REGULAR SEMIGROUPS WHICH ARE EXTENSIONS OF GROUPS

JANET E. MILLS
REGULAR SEMIGROUPS WHICH ARE EXTENSIONS OF GROUPS

JANET E. AULT

A semigroup \( V \) is an (ideal) extension of a semigroup \( T \) by a semigroup \( S \) with zero if \( T \) is an ideal of \( V \) and \( S \) is isomorphic to the Rees quotient \( V/T \). Considered here are those semigroups which can be constructed as an extension of a group by a \( 0 \)-categorical regular semigroup. The multiplication in such a semigroup is determined, along with an abstract characterization of the semigroup.

Let \( G \) be a group and \( S \) a \( 0 \)-categorical regular semigroup. The problem of finding all extensions of \( G \) by \( S \) is essentially that of determining the associative multiplications on the set \( V = G \cup (S\setminus 0) \) which make \( G \) an ideal of \( V \). Such multiplications are characterized here completely in so far as semigroups are concerned. This description is made possible by a new use of the minimal primitive congruence on \( S \) as defined by T. E. Hall in [3].

Finally, having made such extensions, we give a characterization of those semigroups which can be constructed in this manner, that is, as an extension of a group by a \( 0 \)-categorical regular semigroup.

1. Preliminary remarks. For a semigroup \( S \) with zero, let \( S^* \) denote \( S\setminus 0 \), and \( E_S \) be the set of idempotents of \( S \). Letting \( T \) be any semigroup, a function \( \theta: S^* \rightarrow T \) satisfying the condition

\[(a\theta)(b\theta) = (ab)\theta \quad \text{if} \quad ab \neq 0 \quad \text{in} \quad S\]

is called a partial homomorphism of \( S \) into \( T \).

By Theorem 4.19 of [2], every extension of a group by an arbitrary semigroup \( S \) with zero is completely determined by a partial homomorphism of \( S \) into the group. It is our task here to characterize all such functions in the case that \( S \) is a \( 0 \)-categorical regular semigroup.

A subset \( A \) of a semigroup \( S \) is called categorical if for \( a, b, c \) in \( S, abc \in A \) implies that \( ab \in A \) or \( bc \in A \). If \( S \) has a zero and \( \{0\} \) is a categorical subset of \( S \), then \( S \) is called \( 0 \)-categorical or categorical at \( 0 \).

Examples of \( 0 \)-categorical semigroups include Rees matrix semigroups, primitive regular semigroups, \( \omega \)-regular semigroups (see [1]),
and of course, any semigroups having 0 as a prime ideal. For the class of primitive regular semigroups, Theorems 6.39 and 4.22 of [2] can be combined to characterize all partial homomorphisms of any primitive regular semigroup into a group. In the next section, the general problem will be reduced to just this particular case.

2. Construction of extensions. In this section, we let $S$ be a regular semigroup which is categorical at 0. Define $\rho_0$ on $S^*$ by

$$a \rho_0 b \text{ if } ea = eb \neq 0, af = bf \neq 0 \text{ for some } e, f \in E_s .$$

Let $\rho_1$ be the equivalence relation on $S$ generated by $\rho_0$. Define $\rho_2$ on $S$ by

$$a \rho_2 b \text{ if } a = xcy, b = xdy, \text{ for some } c, d .$$

Finally, define $\rho$ on $S$ by

$$a \rho b \text{ if } a \rho_0 a_1 a_2 \cdots a_n \rho_2 b ,$$

for some $a_1, a_2, \ldots, a_n \in S$. Then T. E. Hall has shown in [3] that $\rho$ is a 0-restricted congruence on $S$, that is, $[0]$ is a class of $\rho$, and, more importantly, that $S/\rho$ is a primitive regular semigroup.

A partial congruence $\sigma$ is an equivalence relation on $S^*$ satisfying the property: for $a, b \in S^*$, $a \sigma b$ implies that $axb \sigma bx$ whenever $ax, bx \neq 0$, and $xa \sigma xb$ whenever $xa, xb \neq 0$.

Clearly, every partial homomorphism of $S$ into a group $G$ induces a partial congruence on $S^*$. In fact, since $G$ is cancellative, the partial congruence $\sigma$ is cancellative, that is, if $axb \sigma bx$ or $xa \sigma xb$, then $a \sigma b$.

Let $\rho^* = \rho | S^*$. 

**Lemma.** The partial congruence $\rho^*$ is contained in every cancellative partial congruence on $S^*$.

**Proof.** The proof follows easily from cancellativity of the partial congruence and the fact that $\rho$ is 0-restricted.

**Theorem.** Every extension of a group $G$ by a 0-categorical regular semigroup $S$ is uniquely determined by a partial homomorphism of the primitive regular semigroup $S/\rho$ into $G$.

In particular, a function $\theta: S^* \rightarrow G$ is a partial homomorphism if and only if $\theta = \eta \psi$, where $\eta: S \rightarrow S/\rho$ is the canonical homomorphism, and $\psi: (S/\rho)^* \rightarrow G$ is a partial homomorphism.

**Proof.** Let $\theta: S^* \rightarrow G$ be a partial homomorphism. Define $\sigma$ on
S* by \(a\sigma b\) if \(a\theta = b\theta\). Then \(\sigma\) is a cancellative partial congruence on \(S^*\). Further, let \(\eta: S \to S/\rho\) be the canonical homomorphism and define \(\psi: (S/\rho)^* \to G\) by \(\alpha\psi = a\theta\) if \(a\eta = \alpha\). By the preceding lemma, we see that \(\psi\) is well-defined, and since \(\theta\) is a partial homomorphism, so is \(\psi\). Finally, for \(a \in S^*\) we have \(a\eta \neq 0\) and \(a\eta \psi = (a\eta)\psi = a\theta\).

