \cup X^m$, hence there is $a_m \in Y^0, a_m \in X_0 \cup \cdots \cup X^m$. Also there is $X_m \in S^2$ such that 
\[ \{a_0, \ldots, a_m\} \cap X_m = \{a_0, \ldots, a_m\} \cap Y^0. \n\]
Now clearly if we define $a_\mu = a'$, clearly $\langle a_\alpha \mid \alpha < \omega + 1 \rangle \in A^{\omega+1}$ and is strongly cut by $S$; so the conclusion of theorem holds.

Similarly the conclusion of the theorem holds if

(***)

there is $A^i \subset A$ and $S^i \subset \{Y \cap A^i \mid Y \in S_2\}$ such that $|S^i| > \mu$, and for every $X \in S^i$

\[ |\{Y \cap (A^i - X) \mid Y \in S^i\}| \leq \mu. \n\]

Hence we can assume (*) and (**) do not hold. So there is $X^0 \in S_3$ such that $S_3 = \{Y \cap X^0 \mid Y \in S_2\}$ is of cardinality $> \mu$. (Otherwise, taking $A^i = A$, $S^i = S_3$, (**) holds.) Similarly there is $X^i \in S_s$ such that $S_i = \{Y \cap (X^0 - X^i) \mid Y \in S_2\}$ is of cardinality $> \mu$ (otherwise taking $A^i = X^0, S^i = S_3$, (***) holds). Now $|S_i| > \mu \geq |X^0 - X^i|$, and $S_i$ is a family of subsets of $X^0 - X^i$. Hence there is $\bar{a} \in (X^0 - X^i)^{s}$ which is strongly cut by $S_i$ or by $(X^0 - X^i)(-)S_i$. Taking as $\bar{a}_m, a^i$ or $a^i$ (accordingly), we get a sequence from $A^{\omega+1}$ which is strongly cut by $S$ or $A(-)S$. So we prove Thorem 1.5A.

Naturally the question arises on the finite case. More exactly

**DEFINITION 1.5**. For natural numbers $m, n$ let $f(m, n)$ be the first ordinal $\alpha$ such that $P_3(\alpha, m, n)$ holds.

The result is $f(m, n) = 1 + \sum_{k=0}^{m-1} \binom{m}{k}$. The proof follows from a little more complex result, of Perles and Shelah.

Another natural generalization is the relation $P_4(\lambda, \mu, \chi)$ which is

**DEFINITION 1.5**. $P_4(\lambda, \mu, \chi)$ holds if whenever $|S| = \lambda, |A| = \mu$, and $S$ is a family of subsets of $A$, there exists $B \subset A, |B| = \chi$, such that for every $C \subset B$ there is $x \in S$ such that $X \cap C = B$.

Clearly $P_4(\lambda, \mu, \chi)$ implies $P_3(\lambda, \mu, \chi)$ and $P_3(\lambda, \mu, \alpha)$ for every $\alpha < \chi^+$. The only result known to me is that if $\lambda \geq \text{Ded}(\mu)$, $\lambda$ is regular and $\chi$ is finite, then $P_4(\lambda, \mu, \chi)$ holds. (see Shelah [15]). Perles and I prove that if $\mu$ and $\chi$ are finite $P_4(\lambda, \mu, \chi)$ holds if and only if $\lambda > \sum_{k=0}^{\mu} \binom{\mu}{k}$. Later and independently Sauer [19] proved it.

2. On stable models and theories. In this section we shall apply a combinatorial theorem from § 1 to get results in the theory of models.

