GENERALIZED RAMSEY THEORY FOR GRAPHS. III. SMALL OFF-DIAGONAL NUMBERS

Václav Chvátal and Frank Harary
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The classical Ramsey theory for graphs studies the Ramsey numbers \(r(m, n)\). This is the smallest \(p\) such that every 2-coloring of the lines of the complete graph \(K_p\) contains a green \(K_m\) or a red \(K_n\). In the preceding papers in this series, we developed the theory and calculation of the diagonal numbers \(r(F)\) for a graph \(F\) with no isolated points, as the smallest \(p\) for which every 2-coloring of \(K_p\) contains a monochromatic \(F\). Here we introduce the off-diagonal numbers: \(r(F_1, F_2)\) with \(F_1 \neq F_2\) is the minimum \(p\) such that every 2-coloring of \(K_p\) contains a green \(F_1\) or a red \(F_2\). With the help of a general lower bound, the exact values of \(r(F_1, F_2)\) are determined for all graphs \(F_i\) with less than five points having no isolates.

1. Introduction. The small \((p \leq 4\) points\) graphs \(F_i\) having no isolated points are shown in Figure 1, together with their symbolic names, following the notation for operations on graphs in the book [3, p. 21]. In fact, we follow the terminology and notation of this book throughout.

In [1, 2], we defined the number \(r(F)\) as the minimum \(p\) for which every 2-coloring (of the lines) of \(K_p\) contains a monochromatic subgraph \(F\). The number \(r(F_i, F'_i)\) is the corresponding smallest \(p\).
such that every 2-coloring of $K_p$ contains a green $F_1$ or a red $F_2$. Obviously $r(F) = r(F, F)$, so that the numbers $r(F)$ are diagonal within the $r(F_1, F_2)$.

There is an equivalent formulation of the definition of $r(F_1, F_2)$ in terms of graphical complementation rather than 2-colorings of a complete graph. Namely, $r(F_1, F_2)$ is the minimum $p$ such that whenever a $p$-point graph $G$ does not have $F_1$ as a subgraph, then its complement $\overline{G}$ contains $F_2$. It is convenient to assign numbers to the following immediate consequences of the definition: symmetry, monotonicity, and a crude lower bound,

\begin{align*}
(1) & \quad r(F_1, F_2) = r(F_2, F_1) \\
(2) & \quad F'_1 \subset F_1$ and $F'_2 \subset F_2$ imply $r(F'_1, F'_2) \leq (F_1, F_2) \\
(3) & \quad r(F_1, F_2) \geq \max (p(F_1), p(F_2)).
\end{align*}

When $F_1$ and $F_2$ are both complete graphs, we have specialized to $r(K_m, K_n) = r(m, n)$, the classical Ramsey numbers for graphs. As all the numbers $r(m, n)$ are known for $m, n = 2, 3, 4$, we begin with some information about off-diagonal Ramsey numbers for small $F_1$ and $F_2$. The existence of the diagonal numbers $r(n, n)$ was established by Ramsey [4] himself; that of all the other numbers $r(F_1, F_2)$ follows from (2).

From [3, p. 17], we have the following values of $r(m, n)$:

\[
\begin{array}{c|cccc}
  m & 2 & 3 & 4 \\
\hline
  2 & 2 & 3 & 4 \\
  3 & 6 & 9 \\
  4 & 18 \\
\end{array}
\]

In [2], the numbers $r(F)$ are determined for the 10 graphs of Fig. 1:

\[
\begin{array}{c|cccccccc}
  F & K_2 & P_2 & 2K_2 & K_3 & P_3 & K_{1,3} & C_4 & K_{1,3} + x & K_4 - x & K_4 \\
  r(F) & 2 & 3 & 5 & 6 & 5 & 6 & 6 & 7 & 10 & 18.
\end{array}
\]

It is obvious that $r(K_5, F) = p(F)$, the number of points in $F$.

