Pacific Journal of Mathematics

REGULAR SEMIGROUPS WHICH ARE EXTENSIONS OF GROUPS

JANET E. MILLS

Vol. 41, No. 2 December 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 2, 1972

REGULAR SEMIGROUPS WHICH ARE EXTENSIONS OF GROUPS

JANET E. AULT

A semigroup V is an (ideal) extension of a semigroup T by a semigroup S with zero if T is an ideal of V and S is isomorphic to the Rees quotient V/T. Considered here are those semigroups which can be constructed as an extension of a group by a 0-categorical regular semigroup. The multiplication in such a semigroup is determined, along with an abstract characterization of the semigroup.

Let G be a group and S a 0-categorical regular semigroup. The problem of finding all extensions of G by S is essentially that of determining the associative multiplications on the set $V = G \cup (S\backslash 0)$ which make G an ideal of V. Such multiplications are characterized here completely in so far as semigroups are concerned. This description is made possible by a new use of the minimal primitive congruence on S as defined by T. E. Hall in [3].

Finally, having made such extensions, we give a characterization of those semigroups which can be constructed in this manner, that is, as an extension of a group by a 0-categorical regular semigroup.

1. Preliminary remarks.

For a semigroup S with zero, let S^* denote $S\backslash 0$, and E_S be the set of idempotents of S. Letting T be any semigroup, a function $\theta\colon S^* \to T$ satisfying the condition

$$(a\theta)(b\theta) = (ab)\theta \quad \text{if } ab \neq 0 \text{ in } S$$

is called a *partial homomorphism of S into T.*

By Theorem 4.19 of [2], every extension of a group by an arbitrary semigroup S with zero is completely determined by a partial homomorphism of S into the group. It is our task here to characterize all such functions in the case that S is a 0-categorical regular semigroup.

A subset A of a semigroup S is called *categorical* if for a, b, c in S, $abc \in A$ implies that $ab \in A$ or $bc \in A$. If S has a zero and $\{0\}$ is a categorical subset of S, then S is called *0-categorical* or *categorical at* 0.

Examples of 0-categorical semigroups include Rees matrix semigroups, primitive regular semigroups, ω-regular semigroups (see [1]), and of course, any semigroups having 0 as a prime ideal. For the class of primitive regular semigroups, Theorems 6.39 and 4.22 of [2] can be combined to characterize all partial homomorphisms of any primitive regular semigroup into a group. In the next section, the general problem will be reduced to just this particular case.

2. Construction of extensions. In this section, we let S be a regular semigroup which is categorical at 0. Define ρ_0 on S^* by

$$a\rho_0 b \quad \text{if} \quad ea = eb \neq 0,\, af = bf \neq 0 \quad \text{for some} \quad e, f \in E_S .$$

Let ρ_1 be the equivalence relation on S generated by ρ_0. Define ρ_2 on S by

$$a\rho_2 b \quad \text{if} \quad a = xcy,\, b = xdy, \quad \text{for some } c\rho_1 d .$$

Finally, define ρ on S by

$$a\rho b \quad \text{if} \quad a\rho_2 a_1,\, a_1\rho_2 a_2,\, \cdots,\, a_n\rho_2 b ,$$

for some $a_1, a_2, \cdots, a_n \in S$. Then T. E. Hall has shown in [3] that ρ is a 0-restricted congruence on S, that is, $\{0\}$ is a class of ρ, and, more importantly, that S/ρ is a primitive regular semigroup.

A *partial congruence* σ is an equivalence relation on S^* satisfying the property: for $a, b \in S^*$, $a\sigma b$ implies that $ax\sigma bx$ whenever $ax, bx \neq 0$, and $xa\sigma xb$ whenever $xa, xb \neq 0$.

Clearly, every partial homomorphism of S into a group G induces a partial congruence on S^*. In fact, since G is cancellative, the partial congruence σ is cancellative, that is, if $ax\sigma bx$ or $xa\sigma xb$, then $a\sigma b$.