Let $L$ be a first-order language; $L_{\lambda, \omega}$ will be its extension by permitting conjunctions on sets of $< \lambda$ formulas, provided that in the conjunction, only finitely many variables appear free. $L_{\omega, \omega}$ will be
the class of formulas $\bigcup_{L_{\omega}} L_{\omega}$. $T$ will denote a set of sentences from $L_{\omega}$. $A$ will denote a set of formulas $\varphi(\bar{x})$ from $L_{\omega}$ (more exactly, $A$ is a set of pairs $<\varphi, \bar{x}>$ where $\varphi \in L_{\omega}$, $\bar{x}$ is a finite sequence of variables, and every free variable of $\varphi$ appears in $\bar{x}$). $A$ is closed if it is closed under negation, finite conjunction (hence all connective), adding dummy variables and changing the order of the variables. $A$ is the closure of $A$. $M, N$ shall denote models ($L$-models, if not said otherwise). $|M|$ is the set of elements of $M$. If $A \subseteq |M|$, $p$ is a $(A, m)$-type over $A$ iff $p$ is a set whose elements are of the form $\varphi(\bar{x}, a)$ where $\bar{x} = <x_0, \ldots, x_m>$, $\varphi(\bar{x}, \bar{y}) \in A$ and $\bar{a} \in A$ (or more exactly $\bar{a}_0, \bar{a}_1, \ldots \in A$).

For $\bar{c} \in |M|$, the $A$-type $\bar{c}$ realizes over $A$, $p(\bar{c}, A, M, A)$ is

$$\{\varphi(\bar{x}, \bar{a}): \bar{a} \in A, \varphi(\bar{x}, \bar{y}) \in A, M \models \varphi(\bar{c}, \bar{a})\}.$$ 

Let

$$S^w(A, M, A) = \{p(\bar{c}, A, M, A): \bar{c} \in |M|^w\}.$$ 

The model $M$ is called $(A, \lambda)$-stable if $|A| \leq \lambda$ implies $|S^i(A, M, A)| \leq \lambda$; otherwise $M$ is $(\lambda, A)$-unstable.

Let $\lambda \in \text{Od}_2(M)$ if there is $n < \omega$, and sequences $\bar{a}^i \subseteq |M|^n$, $l < \lambda$; and a formula $\varphi(\bar{x}, \bar{y}) \in A$ such that $M \models \varphi(\bar{a}^k, \bar{a}^l)$ if and only if $k < l$ for every $k, l < \lambda$.

**Theorem 2.1.** Suppose $M$ is $(A, \kappa)$-unstable, $A = \bar{A}$, $\kappa = \sum_{\delta \in \varepsilon_1} (\kappa^\delta + 2^{\kappa^\delta})$ and $\kappa = \kappa^{|A|}$. Then $\lambda \in \text{Od}_2(M)$.

**Proof.** Let $A = \{\varphi_h(\bar{x}, \bar{y})^k): k < |A|\}$, $A_k = \{\varphi_h(\bar{x}, \bar{y}^k)\}$. As $M$ is $(A, \kappa)$-unstable, there is $A \subseteq |M|$, $|A| \leq \kappa$ such that $|S^i(A, M, A)| > \kappa$. If for every $k < |A|$, $|S^i(A, M, A_k)| \leq \kappa$ then

$$\kappa < |S^i(A, M, A)| \leq \left| \prod_{k \leq |A|} S^i(A, M, A_k) \right| = \prod_{k \leq |A|} |S^i(A, M, A_k)| \leq \kappa^{|A|} = \kappa$$

a contradiction. Hence there is $k < \kappa$ such that $|S^i(A, M, A_k)| > \kappa$. Let $\varphi = \varphi_h$. Now clearly $S^i(A, M, A_k)$ is a set of subsets of

$$\varnothing = \{\varphi_h(\bar{x}, \bar{d}): \bar{a} \in A, \bar{a} \text{ is of the length of } \bar{y}^l\}.$$ 

Clearly $|\varnothing| \leq \kappa$. Hence by Theorem 1.2, there are $p_i \in S^i(A, M, A_k)$ $\bar{a}^l, \bar{b}^l \subseteq |A|$ for $l < \lambda$ such that $\varphi(\bar{x}, \bar{a}^i) \in p_j \iff \varphi(\bar{x}, \bar{b}^i) \in p_j$ if and only if $j < l$. Let $p_i = p(\bar{a}^i, A, M, A_k)$, and $\bar{d}^i = \bar{a}^i \sim \bar{b}^i \sim \bar{c}^i$ (the juxtaposition of the three sequences). Clearly $M \models \varphi(\bar{c}^i, \bar{a}^i) \equiv \varphi(\bar{c}^i, \bar{b}^i)$ if and only if $j < l$. As $A = \bar{A}$, we can easily find $\psi(\bar{x}, \bar{y}) \in A$ such that for $k, l < \lambda; M \models \psi(\bar{d}^i, \bar{d}^j)$ if and only if $k < l$. Hence $\lambda \in \text{Od}_2(M)$. 