2. The simplest Ramsey numbers. We now obtain two equations which give the next two rows in Table 1.1, the first for Ramsey numbers involving $2K_2$ and the second for $P_3$.

**Lemma 1.** For any graph $F$ with no isolates,
Proof. First, when $F$ is complete, we have $r(2K_2, F) > p(F) + 1$ because a 2-coloring of $K_{p+1}$ in which the green lines form just one triangle cannot have a red $K_p$. On the other hand, if a 2-coloring of $K_{p+1}$ has no green $2K_2$, then the green lines form either a star or a triangle, so there must be a red $K_p$.

Secondly when $F$ is not complete, it is a subgraph of $K_p$. In an arbitrary 2-coloring of $K_{p+1}$ which does not contain a green $2K_2$, the green lines again form a star or a triangle. When there is a green star, there must be a red $K_p$. And when we have a green triangle, there must appear a green $K_p - x$. Thus $r(2K_2, F) \leq p(F) + 1$. The equality follows from the 2-coloring of $K_p$ with red $K_{p-1}$ and a green star $K_{1,p-1}$.

The next question is a bit more subtle.

**Lemma 2.** For any graph $F$ with no isolates,

$$r(P_3, F) = \begin{cases} p(F) & \text{if } F \text{ has a 1-factor} \\ 2p(F) - 2\beta_3(F) - 1 & \text{otherwise.} \end{cases}$$

Proof. In each 2-coloring of $K_m$ without a green $P_3$, all the green lines are independent. In other words, the green graph is a subgraph of $[m/2]K_2$ or, equivalently, the red graph contains $K_m - [m/2]K_2$. (For $m$ even, this graph has been called a “party graph” by A. J. Hoffman because everyone talks to everyone else with the exception that nobody talks to his own spouse.) Thus, $r(P_3, F)$ is the smallest $m$ such that $F$ is a subgraph of $K_m - [m/2]K_2$.

For any graph $F$ with $p$ points, we have the maximum number of independent lines in the complement of $F$, $\beta_3(\overline{F}) = n$ if and only if $F \subset K_p - nK_1$. Thus, if $\overline{F}$ has a 1-factor, i.e., $\beta_3(\overline{F}) = p/2$, then we have $F \subset K_p - (p/2)K_2$ or $r(P_3, F) \leq p$. The equality follows trivially from (2).

Now, let $\overline{F}$ have no 1-factor, so that $\beta_3(\overline{F}) = n < p/2$. If $m = 2p - 2n - 1$, then any 2-coloring of $K_m$ having no green $P_3$ has a red $K_m - [m/2]K_2 = K_m - (p - n - 1)K_2$. We will show that such a coloring has a red $F$. Starting with the simple inclusion $(p - n - 1)K_2 \cup K_1 \subset nK_2 \cup (p - 2n)K_2$, and taking complements by merely removing the indicated number of independent lines from a complete graph of the proper size, we obtain $K_p - nK_1 \subset K_m - (p - n - 1)K_2$. Thus, we have $r(P_3, F) \leq 2p - 2n - 1$. On the other hand, the 2-coloring of $K_{m-1}$ which has just $(m - 1)/2 = p - n - 1$ green independent lines
and leaves as the remaining red graph $K_{m-1} - ((m - 1)/2)K_2$ already has no green $P_3$. It contains no red $F$ either, for otherwise $((m - 1)/2)K_2 \subseteq \bar{F}$ or equivalently $n = \beta_n(\bar{F}) > (m - 1)/2 = p - n - 1$, contradicting $n < p/2$ and proving Lemma 2.

3. A useful lower bound. For our last lemma, we easily derive a simple lower bound which is not at all sharp in general, but luckily happens to be rather useful in establishing the values of $r(F', F'_2)$ for the 10 small graphs of Fig. 1.

