IRREDUCIBLE CHARACTERS AND SOLVABILITY OF FINITE GROUPS

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The relationship between the degree of an irreducible character $\zeta$ on a finite group $G$ induced from a nilpotent normal subgroup and the structure of the group $G$ are studied when the degree of $\zeta$ is large. In particular if the square of the degree of $\zeta$ is the index of the center of $G$ in $G$ then $G$ is solvable.

Let $\zeta$ be an irreducible (complex) character on the finite group $G$. What conditions on $\zeta$ insure that $G$ is solvable? Of course, if $\zeta$ is a faithful linear character then $G$ is cyclic. We are interested in the other extreme when the degree of $\zeta$ is large, in part because of the relationship to the theory of projective representations and the Schur multiplier. Let $H$ be a nilpotent normal subgroup of $G$, assume $\zeta = \phi^\sigma$ for some character $\phi$ on $H$, and assume for each Sylow $p$-subgroup $S$ of $G$ that $\zeta|_S = m\lambda$ for some irreducible character $\lambda$ on $S$ where $(m, p) = 1$, then $G$ is solvable. If $Z$ is the center of $G$ the last condition always holds if the degree of $\zeta$ is $[G: Z]^{1/2}$, that is, if $G$ is a “group of central type” [2]. It is easy to see that no irreducible character on $G$ can have degree larger than $[G: Z]^{1/2}$. Another upper bound for the degree of an irreducible character on $G$ is $d[G: H]$ where $d = \max \{\text{degree } \rho|\rho \text{ is an irreducible character on } H\}$ ([3] 17.9 p. 570). If $[G, G]$ is the commutator subgroup of $G$ and $Z \cap [G, G]$ contains an element of order $d[G: H]$ then there is an irreducible character $\zeta$ of degree $d[G: H]$ on $G$. Moreover, $\zeta = \phi^G$ for some character $\phi$ on $H$, and for each Sylow $p$-subgroup $S$ of $G$, $\zeta|_S = \sum_{j=1}^n \lambda_j$ where the $\lambda_j$ are irreducible characters on $S$ with $\lambda_j(1)$ equal to the $p$-part of $\zeta(1)$ ($j = 1, \cdots, n$). If $n = 1$ for each prime $p$ dividing $[G: 1]$ then $G$ is solvable. An example showing the necessity of the hypothesis on $n$ is given. The conditions on the character $\zeta$ with respect to the Sylow subgroups $S$ of $G$ restrict the action of $G$ on $S$. To illustrate this we show $G$ is nilpotent if and only if for every Sylow subgroup $S$ of $G$ and every irreducible character $\chi$ on $G$, $\chi|_S = m\lambda$ for some irreducible character $\lambda$ on $S$.

In what follows all groups are finite and all characters and representations are taken over the complex numbers. If $n$ is an integer and $p$ is a prime integer we let $n_p$ denote the largest factor of $n$ which is a power of the prime $p$. Our standard reference is [3] and all unexplained terminology and notation coincides with [3].
THEOREM 1. Let $\zeta$ be an irreducible character on the group $G$ and let $H$ be a nilpotent normal subgroup of $G$. Assume

1. $\zeta = \phi^G$ for some character $\phi$ on $H$.
2. For each Sylow $p$-subgroup $S$ of $G$, $\zeta|_S = m\lambda$ for some irreducible character $\lambda$ on $S$ where $(p, m) = 1$. Then $G$ is solvable.

Proof. A theorem of P. Hall ([3] 1.10 p. 662) asserts that a group is solvable if every Sylow subgroup has a complement, this theorem will be applied to $G/H$. Let $p$ be a prime dividing $[G: 1]$, let $P$ be the Sylow $p$-subgroup of $H$ and $S$ a Sylow $p$-subgroup of $G$. Since $P$ is a characteristic subgroup of $H$, $P$ is a normal subgroup of $G$ and $P \trianglelefteq S$. By Clifford’s Theorem ([3] 17.3 p. 565)

$$\zeta|_P = e(\rho_1 + \cdots + \rho_n)$$

where the $\rho_i$ are inequivalent irreducible characters on $P$ conjugate in $G$. We determine the number $n$. By hypothesis 2, $\zeta|_P = m\lambda|_P$ so $\lambda|_P = e/m(\rho_1 + \cdots + \rho_n)$ and the $\rho_i$ are conjugate in $S$ by Clifford’s Theorem. Now $(\phi, \zeta|_H) \geq 1$ so by relabeling we can say $(\rho_i, \phi|_P) \geq 1$. We claim $\phi|_S = \lambda$ so $e/m = 1$ and $n = [S: P]$. To verify the claim hypothesis 2 says the $p$-part of $\zeta(1)$ is $\lambda(1)$. Also, $\phi|_P = q\phi_1$ (since $H$ is nilpotent) where $\phi_1(1)$ is a power of $p$ so $\phi_1(1)$ divides the $p$-part of $\phi^G(1) = \zeta(1)$. Since $\lambda$ is contained in $\phi|_S$ this implies $\lambda = \phi|_S$ verifying the claim.

