ON $p$-THETIC GROUPS

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The subject of this paper is a class of locally compact abelian (LCA) groups. Let \( p \) be a prime and let \( \mathbb{Z}(p^\infty) \) denote the group of complex \( p \)-th roots of unity equipped with the discrete topology. An LCA group \( G \) is called \( p \)-thetic if it contains a dense subgroup algebraically isomorphic to \( \mathbb{Z}(p^\infty) \). It is shown that a \( p \)-thetic LCA group is either compact or is topologically isomorphic to \( \mathbb{Z}(p^\infty) \). This fact leads to the formulation of a property which characterizes the \( p \)-thetic, the monothetic, and the solenoidal groups. Applications to some purely algebraic questions are presented.

Let us take a paragraph to settle notation. Throughout, all groups are assumed to be LCA Hausdorff topological groups. Some LCA groups which we shall mention frequently are the integers \( \mathbb{Z} \) taken discrete, the additive group \( \mathbb{Q} \) of the rationals taken discrete, the additive group \( \mathbb{R} \) of the real numbers with the usual topology, the circle \( T \), the cyclic groups \( \mathbb{Z}(n) \) of order \( n \), and the quasicyclic groups \( \mathbb{Z}(p^\infty) \), where \( p \) is a prime. Probably the most important group which we shall use is the group of \( p \)-adic integers, where \( p \) is a prime (see [2, §1] or [7, §10] for the definition and notation). The group of \( p \)-adic integers with its usual compact topology is written \( \mathbb{J}_p \); we use \( I_p \) to stand for the \( p \)-adic integers with the discrete topology. If \( G \) is an LCA group, then \( \hat{G} \) stands for the character (or dual) group of \( G \). In [7, 25.2] it is shown that the dual of \( \mathbb{J}_p \) is \( \mathbb{Z}(p^\infty) \). If \( G \) is a group, we let \( B(G) \) denote the torsion subgroup of \( G \), while \( B_p(G) \) denotes the set of elements of \( G \) whose order is a power of a fixed prime \( p \). Topological isomorphism is denoted by \( \cong \).

THEOREM 1. Let \( G \) be a \( p \)-thetic LCA group. Then either \( G \) is compact or else \( G \) is topologically isomorphic to \( \mathbb{Z}(p^\infty) \).

Proof. Since \( G \) is \( p \)-thetic, there is a continuous homomorphism \( f: \mathbb{Z}(p^\infty) \rightarrow G \) having dense image. Hence the transpose map \( f^*: \hat{G} \rightarrow \mathbb{J}_p \) is one-one [7, 24.41]. We wish to show that either \( \hat{G} \) is discrete or \( \hat{G} \cong \mathbb{J}_p \). We first note that \( \hat{G} \) must be totally disconnected, since \( f^* \) is one-one and \( \mathbb{J}_p \) is totally disconnected. Thus \( \hat{G} \) contains a compact open subgroup \( U \). If \( U \) is trivial, then \( \hat{G} \) is discrete. Otherwise, \( f^*(U) \) is a nontrivial compact subgroup of \( \mathbb{J}_p \) and is hence open in \( \mathbb{J}_p \) [7, 10.16(a)]. Now the restriction of \( f^* \) to the compact subgroup \( U \) is a topological isomorphism from \( U \) onto the open subgroup \( f^*(U) \)
of $J_p$. Hence $f^*$ is an open mapping, so that $\hat{G}$ is topologically isomorphic to $f^*(\hat{G})$. Since every closed subgroup of $J_p$ is topologically isomorphic to $J_p$ itself, we conclude that $\hat{G} \cong J_p$. This completes the proof.

Now let $G$ and $H$ be LCA groups. We say that $G$ is $H$-dense if there exists a continuous homomorphism $f: H \rightarrow G$ such that $f(H)$ is a dense subgroup of $G$. Thus the monothetic groups are the $Z$-dense groups, the solenoidal groups are the $R$-dense groups, and the $p$-thetic groups are just the $Z(p^\infty)$-dense groups. As is well known, the LCA monothetic and solenoidal groups are either compact, or else topologically isomorphic to $Z$ and $R$, respectively [7, 9.1]. As we have just proved, a $p$-thetic LCA group is either compact or is topologically isomorphic with $Z(p^\infty)$. These facts lead us to the very natural question: For which LCA groups $H$ is it the case that every $H$-dense LCA group $G$ is either compact or is topologically isomorphic to $H$? Since every $H$-dense group $G$ is automatically compact for compact $H$, the question is of interest only for noncompact $H$. It is not difficult to determine the answer to this question, and our answer will show that, in a sense, the study of the $p$-thetic groups complements the study of the monothetic and solenoidal groups.