**Lemma 4.** Let $F_1$ and $F_2$ be two graphs (not necessarily different) with no isolated points. Let $c$ be the number of points in a largest connected component of $F_1$, and let $\chi$ be the chromatic number of $F_2$. Then the following lower bound holds:

$$r(F_1, F_2) \geq (c - 1)(\chi - 1) + 1.$$  

**Proof.** Consider the graph $G = (\chi - 1)K_{c-1}$. Since $G$ has no component with at least $c$ points, it cannot possibly contain $F_1$. On the other hand, the complement $G$ is $(\chi - 1)$-chromatic and hence cannot contain the $\chi$-chromatic graph $F_2$. The inequality follows at once, as $G$ has $(c - 1)(\chi - 1)$ points.

Remarkably, we shall find that in all but the two instances $r(K_1, C_4) \geq 4$ and $r(K_4 - x, K_4) \geq 10$, this lower bound turns out to yield the exact number for $r(F', F'_2)$.

**Figure 2**

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G:
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[Diagram of a graph G]
Referring to Table 2 below, we next show that better lower bounds than 4 and 10 respectively are given by

\[(4) \quad r(K_{1,3}, C_4) \geq 6\]

\[(5) \quad r(K_4 - x, K_4) \geq 11.\]

Later we will see that (4) and (5) give the correct values of these two Ramsey numbers.

To prove (4) we need only exhibit a graph \(G\) with 5 points such that \(G\) has no \(K_{1,3}\) (i.e., no point of degree exceeding 2) and \(\bar{G}\) has no 4-cycle. Clearly \(G = C_5\) works.

Similarly (5) can be verified by producing \(G\) with 10 points not containing \(K_4 - x\) such that \(\beta_0(G) < 4\). This example is a bit trickier, but we finally found it.

The graph \(G\) of Fig. 2 has just four triangles, no two having a common line. Hence \(G\) does not contain \(K_4 - x\). It is also easily seen that \(G\) has no set of 4 independent points.

4. Forcing forbidden subgraphs. For each pair \(F_1, F_2\) of forbidden graphs, we must argue that when the number \(r\) of points is right, every graph \(G\) with \(r\) points not containing \(F_1\) must have \(F_2\) in its complement. In particular, we will prove the next 8 upper bounds which establish the remaining off-diagonal Ramsey numbers.

\[(6) \quad r(P_4, K_{1,3}) \leq 5\]

\[(7) \quad r(P_4, C_4) \leq 5\]

\[(8) \quad r(K_{1,3}, C_4) \leq 6\]

\[(9) \quad r(K_{1,3} + x, K_4 - x) \leq 7\]

\[(10) \quad r(C_4, K_4 - x) \leq 7\]

\[(11) \quad r(K_{1,3} + x, K_4) \leq 10\]

\[(12) \quad r(C_4, K_4) \leq 10\]

\[(13) \quad r(K_4 - x, K_4) \leq 11.\]

Proof of (6) and (7). By coincidence, both (6) and (7) may be shown at one fell swoop. Let \(G\) have no 4-point path \(P_4\) on its 5 points. There are only two possibilities for such a graph: either \(G \subseteq K_5 \cup K_5\) or \(G \subseteq K_{1,4}\). Taking complements, \(K_{1,3} \subseteq \bar{G}\) or \(K_4 \subseteq \bar{G}\), so that necessarily both \(K_{1,3}\) and \(C_4\) are subgraphs of \(\bar{G}\).

Proof of (8). Taking \(G\) as a 6-point graph with all degrees \(\leq 2\)
forces $\bar{G}$ to have each degree $\geq 3$. Thus, in $\bar{G}$, the neighborhoods of any two nonadjacent points have at least two common points, so that $\bar{G}$ must contain $C_4$.

The next assertion (9) will automatically have several consequences by the monotonicity condition (2).