Now $G$ acts on $\rho_1 \cdots \rho_n$ by conjugation and the inertia group $H^*$ of the action of $G$ on $\rho_i$ has index $n = \lambda(1)/\rho_i(1) = [S: P]$. Also $H^*$ contains $H$ since $H$ is nilpotent so $H^*/H$ is a $p$-complement in $G/H$. The Theorem of P. Hall completes the proof.

We next give a sufficient condition that $\zeta$ satisfy condition 2 of Theorem 1. (See [2] Theorem 2).

THEOREM 2. Let $\zeta$ be an irreducible character on $G$ and let $Z$ be the center of $G$. If $\zeta(1)^Z = [G: Z]$ then for each Sylow $p$-subgroup $S$ of $G$, $\zeta|_S = m\lambda$ for some irreducible character $\lambda$ on $S$ and $(p, m) = 1$.

Proof. By Schur’s lemma $\zeta|_Z = \zeta(1)\psi$ where $\psi$ is a linear character on $Z$. Then by reciprocity $(\zeta, \psi^G) = (\zeta|_Z, \psi) = \zeta(1)$ so by counting degrees, $\zeta(1)\zeta = \psi^G$. Let $S$ be a Sylow $p$-subgroup of $G$ and let $R$ be the subgroup of $G$ generated by $Z$ and $S$. Let $\lambda$ be an irreducible character of $R$ contained in $\psi^G$. By Schur’s lemma $\lambda|_S$ remains irreducible because the elements of $Z$ are represented by Scalar. Since $\lambda$ is contained in $\psi^G$, $\lambda = m\zeta$ for some integer $m$. By counting degrees

$$m = [G: R]\lambda(1)/\zeta(1).$$
Since \( \lambda \) is irreducible on \( S \), \( \lambda(1) = p^a \) for some \( a \), \([G: R]\) is prime to \( p \) since \( R \) contains \( S \). The \( p \)-part of \( \zeta(1)^2 \) is \([S: S \cap Z]\). Thus \( \lambda(1)^2 = [S: S \cap Z] \) and \((\zeta, \lambda^0) = (\zeta|_S, \lambda) = (\zeta|_S, \lambda|_S) = [G: Z]/\lambda(1)^2 \). Thus \( \zeta|_S = m\lambda \) where \( m \) is the largest divisor of \( \zeta(1) \) prime to \( p \). We combine the first two results to obtain.

**Corollary 1.** Let \( \zeta \) be an irreducible character on the group \( G \), and let \( H \) be a nilpotent normal subgroup of \( G \). Assume \( \zeta = \phi \) for some character \( \phi \) on \( H \) and \( \zeta(1)^2 = [G: Z] \) where \( Z \) is the center of \( G \). Then \( G \) is solvable.

The principal theorem of [1] is now an easy consequence of Corollary 1.

**Corollary 2.** Let \( \zeta \) be an irreducible character on the finite group \( G \), and let \( A \) be an abelian normal subgroup of \( G \). If \( \zeta(1)^2 = [G: A] = [G, Z] \) where \( Z \) is the center of \( G \) then \( G \) is solvable.

*Proof.* Let \( \phi \) be a linear constituent of \( \zeta |_A \). Then by reciprocity, \( \zeta \) is a constituent of \( \phi^0 \). But \( \zeta(1) = \phi^0(1) = [G: A] \) so \( \phi^0 = \zeta \). By Corollary 1, \( G \) is solvable.

We now verify some of the hypothesis of Theorem 1 in another situation. We begin by summarizing basic results relating ordinary representations, projective representations, and the Schur Multiplier. Our nontrivial assertions are the contents of 23.3, p. 629 of [3]. Let \( G \) be a finite group with center \( Z \), assume \( n \) is the exponent of \([G, G] \cap Z\) and let \( \widehat{G} = G/Z \). Write

\[
G = \bigcup_{g \in \widehat{G}} ZR(g)
\]

where \( R(g) \) is an element in \( G \) corresponding to \( g \). Then \( R(g_1)R(g_2) = A(g_1, g_2)R(g_1 g_2) \) where \( A(g_1, g_2) \in Z \). Let \( a \in [G, G] \cap Z \) order \( n \) and let \( \theta \) be a linear character on \( Z \) which is faithful on the cyclic group generated by \( a \). Define a 2-cycle \( \alpha \) on \( \widehat{G} \) by

\[
\alpha(g_1, g_2) = \theta(A(g_1, g_2)).
\]

Let \( K^* \) be the multiplicative group of the complex numbers. The element \( \alpha \) represents in the Schur multiplier \( H^2(\widehat{G}, K^*) \) has order \( n \).

