

# Pacific Journal of Mathematics

**HOMOMORPHISMS OF BANACH ALGEBRAS WITH MINIMAL  
IDEALS**

GREGORY FRANK BACHELIS

## HOMOMORPHISMS OF BANACH ALGEBRAS WITH MINIMAL IDEALS

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Let  $A$  be a semi-simple Banach algebra with socle  $F$ , and let  $\nu$  be a homomorphism of  $A$  into a Banach algebra. It is shown that if  $I$  is a minimal one-sided ideal of  $A$ , then the restriction of  $\nu$  to  $I$  is continuous. This is then used to deduce continuity properties of the restriction of  $\nu$  to  $F$ . In particular, if  $F$  has a bounded left or right approximate identity, then  $\nu$  is continuous on  $F$ .

In [1] and [2] we deduced continuity properties of  $\nu|_F$  in case  $A$  was a semi-simple annihilator Banach algebra. In this paper we obtain essentially the same results, but without the hypothesis that  $A$  be an annihilator algebra.

We first show that the restriction of  $\nu$  to any minimal one-sided ideal is continuous. The proof is almost purely algebraic. We then show that there exists a constant  $K$  such that

$$\|\nu(xy)\| \leq K \|x\| \|y\|, \quad x \in F, \quad y \in \bar{F}.$$

As a corollary we obtain that  $\nu|_F$  is continuous if  $F$  has a bounded left or right approximate identity.

1. Preliminaries. Throughout this section we assume that  $A$  is a complex semi-simple Banach algebra. The socle,  $F$ , is defined to be the sum of the minimal right ideals. An idempotent  $e$  is called *minimal* if  $eA$  is a minimal right ideal. We use without reference the basic facts about the socle of a Banach algebra (see e.g. [7, pp. 45–47]).

The following two lemmas, together with the “Main Boundedness Theorem” of Bade and Curtis ([3, Thm. 2.1], [2, Thm. 4.1]) are the basic ingredients in the proofs that follow. The first lemma is due essentially to Barnes.

LEMMA 1.1. *Let  $\{x_1, \dots, x_n\} \subset F$ . Then there exist idempotents  $e$  and  $f$  in  $F$  such that  $\{x_1, \dots, x_n\} \subset eAf$  and  $eAf$  is finite-dimensional.*

*Proof.* By hypothesis, there exist minimal right ideals,  $I_1, \dots, I_m$ , whose sum contains  $\{x_1, \dots, x_n\}$ . By [4, Thm. 2.2], there exists an idempotent  $e \in F$  such that  $eA = I_1 + \dots + I_m$ . Thus  $x_k \in eA$ ,  $1 \leq k \leq n$ .

Similarly, there exists an idempotent  $f \in F$  such that  $x_k \in Af$ ,  $1 \leq k \leq n$ . Hence  $x_k = ex_k f \in eAf$ ,  $1 \leq k \leq n$ .

If  $u$  and  $v$  are minimal idempotents, and  $uAv \neq (0)$ , then  $uAv$  is one-dimensional [9, Lemma 5.1]. Since  $e$  and  $f$  are each the sum of minimal idempotents,  $eAf$  is finite-dimensional.

**LEMMA 1.2.** *Let  $e$  be a minimal idempotent and suppose that  $eA$  is infinite-dimensional. Then there exists a sequence of minimal idempotents  $\{g_n\}$  such that  $g_n g_m = 0$ ,  $n \neq m$ , and  $eAg_n \neq (0)$  for all  $n$ .*

*Proof.* Let  $g_1 = e$ . Assume that  $g_1, \dots, g_n$  have been chosen with the desired properties. Let  $f = g_1 + \dots + g_n$ . Then  $f = f^2$  and  $eAf$  is finite-dimensional. Thus there exists  $x \in eA$  such that  $x(1-f) \neq 0$ . Since  $eA$  is a minimal right ideal, there exists  $w \in A$  such that  $x(1-f)w = e$ . Let  $g_{n+1} = (1-f)wex(1-f)$ . Then  $f g_{n+1} = g_{n+1} f = 0$ , so  $g_k g_{n+1} = g_{n+1} g_k = 0$ ,  $1 \leq k \leq n$ . Also

$$\begin{aligned} g_{n+1}^2 &= (1-f)wex(1-f)(1-f)wex(1-f) \\ &= (1-f)wex(1-f)wex(1-f) \\ &= (1-f)wex(1-f) \\ &= g_{n+1}. \end{aligned}$$

Since  $e$  is minimal,  $g_{n+1}$  is as well. Since

$$exg_{n+1} = ex(1-f) = x(1-f) \neq 0,$$

$eAg_{n+1} \neq (0)$ . The conclusion follows by induction.

**NOTE.** Lemma 1.2 above takes the place of [2, Lemma 2.2] in what follows. Evidently the latter does not hold in this more general situation, since the norm induced in  $eA$  as a subset of  $(Ae)^*$  (the set of bounded linear functionals on  $Ae$ ) need not be equivalent to the given norm on  $eA$  (see Remark 2.5).

**2. The main results.** Throughout this section we assume that  $A$  is a complex semi-simple Banach algebra with socle  $F$  and that  $\nu$  is a homomorphism of  $A$  into a Banach algebra. We first show the following.

**THEOREM 2.1.** *If  $I$  is a minimal one-sided ideal, then  $\nu|I$  is continuous.*

*Proof.* Suppose that  $I$  is a minimal right ideal. Then there exists a minimal idempotent  $e$  such that  $I = eA$ .

