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**VECTOR SPACE DECOMPOSITIONS AND THE ABSTRACT  
IMITATION PROBLEM**

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## VECTOR SPACE DECOMPOSITIONS AND THE ABSTRACT IMITATION PROBLEM

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Let  $\mathcal{S}$  be a Hilbert space,  $\mathcal{P}$  a closed subspace of  $\mathcal{S}$ ,  $L$  an orthogonal projection operator on  $\mathcal{S}$ . The "imitation problem" consists of finding the solutions  $p \in \mathcal{P}$  of the equation

$$p - s = L(p - s)$$

for given  $s \in \mathcal{S}$ . If  $\bar{W}$  is a compact bordered Riemann surface,  $A$  a boundary neighborhood,  $s$  a "singularity differential" defined on  $\bar{A}$ ,  $p$  will be a harmonic exact differential which imitates  $s$  on  $\bar{A}$  in a sense precised by  $L$  (hence the name "imitation problem"). Existence and uniqueness theorems are given for the solution. Some concrete applications are described. The paper ends with a constructive method of solution in the case of  $L^2$ -normal operators.

0. Introduction. The "imitation problem" has originally been formulated by L. Sario (see for instance [1]). It is fundamental in the construction of harmonic functions on a Riemann surface with given singularities and given boundary behavior. It can be formulated as follows: given a "singularity function"  $s$  defined in a boundary neighborhood, and a "normal operator  $L$ ", construct a harmonic function  $p$  defined on the whole Riemann surface and satisfying in the given boundary neighborhood the equation

$$p - s = L(p - s).$$

Sario's original solution uses the sup norm. For problems involving harmonic differentials, the  $L^2$  norm is introduced somewhat more naturally and progress has been made in various directions. (see [5]). In §1 we study the abstract "imitation problem" for an arbitrary Hilbert space and give an existence and uniqueness theorem for the solution. In §2 we consider some decompositions of the vector space  $\mathfrak{S}(\bar{A})$  of harmonic exact differentials defined on a boundary neighborhood  $A$  of a compact bordered Riemann surface  $\bar{R}$  and continuous in  $\bar{A}$ , and study some corresponding "imitation problems". In §3 we return to the  $L^2$  case and give a constructive method of solution when the operator  $L$  is  $L^2$ -normal. The method may be applied to the case of harmonic differentials on a Riemannian manifold of dimension  $> 2$ , and also to open manifolds.

1. The abstract imitation problem in a Hilbert space. Let

$\mathcal{S}$  be a Hilbert space,  $\mathcal{B}(\mathcal{S})$  the algebra of bounded linear operators on  $\mathcal{S}$ . We are given a closed subspace  $\mathcal{P} \subset \mathcal{S}$  corresponding to the orthogonal projection  $F$ . Given the orthogonal projection  $L$  on  $\mathcal{S}$  we want to solve the equation

$$(*) \quad p - s = L(p - s)$$

for  $p \in \mathcal{P}$  given the "singularity"  $s \in \mathcal{S}$ . We assume moreover that  $p = Ts$  where  $T \in \mathcal{B}(\mathcal{S})$ .

We are going to prove the theorem:

**THEOREM.** *Let  $\mathcal{P}$  be a closed subspace of the Hilbert space  $\mathcal{S}$  and let  $F$  denote orthogonal projection on  $\mathcal{P}$ . Let  $L$  be an arbitrary orthogonal projection operator on a subspace of  $\mathcal{S}$ . Then the imitation problem*

$$p - s = L(p - s)$$

*admits a unique solution in  $\mathcal{P}$  of the form*

$$p = Ts$$

*(where  $T$  is a bounded linear operator on  $\mathcal{S}$ ) if and only if*

$$\text{Im } L \perp \text{Im } F = \mathcal{S}.$$

*Proof.* Observe that (\*) may be written as:

$$(I - L)(I - T)s = 0$$

which is true for each  $s \in \mathcal{S}$ .