**Theorem 2.** Let $H$ be a non-compact LCA group. The following are equivalent:

1. Every $H$-dense LCA group $G$ is either compact or is topologically isomorphic to $H$.
2. $H$ is topologically isomorphic with either $Z$, $R$, or $Z(p^\infty)$, where $p$ is a prime.

**Proof.** We have already shown that (2) → (1). For the converse, assume that (1) holds for $H$. We show that any strictly stronger topology on $\hat{H}$ which makes $\hat{H}$ into a locally compact group must be the discrete topology. To this end, let $D$ denote $\hat{H}$ with a strictly stronger locally compact topology. Then the identity map $i: D \rightarrow \hat{H}$ is continuous and one-one, so that the transpose map $i^*: D \rightarrow \hat{D}$ has dense image [7, 24.41]. Since (1) holds, either $\hat{D} \cong H$ or else $\hat{D}$ is compact. Since the first alternative has been ruled out, we conclude that $D$ is discrete, as we wished to show. We now invoke [9, Theorem 2] or [10, Theorem 2.1] to conclude that $\hat{H}$ contains an open subgroup $U$ which is topologically isomorphic with either $T$, $R$ or $J_p$ for some prime $p$. Hence $\hat{U}$ is a quotient $H$ by a closed subgroup. If $\pi: H \rightarrow \hat{U}$ is the projection of $H$ onto $\hat{U}$, we conclude from (1) that either $H \cong \hat{U}$ or else $\hat{U}$ is compact. Since $\hat{U}$ is not compact, we conclude that $H \cong \hat{U}$, so that $H \cong Z$, $H \cong R$, or $H \cong Z(p^\infty)$. Thus (1) → (2), which completes the proof.
Since a compact group is $p$-thetic if and only if its discrete dual is isomorphic to a subgroup of the discrete group $I_p$ of $p$-adic integers, we will do well, before mentioning some examples and simple properties of $p$-thetic groups, to recall a few basic properties of the group $I_p$, all of which may be found in [4] and [5]. The group $I_p$ is a reduced, torsion-free group of cardinality (and hence rank) of the power of the continuum. It contains an isomorphic copy of the group $Q_p$ consisting of all rational numbers with denominators prime to $p$. The group $I_p$ contains no elements of infinite $p$-height, but every element has infinite $q$-height if $q$ is a prime different from $p$ (we say that an element $x$ in an additively written group $G$ has infinite $p$-height if the equation $p^ny = x$ can be solved for $y$ in $G$ for an arbitrary positive integer $n$).

We now mention a few examples of $p$-thetic groups. The circle $T$ is $p$-thetic for all primes $p$. In fact, since $I_p$ has rank the power of the continuum, it contains isomorphic copies of the free abelian group of rank $M$ if $M$ does not exceed the power of the continuum. Thus the torus $T^n$ is $p$-thetic for all $p$ if and only if $M$ does not exceed the power of the continuum. Other examples of $p$-thetic groups are $\hat{Q}_p$ and $\hat{I}_p$. These groups are $p$-thetic for only the one prime $p$.

The group $\hat{I}_p$ (which is the Bohr compactification of $Z(p^\infty)$) is the “largest compact $p$-thetic group” in the sense that every compact $p$-thetic group (where $p$ is a fixed prime) is a quotient of $\hat{I}_p$ by a closed subgroup.

Every compact $p$-thetic group is a connected monothetic group [7, 25.13] and is hence solenoidal [7, 25.14]. Obviously, the torsion subgroup of a $p$-thetic group is dense in the group, but it is easy to give examples of compact solenoidal groups with dense torsion subgroup which are not $p$-thetic for any prime $p$. For example, let $G$ be the dual of the direct sum (taken discrete) of the groups $Q_p$ and $Q_q$, where $p$ and $q$ are distinct primes. It is easy to see that $G$ could not be isomorphic to a subgroup of a $p$-adic integer group (see the remarks above about $p$-height), and the fact that $G$ has dense torsion subgroup follows from [8, Theorem 2] or [1, Proposition 7].

Professor L. Fuchs has kindly informed one of the authors that, to the best of his knowledge, necessary and sufficient conditions for a group to be embeddable in $\hat{I}_p$ are unknown. Therefore we are unable to give intrinsic characterizations of the $p$-thetic groups, as we can for the monothetic and solenoidal groups (in terms of weight, rank, etc.). The remainder of this paper will be concerned with certain special $p$-thetic groups and their application to the theory of infinite abelian groups.