The converse follows since \(\rho\) is 0-restricted and the two functions \(\eta\) and \(\psi\) are partial homomorphisms.

3. Characterization of the resultant semigroups. Now that all extensions of a group by a 0-categorical regular semigroup have been determined, it is natural to ask what semigroups can be constructed in this manner.

**Theorem.** A semigroup \(V\) is an extension of a group by a (0-categorical) regular semigroup if and only if \(V\) is a regular semigroup which contains a (categorical) minimal left ideal which is also a minimal right ideal.

**Proof.** The direct part is clear since a group contains no proper left or right ideals. Conversely, let \(V\) be a regular semigroup with a minimal left ideal \(L\) which is also a minimal right ideal. By regularity, \(L\) contains an idempotent; moreover, \(L\) contains exactly one idempotent. For, if \(e\) and \(f\) are both idempotents in \(L\), then \(Vf = fV = L\) and there exist \(x, y\) in \(V\) so that \(e = xf = fy\). From this it follows that \(ef = xf = e\) and \(fe = fy = e\). By symmetry, \(f = fe = ef\), and thus, \(e = f\).

Since \(L\) is an ideal of \(V\), \(L\) is a regular semigroup with exactly one idempotent, that is, \(L\) is a group.

**Corollary.** A semigroup \(V\) is an extension of a group by a (0-categorical) inverse semigroup if and only if \(V\) is an inverse semigroup containing a (categorical) minimal left ideal.

**Proof.** The first part follows from the previous theorem and the fact that an extension of one inverse semigroup by another is again an inverse semigroup.

To prove the converse, it is sufficient to show that in an inverse semigroup \(V\), a minimal left ideal \(L\) is also a minimal right ideal. Since \(V\) is an inverse semigroup, \(L\) is generated by a unique idempotent, and since \(L\) is minimal, this is the only idempotent in \(L\). By commutativity of idempotents, it is easy to show that \(L\) must be the only minimal left ideal of \(V\).

Now \(L\) is a right ideal, since, for \(s \in V\), \(Ls\) is a minimal left ideal (see Theorem 2.32 of [2]), and thus \(Ls = L\). Because \(L\) has exactly
one idempotent, \( L \) must be a minimal right ideal.

**REFERENCES**


Received February 23, 1971 and in revised form July 21, 1971.

Some of the above results appear in the author's doctoral dissertation, written under the direction of Professor Mario Petrich at The Pennsylvania State University.
Tom M. (Mike) Apostol, *Arithmetical properties of generalized Ramanujan sums* ................................................................. 281
Janet E. Mills, *Regular semigroups which are extensions of groups* ................................................................. 303
Gregory Frank Bachelis, *Homomorphisms of Banach algebras with minimal ideals* ................................................................. 307
John Allen Beachy, *A generalization of injectivity* ................................................................. 313
David Geoffrey Cantor, *On arithmetic properties of the Taylor series of rational functions. II* ................................................................. 329
Václav Chvátal and Frank Harary, *Generalized Ramsey theory for graphs. III. Small off-diagonal numbers* ................................................................. 335
Frank Rimi DeMeyer, *Irreducible characters and solvability of finite groups* ................................................................. 347
Robert P. Dickinson, *On right zero unions of commutative semigroups* ................................................................. 355
John Dustin Donald, *Non-openness and non-equidimensionality in algebraic quotients* ................................................................. 365
John D. Donaldson and Qazi Ibadur Rahman, *Inequalities for polynomials with a prescribed zero* ................................................................. 375
Robert E. Hall, *The translational hull of an N-semigroup* ................................................................. 379
John P. Holmes, *Differentiable power-associative groupoids* ................................................................. 391
Steven Kenyon Ingram, *Continuous dependence on parameters and boundary data for nonlinear two-point boundary value problems* ................................................................. 395
Robert Clarke James, *Super-reflexive spaces with bases* ................................................................. 409
Gary Douglas Jones, *The embedding of homeomorphisms of the plane in continuous flows* ................................................................. 421
Mary Joel Jordan, *Period H-semigroups and t-semisimple periodic H-semigroups* ................................................................. 437
Ronald Allen Knight, *Dynamical systems of characteristic 0* ................................................................. 447
Kwangil Koh, *On a representation of a strongly harmonic ring by sheaves* ................................................................. 459
Hui-Hsiung Kuo, *Stochastic integrals in abstract Wiener space* ................................................................. 469
Thomas Graham McLaughlin, *Supersimple sets and the problem of extending a retracing function* ................................................................. 485
William Nathan, *Open mappings on 2-manifolds* ................................................................. 495
M. J. O’Malley, *Isomorphic power series rings* ................................................................. 503
Sean B. O’Reilly, *Completely adequate neighborhood systems and metrization* ................................................................. 513
Qazi Ibadur Rahman, *On the zeros of a polynomial and its derivative* ................................................................. 525
Russell Daniel Rupp, Jr., *The Weierstrass excess function* ................................................................. 529
Hugo Teufel, *A note on second order differential inequalities and functional differential equations* ................................................................. 537
M. J. Wicks, *A general solution of binary homogeneous equations over free groups* ................................................................. 543