It follows that  $I - T$  belongs to the right annihilator of  $I - L$ . Now  $\mathcal{B}(\mathcal{S})$  being a Baer ring [4] it follows that there exists  $X \in \mathcal{B}(\mathcal{S})$  such that

$$(**) \quad I - T = LX.$$

Moreover, since  $p \in \text{Im } F$  we have  $Ts \in \text{Im } F$  hence

$$(I - F)Ts = 0.$$

We conclude that  $T$  belongs to the right annihilator of  $I - F$  hence there exists  $Y \in \mathcal{B}(\mathcal{S})$  such that

$$(***) \quad T = FY.$$

Adding up (\*\*) and (\*\*\*) we get the equation

$$(\dagger) \quad I = LX + FY$$

where, we recall  $L, F$  are given orthogonal projection operators and  $X,$

$Y$  are unknown elements of  $\mathcal{B}(\mathcal{S})$ . Clearly, if  $\text{Im } L + \text{Im } F \not\subseteq \mathcal{S}$ , the last equation has no solution. We show conversely that if  $\text{Im } L + \text{Im } F = \mathcal{S}$  the problem has always a solution. We need the:

**LEMMA.** *Let  $A, B$  be closed subspaces of a Hilbert space  $\mathcal{S}$  such that  $A + B = \mathcal{S}$  (vector sum). Then, there exist closed subspaces  $A_m \subset A, B_m \subset B$  such that*

$$A_m \dot{+} B_m = \mathcal{S} \quad (\text{direct sum}) .$$

*Proof.* Let  $\{e_\alpha\}$  be a basis for  $A, \{e_\beta\}$  a basis for  $B$ . Then,  $\{e_\alpha, e_\beta\}$  is a set of generators for  $\mathcal{S}$ . It contains a basis  $\{e_{\alpha_i}, e_{\beta_j}\}$  where  $\{e_{\alpha_i}\} \subset \{e_\alpha\}$  and  $\{e_{\beta_j}\} \subset \{e_\beta\}$ . Let then  $A_m$  be the closed span of  $\{e_{\alpha_i}\}, B_m$  be the closed span of  $\{e_{\beta_j}\}$ . Then  $A_m + B_m = \mathcal{S}$  and  $A_m \cap B_m = \{0\}$ .

We apply the lemma to  $A = \text{Im } L, B = \text{Im } F$ . There exist subspaces  $A_m \subset \text{Im } L, B_m \subset \text{Im } F$  such that

$$A_m \dot{+} B_m = \mathcal{S} .$$

Let  $X_0$  and  $Y_0$  be orthogonal projections on  $A_m$  and  $B_m$  respectively. Then

$$I = LX_0 + FY_0$$

and  $(X_0, Y_0)$  is a solution of  $(\dagger)$ . To study uniqueness, let  $(X, Y)$  be another solution of  $(\dagger)$ . One must have:

$$L(X - X_0) = F(Y_0 - Y) .$$

So if  $\text{Im } F \cap \text{Im } L = \{0\}$  then necessarily  $X = X_0, Y = Y_0$ . If  $\text{Im } F \cap \text{Im } L \neq \{0\}$  then, the operators of the form  $L(X - X_0) = F(Y_0 - Y)$  are the elements of the right annihilator of the set  $\{I - L, I - F\}$  hence of the form  $G\mathcal{B}(\mathcal{S})$  for some orthogonal projection  $G$ . The  $T$ 's we are looking for are of the form  $FY = FY_0 - F(Y_0 - Y) = FY_0 - G\mathcal{B}(\mathcal{S})$ .  $G\mathcal{B}(\mathcal{S})$  is non void: if  $\text{Im } F \cap \text{Im } L = \text{Im } M$  where  $M$  is a projection,  $M$  satisfies  $LM = FM$ . In that case uniqueness is lost and we have proved the theorem.