**Theorem 3.** Let $G$ be a compact connected group of dimension one. Then either $G \cong \hat{Q}$ or else $G$ is $p$-thetic for some prime $p$. 
Proof. If $G$ is torsion-free it follows from [7, 24.28 and 25.8] that $G \cong \hat{Q}$. Otherwise $G$ contains an isomorphic copy $H$ of $Z(p^\infty)$ for some prime $p$, by the structure theorem for divisible groups [7, A. 14] and the fact that a connected LCA group is divisible [7, 24.24]. We shall show that the closure $\bar{H}$ of $H$ is dense in $G$. Since $H$ is divisible, it follows that every non-trivial continuous character of $\bar{H}$ has infinite range, so that $(\bar{H})$ is torsion-free. But $(\bar{H}) \cong \hat{G}/A(\hat{G}, H)$, where $A(\hat{G}, H)$ is the annihilator of $H$ in $\hat{G}$ (see [7, 24.5]). Since every proper quotient of a subgroup of $Q$ is a torsion group, and since every group of rank one is isomorphic to a subgroup of $Q$ [7, A.16], it follows that $A(\hat{G}, H) = \{1\}$, so that $\bar{H} = G$, and therefore $G$ is $p$-thetic.

Remark 1. The group $G$ in Theorem 3 may be $p$-thetic for all $p$, e.g. $G = T$. However, the circle is not the only one-dimensional compact group which is $p$-thetic for all $p$. For example, let us define a subgroup $H$ of $Q$ in the following way. Let $p_n$ denote the $n$th prime and let $H_n$ denote the set of rational numbers of the form $k/(p_1p_2\cdots p_n)$, where $k$ is an integer. The sets $H_n$ define an ascending sequence of subgroups of $Q$. If we let $H$ be the union of the $H_n$, then we can show that $H$ is isomorphic to a subgroup of $Q_p$ for each $p$, but that $H$ is not isomorphic to $Z$. Thus if we set $G = \hat{H}$, we have an example of a one-dimensional compact group which is $p$-thetic for all $p$ but is not isomorphic to $T$.

Before proceeding to our next results, we review briefly the concepts of purity and $p$-purity. If $G$ is a group and $n$ a positive integer, we write $nG$ for the set of elements of $G$ of the form $nx$, where $x$ is in $G$. A subgroup $H$ of a group $G$ is called pure if and only if $nH = H \cap nG$ for each positive integer $n$ and $p$-pure if and only if $p^nH = H \cap p^nG$ for each positive integer $n$, where $p$ is a prime. It is easy to see that if $G$ is torsion-free, a subgroup $H$ is pure (respectively, $p$-pure) if and only if $G/H$ is torsion-free (respectively, $B_p(G/H) = \{0\}$).

Definition 1. Let $G$ be a compact $p$-thetic group. We say that $G$ is pure $p$-thetic if and only if $B(G) \cong Z(p^\infty)$ and that $G$ is $p$-pure $p$-thetic if and only if $B_p(G) \cong Z(p^\infty)$.

Before proceeding to justify the use of the terminology of this definition, we need to state a lemma.

Lemma 1. Let $H$ be a $p$-pure subgroup of $I_p$. Then the index of $pH$ in $H$ is $p$.

Proof. First note that since $H$ has no elements of infinite $p$-
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Let $x = (x_0, x_1, \cdots)$ be an element in $H$ but not in $pH$. Note that $x_0 \neq 0$, since otherwise $x$ would be in $pI_p$ and hence in $pH$, since $H$ is $p$-pure. We claim that the coset $x + pH$ is a generator of the quotient group $H/pH$, so that $H/pH \cong Z(p)$. To see this, let $w + pH$ be an element of $H/pH$, where $w = (w_0, w_1, \cdots)$ is in $H$. Let $y_i$ denote the first coordinate of $iw$, for $0 \leq i \leq p - 1$. Then $w_0 = y_i$ for some $i$ between 0 and $p - 1$. Hence $w - ix$ has 0 in its first coordinate, so that $w - ix$ is in $pH$. That is, $w + pH = i(x + pH)$, which completes the proof.

**Theorem 4.** Let $G$ be compact and let $p$ be a fixed prime. The following are equivalent:

1. $G$ is pure $p$-thetic,
2. $\hat{G}$ is isomorphic to a pure subgroup of $I_p$.

**Proof.** Assume (1). Since $G$ is $p$-thetic, there is a subgroup $H$ of $I_p$ such that $\hat{G} \cong H$. Let $G_n$ denote the subgroup of elements of $G$ having order $n$. By (1) it follows that $G_p \cong Z(p)$ and that $G_q$ is trivial for all primes $q \neq p$. We conclude from [7, 24. 22] that $H/pH \cong Z(p)$ and that $qH = H$ for all primes $q \neq p$. Let us assume, for the moment, that there is an element $x = (x_0, x_1, \cdots)$ in $H$ with $x_0 \neq 0$. In this case, we show that $H$ is pure in $I_p$. Clearly, it suffices to show that $H \cap p^nI_p = p^nH$. First, suppose that $py \in H$ for some $y$ in $I_p$. Since $H/pH \cong Z(p)$, we have that the coset $x + pH$ is a generator of $H/pH$. Thus $py + pH = ix + pH$ for some $i$ between 0 and $p - 1$. Hence there exists $z$ in $H$ such that $py = ix + pz$, so that $ix = p(y - z)$. This means that $ix$ has 0 in its first coordinate. This can occur only if $i = 0$, so that $y = z$, and hence $y$ is in $H$. This proves that $H \cap p^nI_p = pH$. That $H \cap p^nH = p^nH$ for all positive $n$ follows by a simple induction argument. Thus, in this case, $H$ is pure in $I_p$.