Form the projective group algebra \( KG_\alpha \) and let \( M \) be a left \( KG_\alpha \)-module. For each \( g \in \widehat{G} \), left multiplication by \( g \) on \( M \) induces a \( K \)-linear transformation \( T(g) \) of \( M \) and

\[
T(g_1)T(g_2) = \alpha(g_1, g_2)T(g_1g_2).
\]

If \( x \in G \) then \( x = z_iR(g_i) \) where \( z_i \in Z \) and \( g_i \in \widehat{G} \). Let left multiplica-
tion by \( x \) on \( M \) be the linear transformation \( T^*(x) = \theta(z_1)T(g_1) \). If \( y = z_2R(g_2) \in G \) then
\[
x y = z_2A(g_1, g_2)R(g_2,g_2)
\]
and
\[
T^*(x)T^*(y) = \theta(z_1)\theta(z_2)T(g_1,g_2) = \theta(z_2)\theta(A(g_1, g_2))T(g_2,g_2) = T^*(xy).
\]
Thus \( M \) can be viewed as a \( KG \)-module. Notice that \( M \) is irreducible over \( KG \) if and only if \( M \) is irreducible over \( KG_a \). Also, note that \( T^*|_x = T^*(1) \). This process can be reversed when \( M \) is a \( KG \)-module giving the \( G \) representation \( T^* \) if \( T^*|_x = T^*(1)\theta \) for the given linear character \( \theta \) on \( Z \). Define a linear character \( \psi \) on \( G \) by the equation \( \psi(x) = \det(T^*(x)) \). Since \( a \in [G, G] \), \( \psi(a) = 1 \). But \( \psi(a) = \theta(a)^m \) where \( m = T^*(1) \) so \( n \) divides \( T^*(1) \).

Let \( S \) be a Sylow \( p \)-subgroup of \( G \) and \( \bar{S} \) the natural image of \( S \) in \( \bar{G} \). The element the restriction of \( \alpha \) to \( \bar{S} \) represents in \( H^2(\bar{S}, K^*) \) is realized by the equation \( \alpha(y_1, y_2) = \theta(A(y_1, y_2)) \) in the group \( S\bar{Z} \). By ([3] 16.21, p. 118) \( \alpha \) represents an element whose order is \( n_p \) in \( H^2(\bar{S}, K^*) \). In the correspondence of ([3] 23.3, p. 629) this implies \( \theta \) is faithful on a cyclic group of order \( n_p \) in \( [S, S] \cap \bar{Z} \). Form the projective group algebra \( K\bar{S}_\alpha \). Now \( M \) can be viewed as a \( K\bar{S}_\alpha \)-module, let \( M = M_1 \oplus \cdots \oplus M_\ell \) where the \( M_i \) are irreducible \( K\bar{S}_\alpha \) modules.

As above, each \( M_i \) affords an ordinary representation \( T_i^* \) on \( SZ \) which is irreducible. The restriction of \( T_i^* \) to \( S \) is also irreducible since each \( T_i^* \) restricted to \( Z \) is \( T_i^*(1)\theta \). Also, \( \theta \) is faithful on a cyclic group of order \( n_p \) in \( [S, S] \cap \bar{Z} \) so arguing as before \( n_p \) must divide the degree of \( T_i^* \).

**Lemma 1.** Let \( G \) be a finite group with center \( Z \), let \( a \in [G, G] \cap Z \) of order \( n \), and let \( \theta \) be a linear character on \( Z \) faithful on the cyclic group generated by \( a \). Then

1. \( \theta^a = \sum_{i=1}^{\ell} \zeta_i(1)\zeta_i \) where \( n|\zeta_i(1) \) and the \( \zeta_i \) are inequivalent irreducible characters of \( G \).
2. If \( \zeta \) is an irreducible character on \( G \) with \( (\theta^a, \zeta) \geq 1 \) and \( S \) is a Sylow \( p \)-subgroup of \( G \) then \( \zeta|_Z = \sum_{j=1}^{\ell} b_j\lambda_j \) where \( n_p|\lambda_j(1) \), the \( \lambda_j \) are inequivalent irreducible characters on \( S \), and the \( b_j \) are positive integers.