*Notes.* (1) there is actually no restriction when dealing with operators  $L$  which are projections: if  $L$  denotes *any* element of  $\mathcal{B}(\mathcal{S})$ ,  $(*)$  becomes  $(I - L)(I - T) = 0$ . So  $I - T$  belongs to the right annihilator of  $I - L$  and therefore  $I - F = \Lambda U$  where  $\Lambda$  is the orthogonal projection generating the right annihilator of  $I - L$ .

(2) The preceding proof can be applied to the Baer ring of linear endomorphisms of a vector space. Orthogonal projections should be replaced by projection operators.

As an example we apply the previous theory to the construction of harmonic differentials on a Riemann surface which “imitate” some singularity differential in the neighborhood of the ideal boundary (whence the name “imitation problem”).

2. Vector space decompositions and the corresponding “imitation problems”. Let  $\bar{R}$  be any compact bordered Riemann surface. We consider the space  $\mathfrak{S}(\bar{R})$  consisting of harmonic exact differentials on  $\text{Int}(\bar{R})$ , which are continuous on  $\bar{R}$ . Let  $\gamma$  be a cycle on  $\bar{R}$ ,  $[\gamma]$  the corresponding homology class. We introduce the space

$$H_{[\gamma]}(\bar{R}) = \left\{ \omega \in \mathfrak{S}(\bar{R}); \int_{\gamma} * \omega = 0 \right\}$$

(see [1]).

Let now  $\bar{W}$  be a compact bordered Riemann surface,  $\bar{A}$  the complement of a regularly embedded domain  $\Omega$ . We use the standard notation

$$\begin{aligned} \alpha &= Bd\bar{\Omega} \\ \beta &= Bd\bar{W} . \end{aligned}$$

In the vector space  $\mathfrak{S}(\bar{A})$  we consider the subspaces

$$\begin{aligned} H_{0\beta}(\bar{A}) &= \{ \omega \in \mathfrak{S}(\bar{A}), \omega = df, df|_{\beta} = 0 \} \\ H_{0\alpha}(\bar{A}) &= \{ \omega \in \mathfrak{S}(\bar{A}), \omega = df, df|_{\alpha} = 0 \} \\ H_{0\beta}^{\star}(\bar{A}) &= \left\{ \omega \in \mathfrak{S}(\bar{A}); \omega = df, *df|_{\beta} = 0, \int_{\alpha_i} *df \right. \\ &\quad \left. = 0, \text{ for each component } \alpha_i \text{ of } \alpha \right\} \\ H_{0\alpha}^{\star}(\bar{A}) &= \left\{ \omega \in \mathfrak{S}(\bar{A}); \omega = df, *df|_{\alpha} = 0, \int_{\beta_i} *df \right. \\ &\quad \left. = 0, \text{ for each component } \beta_i \text{ of } \beta \right\} \\ H'_{0\beta}(\bar{A}) &= H_{0\beta}(\bar{A}) \cap H_{[\beta]}(\bar{A}) \\ H'_{0\alpha}(\bar{A}) &= H_{0\alpha}(\bar{A}) \cap H_{[\beta]}(\bar{A}) . \end{aligned}$$

Observe that:

$$H_{0\beta}^{\star}(\bar{A}) = H_{0\beta}^{\star}(\bar{A}) \cap H_{[\beta]}(\bar{A}) = H_{0\beta}^{\star}(\bar{A}) .$$

Another important subspace will be

$$H_{ext}(\bar{A}) = \{ \omega \in \mathfrak{S}(\bar{A}); \omega = \hat{\omega}|_{\bar{A}} \text{ where } \hat{\omega} \in \mathfrak{S}(\bar{W}) \} .$$

Clearly  $H_{ext}(\bar{A}) \subset H_{[\beta]}(\bar{A})$ .