Let  $J = \{x \in A \mid y \mapsto \nu(xy) \text{ is continuous on } A\}$ . Then one verifies that  $J$  is a two-sided ideal in  $A$  [8, p. 153], and that an idempotent  $g$  is in  $J$  if and only if  $\nu \mid gA$  is continuous.

We may assume  $eA$  is infinite-dimensional, since otherwise the conclusion trivially follows. Choose  $\{g_n\}$  as given by Lemma 1.2. If  $g_n \in J$  then there exists  $x_n \in g_nA$  such that  $\|x_n\| = 1$  and  $\|\nu(x_n)\| > n \|g_n\|$ . Since  $g_n x_n = x_n$  and  $g_m x_n = g_m g_n x_n = 0$ ,  $m \neq n$ , the Main Boundedness Theorem [2, Thm. 4.1] implies that  $g_n \in J$  for some  $n$ . Since  $eA g_n \neq (0)$  and  $J$  is a left ideal, we have that  $eA \cap J \neq (0)$ . But  $J$  is a right ideal and  $eA$  is a minimal right ideal. Thus  $e \in eA \subset J$ , and  $\nu \mid eA$  is continuous.

REMARK 2.2. (cf. [3, p. 597]) If  $I = eA$  is an infinite-dimensional minimal right ideal, then it is always possible to construct a discontinuous homomorphism  $\nu$  of  $eA$  into a Banach algebra. For let  $\phi$  be a discontinuous linear functional on  $eA$ , and define

$$\|x\|_1 = \|x\| + |\phi(x)|, \quad x \in eA.$$

If  $x \in eA$ , then  $x e = \lambda e$ ,  $\lambda$  complex. Thus

$$x^n = (x e)^{n-1} x = \lambda^{n-1} x,$$

so

$$\|x^n\|^{1/n} = |\lambda|^{(n-1)/n} \|x\|^{1/n}.$$

Hence  $|\lambda| = \rho(x)$ , the spectral radius of  $x$ .

If  $x, y \in A$ , then

$$\begin{aligned} \|xy\|_1 &= \|xy\| + |\phi(xy)| \\ &\leq \|x\| \|y\| + |\phi(xey)| \\ &\leq \|x\| \|y\| + \rho(x) |\phi(y)| \\ &\leq \|x\| (\|y\| + |\phi(y)|) \\ &\leq \|x\|_1 \|y\|_1, \end{aligned}$$

so  $\|\cdot\|_1$  is a normed algebra norm on  $eA$ .

Now let  $B$  be the completion of  $eA$  in this norm and define  $\nu: eA \rightarrow B$  by  $\nu(x) = x$ . Then  $\nu$  is a discontinuous homomorphism of  $eA$ . By the above theorem,  $\nu$  does not extend to a homomorphism of  $A$ .

We now have:

THEOREM 2.3. *Let  $A$  be a semi-simple Banach algebra with socle  $F$  and let  $\nu$  be a homomorphism of  $A$  into a Banach algebra. Then there exists a constant  $K$  such that*

$$\|\nu(xy)\| \leq K \|x\| \|y\| \quad x \in F, \quad y \in \bar{F}$$

*Proof.* Since  $F$  is the sum of the minimal right ideals and also the sum of the minimal left ideals, it follows from Theorem 2.1 that, for any  $x \in F$ , the mappings  $y \rightarrow \nu(xy)$  and  $y \rightarrow \nu(yx)$  are continuous on  $A$ . Thus it suffices to show that

$$\sup_{x, y \in F} \frac{\|\nu(xy)\|}{\|x\| \|y\|} < \infty .$$

In addition, if  $e = e^2 \in F$ , then  $\nu|eA$  and  $\nu|Ae$  are continuous. With these observations, the proof is virtually the same as that of [2, Thm. 4.5], with [2, Lemma 2.1] replaced by Lemma 1.1.

**COROLLARY 2.4.** *If  $F$  has a bounded left or right approximate identity, then  $\nu$  is continuous on  $F$ .*

*Proof.* If  $F$  has a bounded left or right approximate identity, then of course so does  $\bar{F}$ . The proof now follows as that of [2, Cor. 4.9], with [1, Cor. 4.9] replaced by Lemma 1.1.

**REMARK 2.5.** Let  $X$  and  $Y$  be Banach spaces with  $Y \subset X$  and such that the inclusion map  $i: Y \rightarrow X$  is continuous and  $i(Y)^\perp = X$ . Let  $B(X, Y)$  denote the bounded operators from  $X$  to  $Y$  and let  $B'(X, Y)$  denote the compact operators from  $X$  to  $Y$ . Then  $B(X, Y)$  is a semi-simple Banach algebra with  $B'(X, Y)$  as a closed two-sided ideal. If  $A$  is a closed two-sided ideal of  $B(X, Y)$  containing  $B'(X, Y)$ , then  $A$  is semi-simple with socle  $F$  consisting of those bounded operators from  $X$  to  $Y$  with finite-dimensional range. Each minimal right ideal of  $A$  is linearly homeomorphic to  $X^*$  and each minimal left ideal is linearly homeomorphic to  $Y$ . Now  $i^*: Y^* \rightarrow X^*$  is one-to-one and continuous, but not bi-continuous if  $Y \neq X$ .

If  $X = Y$  and satisfies the metric approximation property [5, p. 178], then  $F$  has a bounded left approximate identity, so the above corollary applies to  $A$ . If in addition  $X$  has a continued bisection, then Johnson [6, Thm. 3.5] has shown that any homomorphism of  $A$  into a Banach algebra is continuous on  $B'(X, X) (= \bar{F})$ . He has also shown that any homomorphism of  $B(X, X)$  into a Banach algebra is continuous if  $X$  has a continued bisection [6, Thm. 3.3].

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