Proof of (9). Let $G$ be an arbitrary graph of 7 points not containing $K_{1,3} + x$. We assume $\bar{G}$ does not contain $K_4 - x$ and proceed to derive a contradiction. There are two possibilities, depending on whether $G \supseteq K_3$. If $G$ does have a triangle $u_i u_j u_k$, then there can be no line $u_i v_j$ in $G$. Now each pair of the points $v_j$ is forced to be adjacent in $G$, for otherwise $\bar{G}$ would contain $K_4 - x$. Hence the points $v_j$ induce $K_4$ in $G$, a contradiction.

Next, if $G$ has no triangle, then it has 3 independent points $u_i, u_j, u_k$ since $r(K_3, K_3) = r(K_3) = 6$. Again, we denote the remaining four points by $v_j$. Each $v_j$ must be adjacent in $G$ to at least two of the points $u_i$, for otherwise $G \supseteq K_i - x$. If there is even one line $v_i v_j$, then $G$ contains $K_{1,3} + x$, contrary to the hypothesis. Thus $\bar{G}$ is forced to contain $K_4$, and a fortiori $K_4 - x$.

We now apply (2) and the inclusions

$$K_{1,3} + x \supseteq K_{1,3}, P_4, K_3$$

to (9) to obtain at once the lower bounds

(14) \hspace{1cm} r(K_3, K_4 - x) \leq 7

(15) \hspace{1cm} r(P_4, K_4 - x) \leq 7

(16) \hspace{1cm} r(K_{1,3}, K_4 - x) \leq 7.

Similarly $K_4 - x \supseteq K_{1,3} + x, C_4, K_{1,3}, P_4$ and (2) applied to (14) give

(17) \hspace{1cm} r(K_3, P_4) \leq 7

(18) \hspace{1cm} r(K_3, K_{1,3}) \leq 7

(19) \hspace{1cm} r(K_3, C_4) \leq 7

(20) \hspace{1cm} r(K_3, K_{1,3} + x) \leq 7.

Similarly by (15),

(21) \hspace{1cm} r(P_4, K_{1,3} + x) \leq 7,

and by (16),
Proof of (10). Let $G$ be an arbitrary graph with 7 points and no $C_4$. We will assume $G \not\supset\, K_4 - x$ and deduce a contradiction.

In the proof, we distinguish two cases according to whether there is or is not a point $u$ of degree smaller than three. In the first case, we delete the point $u$ together with its neighbors and are left with a subgraph $H$ of $G$ having at least four points. Clearly, $H$ has no $C_4$ because $G$ has none. Thus, as $r(P_3, C_4) = 4$ by Lemma 2, $H$ is forced to contain $P_3$. By definition of $H$, $u$ is adjacent to no point in $H$. Therefore, $G$ contains $K_{4 - x}$, contradicting the assumption.

Next, we consider the second case where each point in $G$ has degree at least three. Now the inequality (9), $r(K_{1,3} + x, K_4 - x) \leq 7$, proved above, implies $K_{1,3} + x \subset G$. A fortiori, $G$ contains a triangle $u_1u_2u_3$. Now, since each point of $G$ has degree at least three and $G$ contains no $C_4$, we conclude that there are three other points $v_1, v_2, v_3$ such that $u_iv_i$ is a line of $G$ for each $i = 1, 2, 3$. In other words, $G$ contains the subgraph shown in Figure 3. Actually, it is easy to check that the graph in Fig. 3 is the subgraph of $G$ induced by $u_1, u_2, u_3, v_1, v_2, v_3$, for the addition of any line to this graph produces $C_4$. But then $G$ contains $K_{4 - x}$ with points $u_1, v_1, v_2, v_3$ again contradicting the assumption.

![Figure 3](image-url)

Proof of (11). Assume there is a graph $G$ with 10 points such that $G$ contains no $K_{1,3} + x$ and $\beta_1(G) < 4$. As $r(K_5, K_4) = r(3, 4) = 9$, $G$ contains a triangle $u_1u_2u_3$. Let the other points in $G$ be $v_j(j = 1, 2, \ldots, 7)$. There cannot be any line $u_iv_j$ for otherwise $G$ would contain a $K_{1,3} + x$. Now, let us consider the subgraph $H$ of $G$ spanned by the
Now we can apply (2) and the inclusions $K_{1,3} + x \supseteq K_{1,3}$, $P_4$ to (11) to obtain two more upper bounds,

\begin{align}
(23) & \quad r(K_{1,3}, K_4) \leq 10 \\
(24) & \quad R(P_4, K_4) \leq 10.
\end{align}

It is quite convenient to have another lemma for the proof of (12).