Let now  $\Gamma(\bar{A})$  be the space of square integrable harmonic differentials on  $\bar{A}$ . We denote

$$\begin{aligned} h_{[\beta]}(\bar{A}) &= \text{closure in } \Gamma(\bar{A}) \text{ of } H_{[\beta]}(\bar{A}) \\ h'_{0\gamma}(\bar{A}) &= \text{closure in } \Gamma(\bar{A}) \text{ of } H'_{0\gamma}(\bar{A}) \\ h_{0\gamma}^*(\bar{A}) &= \text{closure in } \Gamma(\bar{A}) \text{ of } H_{0\gamma}^*(\bar{A}) \end{aligned}$$

where  $\gamma$  stands for  $\alpha$  or  $\beta$ . We have the following vector space decompositions:

PROPOSITION.

$$\begin{aligned} h_{[\beta]}(\bar{A}) &= h'_{0\alpha}(\bar{A}) \oplus h_{0\beta}^*(\bar{A}) \\ h_{[\beta]}(\bar{A}) &= h_{0\alpha}^*(\bar{A}) \oplus h'_{0\beta}(\bar{A}) . \end{aligned}$$

*Proof.* We prove the first equality. The second is obtained by symmetry. First, we show

$$h_{[\beta]}(\bar{A}) = H_{0\alpha}(\bar{A}) \oplus H_{0\beta}^*(\bar{A}) .$$

Observe that  $H_{0\alpha}(\bar{A})$  is orthogonal to  $h_{0\beta}^*(\bar{A})$ : let  $df \in H_{0\alpha}(\bar{A})$ ,  $dg \in H_{0\beta}^*(\bar{A})$ . The inner product on the Hilbert space  $\Gamma(\bar{A})$  induces an inner product on  $H_{[\beta]}(\bar{A})$ . So,

$$(df, dg)_{\bar{A}} = \int_{\beta-\alpha} f * \bar{d}g = \int_{\beta} f * \bar{d}g = 0 .$$

Let now  $dk$  be an element of  $H_{[\beta]}(\bar{A})$ . We want to find  $df \in H'_{0\alpha}$  and  $dg \in H_{0\beta}^*(\bar{A})$  such that

$$dk = df + dg .$$

We must have  $dg|_{\alpha} = dk|_{\alpha}$ ,  $*dg|_{\beta} = 0$ ,  $\int_{\alpha_i} *dg = 0$  for each component  $\alpha_i$  of  $\alpha$ . Also  $df|_{\alpha} = 0$ ,  $*df|_{\beta} = *dk|_{\beta}$  and  $\int_{\alpha_i} *df = \int_{\alpha_i} *dh$  for each component  $\alpha_i$  of  $\alpha$ . Such a problem has a unique solution.

We now take closures in  $\Gamma(\bar{A})$ . Observe that  $h'_{0\alpha}(\bar{A})$  and  $h_{0\beta}^*(\bar{A})$  are orthogonal since  $H'_{0\alpha}(\bar{A})$  and  $H_{0\beta}^*(\bar{A})$  are dense and orthogonal. It follows that

$$h_{[\beta]}(\bar{A}) = h'_{0\alpha}(\bar{A}) + h_{0\beta}^*(\bar{A}) .$$

We now consider some orthogonal projections in the space  $h_{[\beta]}(\bar{A})$ , which may be used as operators  $L$  of §1.

(1) Let  $A_0$  be orthogonal projection on  $h_{0\beta}^*(\bar{A})$ . We have

$$\ker A_0 = h'_{0\alpha}(\bar{A}) .$$

In particular  $*A_0 df \in h_{0\beta}(\bar{A})$  and hence  $A_0 df$  has “vanishing normal derivative” on  $\beta$ . Moreover  $(I - A_0) df \in h'_{0\alpha}$ . So  $A_0 df|_{\alpha} = df|_{\alpha}$  and  $A_0$  has the property of Sario’s “ $L_0$  operator”.