DEFINITION 2.1. Let $A, C \subseteq |M|$. $C$ is $\Delta$-indiscernible over $A$ in $M$ if for every $n$, and every $n$ different elements $c_0, \ldots, c_{n-1}$ of $C$, and every additional $n$ different elements $c^n_0, \ldots, c^{n-1}$ of $C$

$$p(\langle c_0, \ldots, c_{n-1} \rangle, A, M, \Delta) = p(\langle c^n_0, \ldots, c^{n-1} \rangle, A, M, \Delta).$$

THEOREM 2.2. Suppose $M$ is $(\bar{\Delta}, \lambda)$-stable, $\lambda \in Od_2(M)$, $A \subseteq |M|$, $C \subseteq |M|$, $|A| \leq \lambda < |C|$, and the cofinality of $\lambda$ is greater than $|\Delta|$. Then there exists $C_i \subseteq C$, $|C_i| > \lambda$ such that $C_i$ is $\Delta$-indiscernible in $M$ over $A$.

REMARK. Taking a Souslin tree, we can see that the condition $\lambda \notin Od_3(M)$ is necessary. (More exactly, this is consistent with $ZF + AC$.) Instead of $(\lambda)$ we can demand $\exists \mu < \lambda$, $\mu \inOd_3(M)$.

Morley in [9] Theorem 4.6 proved a similar theorem for models of a complete, first-order, countable, totally transcendental theory. In [12] this was generalized to models of stable theories, and in [13], Theorem 3.1 to models with stable finite diagram. Another generalization is Theorem 5.9A of Shelah [15]. Theorem 2.2, in fact, implies all these theorems. (For 5.9A [15] we should note that if $\Delta$ is finite, then there is a finite $\Delta_i$, $\Delta \subseteq \Delta_i \subseteq \bar{\Delta}$, such that for any $M, \lambda$; $M$ is $(\Delta_i, \lambda)$-stable if and only if it is $(\bar{\Delta}, \lambda)$-stable.)

Proof. As the proof is very similar to the proof of Theorem 3.1 [13], we omit it.

DEFINITION 2.2. $T$ is $(\Delta, \lambda)$-stable if every model of $T$ is $(\Delta, \lambda)$-stable. $T$ is $\Delta$-stable, if for at least one $\lambda$ it is $(\Delta, \lambda)$-stable, $T$ is $(\Delta, \lambda)$-unstable [$\Delta$-unstable] if it is not $(\Delta, \lambda)$-stable [$\Delta$-stable]. Let $\lambda \in Od_3(T)$ if for at least one model $M$ of $T$, $\lambda \in Od_3(M)$. $T$ is stable if it is $\Delta$-stable for every $\Delta$; otherwise-unstable.

REMARK. If $T$ has no model of cardinality $> \lambda$, then it is $(\Delta, \lambda)$-stable, and hence stable.

THEOREM 2.3. Suppose $T$, $\Delta \subseteq L_{\lambda+, \omega}$, $|T| \leq \lambda$, $|L| \leq \lambda$, $T$ is $(\Delta, \kappa)$-unstable, $\kappa^{\omega(2)} = \kappa$. Then $T$ is $\Delta$-unstable.

REMARK. (1) $\mu(\lambda)$ is the first cardinality such that if a sentence of a language $L^{+, \omega}_{\lambda}$ has a model of cardinality $\mu(\lambda)$, it has models in any cardinality $\geq \lambda$.

(2) We can demand only: $T$, $\Delta \subseteq L^{+, \omega}_{\lambda}$, $|T| + |\Delta| \leq \lambda$, and for every $\mu < \mu(\lambda)$ there is $\kappa = \kappa^\mu$ such that $T$ is $(\Delta, \kappa)$-unstable.