**Lemma 3.** If a graph $G$ with $p$ points has minimum degree $d$ and $d(d - 1) > p - 1$, then $G$ contains $C_4$.

**Proof.** Let $n$ be the total number of paths $P_3$ contained in $G$. There are exactly $p$ choices for the midpoint of $P_3$, and for each fixed midpoint at least $\binom{d}{2}$ choices of the endpoints. Therefore $n \geq p \binom{d}{2} > \binom{p}{2}$ so there must be two distinct paths $P_3$ in $G$ with the same pair of endpoints, and hence a cycle $C_4$.

**Proof of (12).** Let $G$ be a graph with 10 points such that the point independence number $\beta_0(G) < 4$. Then necessarily the chromatic number $\chi(G) \geq 4$. Hence by Brooks’ Theorem, see [3, p. 128], either $K_4$ (and hence $C_4$) is contained in $G$, or the degree of each point of $G$ is at least four in which case the conclusion follows from Lemma 3.

**Proof of (13).** We have to show that there is no graph $G$ with 11 points such that $K_4 - x \not\subseteq G$ and $\beta_0(G) < 4$, so again we assume the contrary. Our first aim is to show that $G$ must be regular of degree 4. This will be done by degrees, considered as possible separate cases.

**Case 1.** $G$ has a point $u$ of degree $\geq 7$. Then the neighborhood subgraph $H$ of $u$ (induced by the neighborhood of $u$) has at least 7 points and clearly contains no set of four independent points. By Lemma 2, $r(P_3, K_4) = 7$, so $H$ must contain $P_3$, which on joining $u$ implies $K_4 - x \subseteq G$. This contradiction proves the impossibility of Case 1.

**Case 2.** $G$ has a point $u$ of degree 6. Then the neighborhood
subgraph $H$ of $u$ has exactly six points, no four of them being independent. As $G$ contains no $K_4 - x$, $H$ cannot contain $P_3$. It is easy to see that these conditions imply $H = 3K_2$; let the three independent lines of $H$ be $v_iw_i, v_iw_2$, and $v_iw_3$. There are four other points in $G$; call one of them $u_i$. This point cannot be adjacent to both $v_i$ and $w_i$ for some $i \in \{1, 2, 3\}$ since otherwise $G$ would contain $K_4 - x$. Thus, we may assume $u_i$ not adjacent to $v_i, v_2, v_3$. But then the points $u_0, v_1, v_2, v_3$ are independent contradicting $\beta_0(G) < 4$. Hence the assumption of Case 2 is false.

**Case 3.** $G$ has a point $u$ of degree 5. Similarly as above, we can prove that the neighborhood graph $H$ of $u$ must be $2K_2 \cup K_2$. Let its two lines be $u_iu_j$ and $u_2v_3$, and let its fifth point be $w$. There are five other points in $G$. If all of them are adjacent to $w$, then the degree of $w$ equals six. As we saw, this assumption led to a contradiction in Case 2. Thus there is a point $u_0$ adjacent neither to $u$ nor to $v$. Clearly, $u_0$ cannot be adjacent to both $u_i$ and $v_i$ (nor to both $u_2$ and $v_3$) as otherwise $G$ would contain $K_4 - x$. Thus, we may assume $u_0$ not adjacent to $u_1, u_2$. But then $u_0, u, v_1, v_2$ form a set of four independent points, contradicting $\beta_0(G) < 4$.