(2) Let  $A_1$  be orthogonal projection on  $h'_{0\beta}(\bar{A})$ . We have

$$\ker A_1 = h_{0\alpha}(\bar{A}) .$$

So  $A_1 df|_{\beta} \in h'_{0\beta}$  and “ $A_1 df$  vanishes on  $\beta$ ” However  $*(I - A_1)df|_{\alpha} \in h'_{0\alpha}$  hence

$$*A_1 df|_{\alpha} = *df|_{\alpha}$$

and  $A_1$  differs from Sario’s “ $L_1$  operator” by its behavior on  $\alpha$ . Some other vector space decompositions will be of interest:

PROPOSITION.  $H_{[\beta]}(\bar{A}) = H_{ext}(\bar{A}) \oplus H'_{0\beta}(\bar{A})$ .

*Proof.* Observe that  $H_{ext}(\bar{A}) \cap H_{0\beta}(A) = \{0\}$ . This is a consequence of the fact that on  $\bar{W}$ ,  $\Gamma_{hse}^* \cap \Gamma_{he}$  is orthogonal to  $\Gamma_{he} \cap \Gamma_{ho}$ .

Now consider any  $df \in H_{[\beta]}(\bar{A})$ . Let  $\hat{d}f$  be the unique harmonic exact differential on  $\bar{W}$  which has same boundary values as  $df$ . Now:

$$df = \hat{d}f|_{\bar{A}} + (df - \hat{d}f)|_{\bar{A}}$$

and

$$\hat{d}f|_{\bar{A}} \in H_{ext}(\bar{A}), (df - \hat{d}f)|_{\bar{A}} \in H'_{0\beta} .$$

which proves the validity of the direct sum decomposition.

We shall denote by  $K_1$  the corresponding projection on  $H_{ext}(\bar{A})$  and by  $L_1$  the corresponding projection on  $H'_{0\beta}(\bar{A})$ .

PROPOSITION.  $H_{[\beta]}(\bar{A}) = H_{ext}(\bar{A}) \oplus H_{0\beta}^*(\bar{A})$  .

*Proof.*  $H_{ext}(\bar{A}) \cap H_{0\beta}^*(\bar{A}) = \{0\}$ . Thus assume  $\omega = df \in H_{ext}(\bar{A})$  and  $*df|_{\beta} = 0$ . By the uniqueness of the solution to the Neumann problem  $df = 0$ . Consider now any  $df \in H_{[\beta]}(\bar{A})$ . Let  $df$  be the harmonic exact differential on  $\bar{W}$  such that  $*(\hat{d}f)|_{\beta} = *df|_{\beta}$  and  $\int_{\alpha_i} *\hat{d}f = \int_{\alpha_i} *df$  for each component  $\alpha_i$  of  $\alpha$ . We can write

$$df = \hat{d}f|_{\bar{A}} + (df - (\hat{d}f))|_{\bar{A}}$$

where

$$\hat{d}f|_{\bar{A}} \in H_{ext}(\bar{A}) , \quad (df - (\hat{d}f))|_{\bar{A}} \in H_{0\beta}^*(A) ,$$

which proves the validity of the direct sum decomposition. We denote by  $K_0$  the projection on  $H_{ext}(\bar{A})$  and by  $L_0$  the projection on  $H_{0\beta}^*(\bar{A})$ .

*Application.* Solution to the “imitation problem” for harmonic differentials in  $H_{[\beta]}(\bar{A})$ . (cf. §1. note 2).

Assume that we have a decomposition:

$$H_{[\beta]}(\bar{A}) = H_{ext}(\bar{A}) \oplus \check{H}(\bar{A}) . \quad (\text{direct sum}) .$$

We denote by  $L$  the corresponding projection onto  $\check{H}(\bar{A})$  and by  $K$  the corresponding projection onto  $H_{ext}(\bar{A})$ .