**Proof.** Let \( \zeta \) be an irreducible character on \( G \). By Schur’s lemma \( \zeta|_Z = \zeta(1)\psi \) for a linear character \( \psi \) on \( Z \). Now \( (\zeta, \psi^a) = (\zeta|_Z, \psi) = \zeta(1) \). This shows \( \theta^a = \sum_{i=1}^{\ell} \zeta_i(1)\zeta_i \) where the \( \zeta_i \) are inequivalent irreducible characters of \( G \). If \( T_i \) is the representation affording \( \zeta_i \) then \( \det T_i \) is a linear character on \( G \). Since \( a \in [G, G] \), 1 =
det \((T_i(a)) = \det[\theta(a)T_i(1)] = \theta(a)\zeta_w\).
Therefore \(n|\zeta_i(1)\).

To prove (2) we need the analysis which preceded the lemma. Let \(T^*\) be the ordinary representation on \(G\) which affords \(\zeta\) and \(T\) the corresponding projective representation on \(\tilde{G}\). In this situation we showed \(T^*_S = T^*_1 + \cdots + T^*_s\) where the \(T^*_i\) are irreducible and their degree are divisible by \(n_p\). Let \(\lambda_1, \cdots, \lambda_i\) be a full set of inequivalent characters afforded by the \(T_i^*, \cdots, T^*_s\). Then \(\zeta|_S = \sum_{i=1}^s b_i \lambda_i\) where \(b_i\) is the multiplicity of \(\lambda_i\) in \(\zeta|_S\) and \(\lambda_i(1)\) is the degree of some \(T^*_i\) and so is divisible by \(n_p\). We can now prove

**THEOREM 3.** Let \(G\) be group with center \(Z\). Let \(H\) be a normal nilpotent subgroup of \(G\) and let \(d = \max\{\rho(1) | \rho\) is an irreducible character of \(H\)\). If \([G, G] \cap Z\) contains an element of order \(d[G : H]\) then there is an irreducible character \(\zeta\) on \(G\) so that \(\zeta = \phi^o\) for some character \(\phi\) on \(H\), and for each Sylow \(p\)-subgroup \(S\) of \(G\), \(\zeta|_S = \sum_{i=1}^s b_i \lambda_i\) where \(\lambda_i(1) = \zeta(1)_p\). If \(n = 1\) for each \(p\) then \(G\) is solvable.

**Proof.** Let \(n = d[G : H]\) and let \(a \in [G, G] \cap Z\) of order \(n\). Let \(\theta\) be a linear character on \(Z\) which is faithful on the cyclic group generated by \(a\). By the first part of **LEMMA 1**

\[\theta^o = \sum_{i=1}^s \zeta_i(1) \zeta_i\]

where \(n|\zeta_i(1)\) and the \(\zeta_i\) are inequivalent irreducible characters of \(G\). We will show each of the \(\zeta_i\) satisfy the conclusion of the Theorem. By 17.9 p. 570 of [3], \(n\) is the largest possible degree of an irreducible character on \(G\) so \(n = \zeta_i(1) (i = 1, \cdots, s)\) and \(H\) is a maximal nilpotent normal subgroup of \(G\) so \(Z \subseteq H\). Now \(\theta^o(1) = [G : Z]\) so \([G : Z] = sn^2\) where \(s\) is the number of inequivalent \(\zeta_i\) in \(\theta^o\). By Clifford's Theorem (17.3 p. 565, [3])

\[\zeta_i|_H = e(\phi^i_1 + \cdots + \phi^i_m)\]

where the \(\phi^i(j = 1, \cdots, m)\) are inequivalent irreducible characters on \(H\) conjugate in \(G\). Now \(\zeta_i(1)\) is a constituent of \((\phi^i)^o\) and \((\phi^i)^o(1) \leq d[G : H] = \zeta_i(1)\) so for each \(j\), \(\phi^i(j) = d\) and \((\phi^i)^o = \zeta_i\). This verifies the first conclusion of **Theorem 3** for each \(i(i = 1, 2, \cdots, s)\).