Let $\rho^* = \rho | S^*$.

LEMMA. *The partial congruence ρ^* is contained in every cancellative partial congruence on S^*.*

Proof. The proof follows easily from cancellativity of the partial congruence and the fact that ρ is 0-restricted.

THEOREM. *Every extension of a group G by a 0-categorical regular semigroup S is uniquely determined by a partial homomorphism of the primitive regular semigroup S/ρ into G.*

In particular, a function $\theta: S^* \to G$ is a partial homomorphism if and only if $\theta = \eta\psi$, where $\eta: S \to S/\rho$ is the canonical homomorphism, and $\psi: (S/\rho)^* \to G$ is a partial homomorphism.

Proof. Let $\theta: S^* \to G$ be a partial homomorphism. Define σ on

S^* by $a\sigma b$ if $a\theta = b\theta$. Then σ is a cancellative partial congruence on S^*. Further, let $\eta: S \to S/\rho$ be the canonical homomorphism and define $\psi: (S/\rho)^* \to G$ by $\alpha\psi = a\theta$ if $a\eta = \alpha$. By the preceding lemma, we see that ψ is well-defined, and since θ is a partial homomorphism, so is ψ. Finally, for $a \in S^*$ we have $a\eta \neq 0$ and $a\eta\psi = (a\eta)\psi = a\theta$.

The converse follows since ρ is 0-restricted and the two functions η and ψ are partial homomorphisms.

3. Characterization of the resultant semigroups.

Now that all extensions of a group by a 0-categorical regular semigroup have been determined, it is natural to ask what semigroups can be constructed in this manner.

THEOREM. *A semigroup V is an extension of a group by a (0-categorical) regular semigroup if and only if V is a regular semigroup which contains a (categorical) minimal left ideal which is also a minimal right ideal.*

Proof. The direct part is clear since a group contains no proper left or right ideals. Conversely, let V be a regular semigroup with a minimal left ideal L which is also a minimal right ideal. By regularity, L contains an idempotent; moreover, L contains exactly one idempotent. For, if e and f are both idempotents in L, then $Vf = fV = L$ and there exist x, y in V so that $e = xf = fy$. From this it follows that $ef = xf = e$ and $fe = fy = e$. By symmetry, $f = fe = ef$, and thus, $e = f$.

Since L is an ideal of V, L is a regular semigroup with exactly one idempotent, that is, L is a group.

COROLLARY. *A semigroup V is an extension of a group by a (0-categorical) inverse semigroup if and only if V is an inverse semigroup containing a (categorical) minimal left ideal.*

Proof. The first part follows from the previous theorem and the fact that an extension of one inverse semigroup by another is again an inverse semigroup.

To prove the converse, it is sufficient to show that in an inverse semigroup V, a minimal left ideal L is also a minimal right ideal. Since V is an inverse semigroup, L is generated by a unique idempotent, and since L is minimal, this is the only idempotent in L. By commutativity of idempotents, it is easy to show that L must be the only minimal left ideal of V.

Now L is a right ideal, since, for $s \in V$, Ls is a minimal left ideal (see Theorem 2.32 of [2]), and thus $Ls = L$. Because L has exactly one idempotent, L must be a minimal right ideal.

References

1. Janet Ault and Mario Petrich, *The structure of ω-regular semigroups*, Bull. Amer. Math. Soc., **77** (1971), 196-199.
2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys No. 7, Amer. Math. Soc., **I** (1961), **II** (1967).
3. T. E. Hall, *Primitive homomorphic images of semigroups*, J. Aust. Math. Soc., **8** (1968), 350-354.

Received February 23, 1971 and in revised form July 21, 1971.

Some of the above results appear in the author's doctoral dissertation, written under the direction of Professor Mario Petrich at The Pennsylvania State University.

UNIVERSITY OF FLORIDA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics
Vol. 41, No. 2 December, 1972