Finally, to rule out any degree other than 4, we consider

**Case 4.** $G$ contains a point $u$ of degree $\leq 3$. Then there is a set $S$ of seven points in $G$ which are distinct from $u$ and not adjacent to $u$. The subgraph $\langle S \rangle$ of $G$ induced by $S$ contains no $K_4 - x$. Since by (14), $r(K_4 - x, K_3) \leq 7$, $\langle S \rangle$ necessarily contains three independent points $u_1, u_2, u_3$ and hence $G$ contains four independent points, namely $u, u_1, u_2, u_3$ contradicting $\beta_0(G) < 4$.

We have shown that each of the Cases 1–4 leads to a contradiction. Therefore, $G$ must be regular of degree 4. Clearly, every line of $G$ is contained in at most one triangle, for otherwise $G$ would contain $K_4 - x$. On the other hand, if every line of $G$ is in exactly one triangle, then the number of lines of $G$ would be divisible by three. However, $G$ has 22 edges and so it has a line, say $uv$, contained in no triangle. Let the other three neighbors of $u$ be $u_1, u_2, u_3$ and let the other three neighbors of $v$ be $v_1, v_2, v_3$. As $uv$ is contained in no triangle, all these are distinct. Now, we show that the subgraph of $G$ spanned by $u_1, u_2, u_3$ must contain exactly one line. For if it has none, then the points $u_1, u_2, u_3, v$ would be independent; if it has more than one, then $G$ would contain $K_4 - x$ with points $u, u_1, u_2, u_3$. Similarly, the subgraph of $G$ spanned by $v_1, v_2, v_3$ also contains exactly one line. Let these two lines be $u_1u_2$ and $v_1v_2$. Next, let $w$ be one of the remaining three points $w_1, w_2, w_3$ in $G$. This point cannot be adjacent to both $u_1$ and $u_2$ for $G$ would then contain $K_4 - x$.  

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Thus, we may assume $w$ not adjacent to $u_\gamma$. If $w$ is not adjacent to $u_3$, then $u, u_3, v, w$ are four independent points, contradicting $\beta_v(G) < 4$. So $w$ must be adjacent to $u_3$. As $w$ is arbitrary, we conclude that each of the points $w_1, w_2, w_3$ is adjacent to $u_3$. By a symmetry argument, each of $w_i, w_2, w_3$ is adjacent to $v_3$. Then there can be no line $w_iw_j$ in $G$, for otherwise $F$ would contain $K_4 - x$ with points $u_3, v_3, w_i, w_j$. Thus the points $w_1, w_2, w_3$ are independent. But then the points $u, w_1, w_2, w_3$ are independent, contradicting $\beta_0 < 4$.

5. Conclusions. The following table summarizes the results obtained (for both diagonal and off-diagonal) generalized Ramsey numbers.

<table>
<thead>
<tr>
<th></th>
<th>$K_2$</th>
<th>$P_3$</th>
<th>$2K_2$</th>
<th>$K_3$</th>
<th>$P_4$</th>
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Notice the irregularity of the behavior of $r(F_1, F_2)$:

$$r(P_4, K_3) > r(P_4, P_4), r(K_3, K_3).$$

On the other hand,

$$r(P_3, P_3) < r(P_3, K_3) < r(K_3, K_3)$$

(inequalities which continue to hold when all subscripts are increased to 4). These suggest the following

Conjecture. For any graphs $F_1, F_2$ with no isolates,

$$r(F_1, F_2) \geq \min (r(F_1), r(F_2)) .$$

It would be a formidable task indeed to extend this table to all 23 of the 5-point graphs with no isolates. In particular this would include the determination (exact, of course) of $r(5, 5)$ which appears not intractable, but extremely complicated. Our experience show that some of these 5-point graphs will be more delicate to handle than
others. Unless and until some more analytic, powerful, and automatic method is found for calculating the numbers $r(F_1, F_2)$, it is highly unlikely that these will be found for all the 6-point graphs and larger ones.

**REFERENCES**


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