Let \(S\) be a Sylow \(p\)-subgroup of \(G\). By the second part of **LEMMA 1**, \(\zeta_i|_S = \sum_{j=1}^1 b_j \lambda_j^i\)

where the \(\lambda_j^i\) are inequivalent irreducible characters on \(S\) and \(n_p\) divides \(\lambda_j^i(1)\). Since \(H\) is nilpotent, \(d_p = \max\{\gamma(1) | \gamma\) is an irreducible character on \(P\}\). If \(\lambda\) is an irreducible constituent of \(\lambda_j^i|_p\) then
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\( \gamma^s(1) \leq \lambda_j^s(1) \) so \( \gamma^s = \lambda_j^s \) and \( \lambda_j^s(1) = d_p[S: P] = n_p \). This verifies the second conclusion of Theorem 3. If \( n = 1 \) for each \( p \) then \( \zeta|_S = b_\lambda \) and \( \zeta(1) = b_\lambda \). But \( \zeta(1)_p = \lambda(1)_p \) so \( (p, b_\lambda) = 1 \) and by Theorem 1, \( G \) is solvable. This completes the proof.

For an example to show the necessity of Condition 2 in Theorem 1 let \( H \) be any group of order \( n \) and \( J_n(H) \) the group algebra of \( H \) over the ring \( J_n \) of integers modulo \( n \). Let \( A = J_n(H) \) viewed as an additive group and let \( H \) act as a group of automorphisms of \( A \) by

\[
h(ax) = ahx \quad \text{(regular representation)} \quad x, h \in H, a \in J_n.
\]

Let \( G \) be the semi-direct product of \( A \) by \( H \) with respect to this action. Let \( \phi \) be the linear character defined on \( A \) by \( \phi(\sum a_h h) = \xi^a \) where \( \xi \) is a primitive \( n^\text{th} \) root of 1 and \( a \) is an integer representing the coefficient in \( J_n \) of the identity \( e \) of \( H \). One checks that \( [G, A] \cap Z = Z \) where \( Z \), the center of \( G \), is \( \{ \sum a_h h | a_h = a_k \text{ all } h, k \in H \} \) and has exponent \( n \). Also \( \phi \) is distinct from all its conjugates so \( \phi^g = \zeta \) is irreducible. Yet \( G \) need not be solvable. The problem is that the restriction of \( \zeta \) to a Sylow subgroup does not behave properly. For example, if \( H = A_5 \) (the simple group of order 60), and \( S \) is the Sylow 5-subgroup of \( G \) then \( \zeta|_S = \sum \lambda_i \) where the \( \lambda_i \) are 12 distinct irreducible characters on \( S \) of degree 5.

If \( G \) is a finite group with center \( Z \) and \( \zeta \) is a faithful irreducible character on \( G \) with \( \zeta|_S = m\lambda \) for some Sylow subgroup \( S \) and irreducible character \( \lambda \) on \( S \) then the center of \( S \) is \( Z \cap S \). The proof of this observation also proves

**Theorem 4.** The group \( G \) is nilpotent if and only if for each irreducible character \( \zeta \) on \( G \) and each Sylow subgroup \( S \) of \( G \), \( \zeta|_S = m\lambda \) for some irreducible character \( \lambda \) on \( S \).

**Proof.** Assume \( G \) is nilpotent, let \( \zeta \) be an irreducible character on \( G \) and \( S \) a Sylow subgroup. Then \( S \) is normal in \( G \) so by Clifford’s Theorem

\[
\zeta|_S = e(\phi_1 + \cdots + \phi_m)
\]

with the \( \phi_i \) distinct conjugate irreducible characters on \( S \). If \( g \in G \) then \( g = g_sg_s \) where \( g_s \) centralizes \( S \) and \( g_s \in S \). Then, \( \phi_i^g = \phi_i^g = \phi_i^g = \phi \). So \( m = 1 \).

Conversely, let \( S \) be a Sylow subgroup of \( G \) and let \( a \) be an element of the center of \( S \). Let \( \zeta \) be an irreducible character on \( G \), then \( \zeta|_S = m\lambda \) where \( \lambda \) is an irreducible character on \( S \). Let \( Z(S) \) be the center of \( S \). Then by Schur’s lemma, \( \lambda|_{Z(S)} = \lambda(1)\theta \) for some linear character on \( Z(S) \). Thus \( \zeta(a) = \zeta(1)\theta(a) \) so \( a \) is an element of
the center of $G/\ker \zeta$. Since this is true for all irreducible characters on $G$, $a$ is an element of the center of $G$. If $\langle a \rangle$ is the central subgroup of $G$ generated by $a$ then the irreducible characters of $G/\langle a \rangle$ correspond to the irreducible characters of $G$ with kernel $\langle a \rangle$. Thus $G/\langle a \rangle$ satisfies the same hypothesis $G$ does so by induction $G$ is nilpotent.

References


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