(3) We can demand only $T$, $\Delta \subseteq L^{+, \omega}_{\lambda}$, $|T| \leq \lambda$, $|L| < \mu(\lambda)$, $\kappa =
Here we use Ehrenfeucht-Mostowski models (see [2]) and the method of Morley [10]. All the results we use appeared in Chang [1]. As $T$ is $(\mathcal{A}, \kappa)$-unstable, $T$ has a model $M$ and $\forall \alpha \in |M|$ such that $|S'(A, M, \mathcal{A})| > \kappa \geq |A|$. It is well known that $\chi < \mu(\lambda)$ implies $2^\chi < \mu(\lambda)$; hence $\chi < \mu(\lambda)$ implies $2^\chi < \mu(\lambda)$. So $\kappa = \sum_{\alpha < \mu(\lambda)} (\kappa^\alpha + 2^\chi)$. As $|\mathcal{A}| \leq |\mathcal{A}|$, exactly as in the proof of Theorem 2.1, this implies that there are sequences $\mathcal{A}_k$, $\mathcal{B}_k$, $k < \mu(\lambda)$ from $A$ and $c_k \in |M|$, $k < \mu(\lambda)$, and a formula $\varphi(x, \overline{y}) \in \mathcal{A}$ such that:

for every $k, l < \mu(\lambda)$, $M \models \varphi(c_k, \mathcal{A}) \equiv \varphi(c_l, \mathcal{B})$ if and only if $l < k$.

Now we add to $M$ the one place relation $P^\mathcal{A} = \{c_k: k < \mu(\lambda)\}$, and the functions $F_1^\mathcal{A}$, $F_2^\mathcal{A}$ defined by $F_1^\mathcal{A}(c_k) = c_k$, $F_2^\mathcal{A}(\mathcal{B}_k) = c_k$, and otherwise $F_1^\mathcal{A}(\mathcal{A}) \in P^\mathcal{A}$, $F_2^\mathcal{A}(\mathcal{B}) \in P^\mathcal{A}$.

Now using Morley's method we get (in fact we need an improvement of Chang [1]):

(*) for every ordered set $I$, there is a model $M_i$ of $T$, in which there are $\mathcal{A}_i$, $\mathcal{B}_i$ for every $s \in I$ such that: for every $s, t \in I$

$M_i \models \varphi(c_i, \mathcal{A}_i) \equiv [c_i, \mathcal{A}_i]$ if and only if $t < s$.

Let $\chi$ be any cardinality, and we shall prove $T$ is $(\mathcal{A}, \chi)$-unstable. We can find easily an ordered set $I$, $|I| > \chi$, with a dense subset $J$, $|J| \leq \chi$. (If $\chi = \inf \{\chi_i: 2^\chi_i > \chi\}$, then $I$ can be the set of sequences of ones and zeroes of length $\chi_i$, ordered lexicographically.) Let $M = M_I$, and let $A = \bigcup \{\text{Rang } \mathcal{A}_s \cup \text{Rang } \mathcal{B}_s: s \in J\}$. Clearly $|A| \leq \mathfrak{K} + |J| \leq \chi$. On the other hand we shall show that $t_i \neq t_j$, $t_i, t_j \in I$ implies $p(c_{t_i}, A, M, \mathcal{A}) \neq p(c_{t_j}, A, M, \mathcal{A})$. Hence $|S'(A, M, \mathcal{A})| > \chi$, so $T$ is $(\mathcal{A}, \chi)$-unstable.

Suppose $t_i \neq t_j$, $t_i, t_j \in I$. Without loss of generality suppose $t_i < t_j$. As $J$ is a dense subset of $I$, there is $s \in J$, $t_i < s < t_j$. By the definition of $M_I$,

$M \models \varphi(c_{t_i}, \mathcal{A}_s) \equiv [c_{t_i}, \mathcal{B}_s]$

$M \models \varphi(c_{t_j}, \mathcal{B}_s) \equiv \varphi(c_{t_j}, \mathcal{B}_s)$.

Hence

$\varphi(x, \mathcal{A}_s) \in p(c_{t_i}, A, M, \mathcal{D})$ if and only if $\varphi(x, \mathcal{B}_s) \in p(c_{t_j}, A, M, \mathcal{D})$

and

$\varphi(x, \mathcal{A}_s) \in p(c_{t_j}, A, M, \mathcal{D})$ if and only if $\varphi(x, \mathcal{B}_s) \in p(c_{t_i}, A, M, \mathcal{D})$.