The “imitation problem” consists in studying the solutions  $\omega \in H_{ext}(\bar{A})$  of the equation:

$$(*) \quad \omega - s = L(\omega - s) , \quad s \in H_{[\beta]}(\bar{A}) .$$

One can apply the existence and uniqueness theorem of §1, or check directly.

*Uniqueness of the solution:* let  $\omega_1, \omega_2$  be two solutions of (\*) then

$$\omega_1 - \omega_2 = L(\omega_1 - \omega_2)$$

or

$$(I - L)(\omega_1 - \omega_2) = 0$$

i.e.  $\omega_1 - \omega_2 \in \text{Ker}(I - L) = \text{Im}L .$

Now  $\omega_1 - \omega_2 \in \text{Im}K$  and  $\text{Im}K \cap \text{Im}L = \{0\}$ . It follows that  $\omega_1 - \omega_2 = 0$  and the solution is unique.

*Existence of the solution.* To solve  $(I - L)(\omega - s) = 0$  set  $\omega = Ks$ . We then get:

$$(I - L)(I - K)s = 0$$

which is verified for all  $s \in H_{[\beta]}(\bar{A})$  since

$$\text{Im}(I - K) = \text{Ker}L ;$$

from the direct sum decomposition.

**EXAMPLES.**

(1)  $L_1$  and  $K_1$ . The unique solution to

$$\omega - s = L_1(\omega - s)$$

is given by  $\omega = K_1s$ . Such a  $\omega$  has the same boundary behavior as  $s$ .

(2)  $L_0$  and  $K_0$ . The unique solution to

$$\omega - s = L_0(\omega - s)$$

is given by  $\omega = K_0s$  and  $*\omega$  and  $*s$  have same boundary behavior.

**3.  $-L^2$ -normal operators and the “imitation problem”.** We



now return to the  $L^2$  theory and show a constructive method of solution. We consider the Hilbert space  $\mathfrak{S}_1$  defined as the closure in the  $L^2$ -norm on  $\bar{A}$  of the space of harmonic exact differential on  $\bar{A}$ . We are considering operators

$$L: \mathfrak{S}_1 \longrightarrow \mathfrak{S}_1$$

such that

(i)  $L$  is an orthogonal projection operator. (in particular  $L^2 = L$  and  $\|L\| = 1$ )

(ii)  $\text{Im}(I - L) \cap H_{\text{ext}}(\bar{A}) = \{0\}$ .

Such operators will be called  $L^2$ -normal.

We consider in particular the operator

$$K: \mathfrak{S}_1 \longrightarrow \mathfrak{S}_1$$

where  $K$  denotes orthogonal projection onto the subspace  $\mathfrak{R}$  of exact harmonic differentials in  $\mathfrak{S}_1$  which admit a harmonic extension to all of  $\bar{W}$ . The next generalized  $q$ -lemma shows that  $\mathfrak{R}$  is closed.

**GENERALIZED  $q$ -LEMMA.** *There exist numbers  $q(\bar{A})$  and  $q'(\bar{A})$  lying between 0 and 1 such that for each  $\omega \in \Gamma_{he}(\bar{W})$ .  $q'(\bar{A})\|\omega\|_{\bar{W}} \leq \|\omega\|_{\bar{A}} \leq q(\bar{A})\|\omega\|_{\bar{W}}$ .*

*Proof.* We know that  $\Gamma_{he}(\bar{W})$  has the Montel property. Consider the subset  $S \subset \Gamma_{he}(\bar{W})$  defined as

$$S = \{\omega \in \Gamma_{he}(\bar{W}) : \|\omega\|_{\bar{W}} = 1\}.$$

We first want to show that then exists  $q(\bar{A})$ ,  $0 < q(\bar{A}) < 1$  such that

$$\|\omega\|_{\bar{A}} \leq q(\bar{A})$$

for each  $\omega \in S$ .