Finally, to show that the assumption about $x$ may always be made, we need only consider an appropriate subgroup $L_k$ of $I_p$, where $L_k$ consists of all sequences $x = (x_0, x_1, \cdots)$ in $I_p$ with $x_n = 0$ for $n$ less than $k$, and use the fact that $L_k \cong I_p$. This completes the proof that (1) $\implies$ (2).

Conversely, assume (2). Let $H$ be a pure subgroup of $I_p$ such that $\hat{G} \cong H$. Then $G$ is $p$-thetic, and it remains only to show that $B(G) \cong Z(p^\infty)$. By Lemma 1, $H/pH \cong Z(p)$, since a pure subgroup is automatically $p$-pure. Hence $G_p \cong Z(p)$, by [7, 24. 22]. Similarly, since $qH = H$ for all primes $q \neq p$ (since $H$ is pure in $I_p$), it follows that $G_q$ is trivial for $q \neq p$. Hence $B(G) \cong Z(p^\infty)$, so that $G$ is pure $p$-thetic, i.e. (2) $\implies$ (1).

**Remark 2.** The authors of [6] (see [4, Exercise 24 on p. 202])
show, without use of duality, that a reduced torsion-free group $H$ has a unique maximal subgroup if and only if $H$ is isomorphic to a pure subgroup of some group $I_p$. This can be deduced from Theorem 4 above in the following way. Let $H$ be as indicated. It follows from [8, Theorem 2] or [1, Proposition 7] that $B(G)$ is dense in $G$, where $G = \hat{H}$. Since $G$ must have unique minimal closed subgroup, and since $B(G)$ is divisible, it follows that $B(G) \cong \mathbb{Z}(p^\infty)$ for some prime $p$, so that $G$ is pure $p$-thetic. Hence $H$ is isomorphic to a pure subgroup of $I_p$ by Theorem 4. The converse is straightforward. Of course, it should be pointed out, going in the contrary direction, that our Theorem 4 can be deduced, via duality, from the result mentioned in [6].

**Theorem 5.** Let $G$ be compact and let $p$ be a fixed prime. The following are equivalent:

1. $G$ is $p$-pure $p$-thetic,
2. $G$ is isomorphic to a $p$-pure subgroup of $I_p$.

**Proof.** The proof of the implication $(1) \Rightarrow (2)$ follows along the same lines as the corresponding proof in Theorem 4, so that we omit it. Next, assume $(2)$. Thus $G$ is $p$-thetic, and it only remains to show that $B_p(G) \cong \mathbb{Z}(p^\infty)$. But this follows from Lemma 1, as in the proof of Theorem 4. Hence $(2) \Rightarrow (1)$, completing the proof.

**Remark 3.** In [2] Armstrong has shown, by a study of the extensibility of endomorphisms of $p$-pure subgroups of $I_p$, that a $p$-pure subgroup of $I_p$ must be indecomposable. We can provide an altogether different proof of this fact by using Theorem 5 above. We need only observe that a $p$-pure $p$-thetic group $G$ cannot be written as the the topological direct sum of two of its proper closed subgroups, since each summand would be $p$-thetic, whereas $B_p(G) \cong \mathbb{Z}(p^\infty)$.

In closing, we mention a criterion for a compact connected group to be $p$-pure $p$-thetic. This criterion is a direct translation, via duality, of a theorem due to Armstrong (see [3, Proposition 2]).

**Proposition 1.** Let $G$ be compact and connected, and let $p$ be a fixed prime. Then $G$ is $p$-pure $p$-thetic if and only if

1. $B_p(G)$ is dense in $G$,
2. $G$ is topologically indecomposable and $G/H$ is topologically indecomposable for every closed subgroup $H$ of $G$ such that $pH = H$.

**Proof.** This follows by duality from Armstrong's result mentioned above and the fact that if $H$ is a torsion-free abelian group, then a
subgroup \( U \) of \( H \) is \( p \)-pure if and only if its annihilator in \( \hat{H} \) is \( p \)-divisible.

**Remark 4.** It follows from the above proposition that the \( p \)-thetic group \( G \) defined in Remark 1 is \( p \)-pure \( p \)-thetic for each prime \( p \), since condition (1) holds, as shown in Remark 1, and condition (2) follows from the fact that \( G \) is of dimension one, so that it and all its quotients are topologically indecomposable.

**References**


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