So $p(c_{t_i}, A, M, \mathcal{D}) \neq p(c_{t_j}, A, M, \mathcal{D})$, and as noted before this implies $T$
is \((\Delta, \chi)\)-unstable, for every \(\chi\).

Similarly we can prove

**Theorem 2.4.** (1) If \(T, \Delta \subseteq L_{\lambda^+, \omega}; |T| + |\Delta| \leq \lambda\), and for every 
\(\kappa < \mu(\lambda), \kappa \in Od(T)\), then every \(\kappa \in Od(T)\).

(2) If every \(\kappa \in Od(T)\), then \(T\) is \(\Delta\)-unstable.

**Remark.** In 2.4.2 we use the following fact: if \(M\) is \((I, \lambda)\)-stable,
\(A \subseteq |M|, |A| \leq \lambda, m < \omega\) then \(S^m(A, M, \Delta) \leq \lambda\).

**Theorem 2.5.** Suppose \(T \subseteq L_{\lambda^+, \omega}, |T| \leq \lambda, |L| \leq \lambda,\) and \(T\) is unstable. Then there exists \(\Delta \subseteq L_{\lambda^+, \omega}, |\Delta| \leq \lambda\) such that \(T\) is \(\Delta\)-unstable.

**Proof.** As in the proof of Theorem 2.3, we depend on the method of Morley [10], Chang [1]. So let \(T\) be \(\Delta\)-unstable. Without loss of
generality, let \(\Delta = \tilde{\Delta}\) and \(\Delta \subseteq L_{\lambda^+, \omega}\). From Theorem 2.1 it follows
that every \(\mu \in Od(T)\) \([\text{as } T \text{ is } (\Delta, \tilde{\Delta})\text{-unstable}]. \) Let \(\lambda_1 =\mu(\lambda + |T| + \kappa + |\Delta| + |L|)\). So \(T\) has a model \(M\) such that \(\lambda_1 \in Od(M)\).

We expand now \(M\) to \(M^1\) in the following way:

(1) For every subformula \(\varphi(\bar{x})\) of a formula from \(T \cup \Delta\) (including the formulas form \(\Delta\) themselves) we add to \(M\) the relation \(R_{\varphi} = \{\bar{a}: M \models \varphi(\bar{a})\} \).

(2) \(M^1\) has Skolem function for every first-order formula in its
language.

Let \(L^1 = L(M^1)\) be the first-order language associated with \(M^1\).
Clearly \(|L(M^1)| \leq |L| + |T| + |\Delta| + \kappa + \lambda\). As \(\lambda_1 \in Od(M)\), there
are \(\bar{a}^k, k < \lambda_1\) from \(M^1\) and there is \(\varphi_\lambda(\bar{x}, \bar{y}) \in \Delta\) such that \(M^1 \models \varphi_\lambda(\bar{a}^k, \bar{a}^l)\)
if and only if \(k < l\). For simplicity we shall assume the sequences \(\bar{a}^k\)
are of length one, and \(\bar{a}^k = \langle a_k \rangle\).

Hence there is a model \(N\) and \(a_s \in |N|\) for \(s \in I\), which satisfy
the following properties:

(1) the first-order language associated with \(N\) is \(L^1\).

(2) \(N, M^1\) are elementarily equivalent.

(3) \(N\) is a model of \(T\), and for every subformula \(\varphi(\bar{x})\) of a
formula from \(T \cup \Delta, N \models (\forall \bar{x})[\varphi(\bar{x}) \equiv R_\varphi(\bar{x})]\).

(4) \(I\) is an ordered set isomorphic to the rationals \((s, t\) will
denote elements of \(I\).

(5) for each \(s, t \in I; N \models \varphi_\lambda[a_s, a_t]\) if and only if \(s < t\).