If this is not the case, there is a sequence  $(\omega_n)$  from  $S$  such that  $\|\omega_n\|_{\bar{A}} \rightarrow 1$ .

By the Montel property,  $(\omega_n)$  has a convergent subsequence  $(\omega_{n_i})$  and  $\omega_{n_i} \rightarrow \hat{\omega} \in S$ . (since  $S$  is closed). Now  $\|\omega_{n_i}\|_{\bar{A}} \rightarrow 1$  and hence  $\|\hat{\omega}\|_{\bar{A}} = 1$  and so  $\text{supp } \hat{\omega} \subseteq \bar{A}$ . But no element of  $\Gamma_{he}(\bar{W})$  has support contained in  $\bar{A}$  a proper subset of  $\text{Int } \bar{W}$ . ([3] p. 186).

Hence there exists  $q(\bar{A})$ ,  $0 < q(\bar{A}) < 1$  such that

$$\|\omega\|_{\bar{A}} \leq q(\bar{A})\|\omega\|_{\bar{W}}.$$

To get the second inequality, consider  $\bar{Q}$ :

$$\|\omega\|_{\bar{Q}} \leq q(\bar{Q})\|\omega\|_{\bar{W}}$$

hence

$$\|\omega\|_{\bar{W}} - \|\omega\|_{\bar{A}} \leq q(\bar{Q}) \|\omega\|_{\bar{W}}$$

or

$$(1 - q(\bar{Q})) \|\omega\|_{\bar{W}} \leq \|\omega\|_{\bar{A}}$$

and we have  $q'(\bar{A}) = 1 - q(\bar{Q})$ . Which proves the lemma.

NOTE. We have  $1 - q(\bar{Q}) \leq q(\bar{A})$ . So  $q(\bar{A}) + q(\bar{W} - \bar{A}) \geq 1$ .

COROLLARY.  $\mathfrak{R}$  is a closed subspace of  $\mathfrak{S}_1$ .

*Proof.* We show  $\mathfrak{R}$  contains all the limits of its Cauchy sequences. Let  $(\omega_n)$  be Cauchy in  $\mathfrak{R}$ . Let  $(\hat{\omega}_n)$  be the corresponding sequence in  $\Gamma_{h\epsilon}(\bar{W})$  (such that  $\hat{\omega}_n|_{\bar{A}} = \omega_n$ ). Now  $(\hat{\omega}_n) \rightarrow \hat{\omega} \in \Gamma_{h\epsilon}(\bar{W})$  in the  $L^2$  norm on  $\Gamma_{h\epsilon}(\bar{W})$ . Since the  $L^2$ -norms on  $\Gamma_{h\epsilon}(\bar{W})$  and  $\mathfrak{R}$  are equivalent. It follows that

$$(\omega_n) \longrightarrow \hat{\omega}|_{\bar{A}}$$

in the  $L^2$  norm on  $\mathfrak{R}$  and hence  $\mathfrak{R}$  is closed. We now prove:

**THEOREM.** Let  $L$  be a  $L^2$ -normal operator on  $\mathfrak{S}_1$ . Then the equation  $\omega - s = L(\omega - s)$  admits a solution  $\omega \in \mathfrak{R}$ . The solution is unique provided  $\mathfrak{R} \cap \text{Im } L = (0)$ .

*Proof.* Assume there exists  $p \in \mathfrak{S}_1$  such that

$$(^+) \quad -Kp - s = L(p - s) .$$

We then have

$$L(-Kp - s) = L^2(p - s) = L(p - s) = -Kp - s .$$

Setting  $\omega = -Kp$  we obtain an element of  $\mathfrak{R}$  such that

$$\omega - s = L(\omega - s) .$$

It then suffices to solve  $(^+)$ . We rewrite it as:

$$(^{++}) \quad [I - (I - (K + L))]p = -(I - L)s .$$

The latter admits a solution  $p \in \mathfrak{S}_1$  (which can be written as a Neumann series) if

$$\|I - (K + L)\| < 1$$

or, what is the same, if the aperture

$$\theta(\text{Im } (I - K), \text{Im } (K)) < 1 .$$

(For the definition and properties of the aperture see [2] p. 69.) Now

$$\begin{aligned} &\theta(\text{Im}(I - L), \text{Im}(K)) \\ &= \max \{ \text{dist} [S(\text{Im}(I - L)), \text{Im} K], \text{dist} [S(\text{Im} K), \text{Im}(I - L)] \} \end{aligned}$$