(6) for each \(c \in N\), there are \(s_1 < \cdots < s_n(\in I)\) and a term \(B\)
of \(L^1\) such that
\[N \models c = B[a_{s_1}, \ldots, a_{s_n}]\]

(7) for every \(\varphi(x_1, \ldots, x_n) \in L^1, s_1 < \cdots < s_n,\) and \(t_1 < \cdots < t_n\)
the following holds:
\[ N \models \varphi[a_1, \ldots, a_n] \text{ if and only if } N \models \varphi[a_1, \ldots, a_n]. \]

As \( I \) is dense, by \([7], [17]\), this holds also for every \( \varphi \in L_{\omega, \omega} \).

Let \( \bar{x}_0 = \langle x_0, x_i \rangle, \bar{x}_1 = \langle x_2, x_3 \rangle \).

Let \( \{ \varphi_{k,n}(\bar{x}_0, \bar{x}_1, y_0, \ldots, y_{n-1}) \mid n < \omega, k < |L| \} \) be the list of the atomic formulas of \( L \). Let
\[
\Phi_n(\bar{x}_0, \bar{x}_1, y_0, \ldots, y_{n-1}, z_0, \ldots, z_{n-1}) = \bigwedge_{k < |L|} (\varphi_{k,n}(\bar{x}_0, \bar{x}_1, y_0, \ldots, y_{n-1}) \equiv \varphi_{k,n}(\bar{x}_0, \bar{x}_1, z_0, \ldots, z_{n-1}))
\]
\[
\Phi(\bar{x}_0, \bar{x}_1) = (\exists y_0 \forall x_0 \exists z_0 \forall y_1, \exists y_0 \forall x_0 \exists z_0 \forall y_3, \ldots, \exists y_{2m} \forall x_0 \exists z_0 \forall y_{2m+1}, \ldots)_{m < \omega} \left[ \neg \bigwedge_{n < \omega} \Phi_n(\bar{x}_0, \bar{x}_1, y_0, \ldots, y_{n-1}, z_0, \ldots, z_{n-1}) \right].
\]

By Shelah [14], for every \( L \)-model \( M_i \), and \( \bar{a}, \bar{b} \in |M_i|^\lambda, M_i \models \Phi[\bar{a}, \bar{b}] \) if and only if \( \bar{a} \) and \( \bar{b} \) realizes different \( L_{\omega, \omega} \)-types (i.e., there is \( \varphi(\bar{x}_0) \in L_{\omega, \omega} \) such that
\[ M_i \models \varphi[\bar{a}], M_i \models \neg \varphi[\bar{b}]. \]

**Remark.** The definition of the satisfaction of \( \Phi[\bar{a}, \bar{b}] \) is self-evident. Discussion about languages with such expressions can be found in Keisler [6].

Hence we can find functions \( F_1, \ldots, F_n, \ldots \) whose domains and ranges are \(|N|\), each with a finite number of places such that:

(*) if \( N_i \) is a submodel of a reduct of \( N \), whose associated first order language include \( L \), and \(|N_i|\) is closed under the functions \( \{ F_n : n < \omega \} \) then for every \( \bar{a}, \bar{b} \in |N_i|^\lambda, N_i \models \Phi[\bar{a}, \bar{b}] \) implies \( N_i \models \Phi[\bar{a}, \bar{b}] \).

Now as in the downward Lowenheim-Skolem theorem, we can find a model \( N_1 \) such that:

- \((A)\) \(|N_1| \subset |N|, (a_s : s \in I) \subset |N_1|, ||N_1|| \leq \lambda \) and \( N_1 \) is a submodel of a reduct of \( N \).
- \((B)\) \(|N_1| \) is closed under \( \{ F_n : n < \omega \} \)
- \((C)\) if \( \bar{a} \in |N_1|, \varphi(x, y) \) is a subformula of \( \psi \in T \), and \( N \models (\exists x)\varphi(x, \bar{a}) \), then for some \( b \in |N_1|, N \models \varphi[b, \bar{a}] \). Hence \( N_1 \) is a model of \( T \).
- \((D)\) if \( s_1 < \cdots < s_n, t_1 < \cdots < t_n, B \) is a term from \( L^i \), and \( B^\psi[a_{s_1}, \ldots, a_{s_n}] \in |N_1| \), then \( B^\psi[a_{s_1}, \ldots, a_{s_n}] \in |N_1| \).

**Remark.** Notice that by property \((7)\) of \( N \), if \( B_1^\psi[a_{s_1}, \ldots, a_{s_n}] = B_2^\psi[a_{s_1}, \ldots, a_{s_n}] \) then \( B_1^\psi[a_{s_1}, \ldots, a_{s_n}] = B_2^\psi[a_{s_1}, \ldots, a_{s_n}] \).

- \((E)\) The language of \( N_1, L^i \), contains, \( L \), is of cardinality \( \lambda \), is contained in \( L^i \), and for each \( c \in |N_1| \) there is a term \( B \) from \( L^i \) such that \( c = B^\psi[a_{s_1}, \cdots, a_{s_n}] \) for some \( s_1 < \cdots < s_n \).
It is easy to prove that $N_t$ satisfies properties (6) and (7) of $N$, with $L'$ replaced by $L'$. It is also clear, by (C), that $N_t$ is a model of $T$. Let $s < t$, we know that $N \models \varphi[a_s, a_t]$, but $N \models \neg \varphi[a_s, a_t]$. Hence $\langle a_s, a_t \rangle$, $\langle a_t, a_s \rangle$ do that satisfy the same $L_{\omega \omega}$-type in $N$. By (⋆) and (B), $\langle a_s, a_t \rangle$, $\langle a_t, a_s \rangle$ also do not realize the same $L_{\omega \omega}$-type in $N$. As $||N_t|| \leq \lambda$, by Chang [1] it follows that $\langle a_s, a_t \rangle$, $\langle a_t, a_s \rangle$ do not realize the same $L_{\omega \omega}$-type in $N_t$. So there is a formula $\varphi(x, y) \in L_{\omega \omega}$ such that $N_t \models \varphi[a_s, a_t]$, $N_t \models \neg \varphi[a_t, a_s]$. Let $A^t = \{ \varphi(x, y) \}$, $A_t = \tilde{A}$. We shall prove that $T$ is $\Delta_t$-unstable, and so prove the theorem.

By Theorem 2.4.2 it suffices to prove that for every $\kappa$, $\kappa \in Od_{\lambda}(T)$. Let $\kappa$ be any cardinal, and $J$ a dense order set, $I \subset J$, and $J$ contain a subset with order-type $\kappa$. We shall define now $N_2$ as an extension of $N_t$ such that:

\( (\alpha) \quad \{ a_s : s \in J \} \subset N_2 \)

\( (\beta) \quad \) for every element $c$ of $N_2$ there are $s_1 < \cdots s_n \in J$ and term $B \in L_2$ such that

\[ c = B^{\pi}[a_{s_1}, \cdots, a_{s_n}] \]

\( (\gamma) \quad \) if $\varphi(x_1, \cdots, x_n)$ is an atomic formula, $s_1 < \cdots < s_n \in J$, $t_1 < \cdots < t_n \in J$ then

\[ N_2 \models \varphi[a_{s_1}, \cdots, a_{s_n}] \text{ if and only if } N_2 \models \varphi[a_{t_1}, \cdots, a_{t_n}] \]

It can be easily seen that $N_2$ exists. We can also show by induction on formulas of $L_{\omega \omega}$ that $N_2$ is an $L_{\omega \omega}$-elementary extension of $N_t$. (See [7], [17].) Hence $N_2$ is a model of $T$. It is also clear that for every $s, t \in J$, $N_2 \models \varphi[a_s, a_t]$ if and only if $s < t$. By the definition of $J$ and $A_t$ this implies $\kappa \in Od_{\lambda}(N_2)$ hence $\kappa \in Od_{\lambda}(T)$, and by 2.4.2, this implies $T$ is $\Delta_t$-unstable, where $|\Delta_t| \leq \lambda$, $|\Delta_t| \subset L_{\omega \omega}$.

**Theorem 2.6.** If $T$ is unstable, $T \subset L_{\omega \omega}$, $\mu > \lambda + |T|$, then $T$ has exactly $2^\mu$ non-isomorphic models of cardinality $\mu$. (For most cases it suffices to demand $\mu \geq \lambda + |T| + \aleph_1$.)

**Proof.** By Theorem 2.5, and Shelah [16].

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