(where  $S(V)$  denotes the unit sphere in the subspace  $V$ ).

Now the unit spheres in  $\text{Im}(I - L)$ ,  $\text{Im} K$  are closed and bounded hence compact since  $\mathfrak{S}_1$  has the Montel property.

Assume that the max is given by the first term; let  $x \in S(\text{Im}(I - L))$ . The projection of  $x$  on  $\text{Im} K$  lies in the unit ball of  $\text{Im} K$  which is compact. Hence we can consider in the computation of  $\theta$  the distance from  $S(\text{Im}(I - L))$  to the unit ball of  $\text{Im} K$  and the distance is thus attained.

Let

$$df \in S(\text{Im}(I - L)), \quad dg \in \text{Im} K$$

be corresponding points. One has

$$\theta = \frac{|(df, dg)|}{\|df\| \|dg\|}.$$

If now  $\theta = 1$ , then  $|(df, dg)| = \|df\| \|dg\|$  and hence  $df = \lambda dg$  where  $\lambda$  is a constant, and also  $df = (I - L)dh$ .

Now  $dg$  is extendable and  $df \in \text{Im}(I - L)$ . It follows that  $df = 0$ , a contradiction.

(A similar reasoning is valid in case the max in the definition of  $\theta$  is given by the second term.)

It follows that  $\theta < 1$  and  $(^{++})$  has the solution.

$$\omega = -Kp = K \sum_{n=0}^{\infty} [I - (K + L)]^n (I - L)s.$$

NOTE. Instead of  $\mathfrak{S}_1$  one could work in a closed subset of  $\mathfrak{S}_1$  e.g.  $h_{[\beta]}(\bar{A})$ .

The uniqueness is discussed as before: we get uniqueness provided

$$\text{Im} K \cap \text{Im} L = \{0\}.$$

i.e. no differential in the image of  $L$  is extendable to  $\bar{W}$ .

If  $\omega_1$  and  $\omega_2$  are solutions, then

$$(1 - L)(\omega_1 - \omega_2) = 0.$$

Now

$$\omega_i = -Kp_i \quad i = 1, 2.$$

So

$$(I - L)K(p_1 - p_2) = 0 .$$

Hence if

$$\text{Im } K \cap \text{Im } L = \{0\} , \quad p_1 = p_2 \quad \text{and} \quad \omega_1 = \omega_2 .$$

Conversely, if there is a differential  $\tau \in \mathfrak{S}_1$  such that

$$\tau = L\mu = K\nu$$

then if  $\omega$  is a solution in  $\mathfrak{R}$  of

$$\omega - s = L(\omega - s)$$

we have

$$\omega + \tau - s = \tau + L(\omega - s) = L(\mu + \omega - s) = L(\tau + \omega - s)$$

and uniqueness is lost.

As examples we could take:

(i)  $L = A_0$  orthogonal projection on  $h_{0\beta}^*(\bar{A})$  .

Then

$$\text{Im } (I - L) = h'_{0\alpha}(\bar{A})$$

and

$$\text{Im } (I - L) \cap H_{ext}(\bar{A}) = \{0\} \quad \text{and} \quad \text{Im } L \cap H_{ext}(\bar{A}) = \{0\} ;$$

(ii)  $L = A_1$  orthogonal projection on  $h'_{0\beta}(\bar{A})$ . Similar results are valid.

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