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The relationship between the degree of an irreducible character ζ on a finite group G induced from a nilpotent normal subgroup and the structure of the group G are studied when the degree of ζ is large. In particular if the square of the degree of ζ is the index of the center of G in G then G is solvable.

Let ζ be an irreducible (complex) character on the finite group G. What conditions on ζ insure that G is solvable? Of course, if ζ is a faithful linear character then G is cyclic. We are interested in the other extreme when the degree of ζ is large, in part because of the relationship to the theory of projective representations and the Schur multiplier. Let H be a nilpotent normal subgroup of G, assume $\zeta = \phi^G$ for some character ϕ on H, and assume for each Sylow p-subgroup S of G that $\zeta|_S = m\lambda$ for some irreducible character λ on S where $(m, p) = 1$, then G is solvable. If Z is the center of G the last condition always holds if the degree of ζ is $[G: Z]^{1/2}$, that is, if G is a "group of central type" [2]. It is easy to see that no irreducible character on G can have degree larger than $[G: Z]^{1/2}$. Another upper bound for the degree of an irreducible character on G is $d[G: H]$ where $d = \max\{\text{degree } \rho \mid \rho \text{ is an irreducible character on } H\}$ ([3] 17.9 p. 570). If $[G, G]$ is the commutator subgroup of G and $Z \cap [G, G]$ contains an element of order $d[G: H]$ then there is an irreducible character ζ of degree $d[G: H]$ on G. Moreover, $\zeta = \phi^G$ for some character ϕ on H, and for each Sylow p-subgroup S of G, $\zeta|_S = \sum_{j=1}^{n} \lambda_j$ where the λ_j are irreducible characters on S with $\lambda_j(1)$ equal to the p-part of $\zeta(1)$ $(j = 1, \cdots, n)$. If $n = 1$ for each prime p dividing $[G: 1]$ then G is solvable. An example showing the necessity of the hypothesis on n is given. The conditions on the character ζ with respect to the Sylow subgroups S of G restrict the action of G on S. To illustrate this we show G is nilpotent if and only if for every Sylow subgroup S of G and every irreducible character χ on G, $\chi|_S = m\lambda$ for some irreducible character λ on S.

In what follows all groups are finite and all characters and representations are taken over the complex numbers. If n is an integer and p is a prime integer we let n_p denote the largest factor of n which is a power of the prime p. Our standard reference is [3] and all unexplained terminology and notation coincides with [3].

THEOREM 1. *Let ζ be an irreducible character on the group G and let H be a nilpotent normal subgroup of G. Assume*

1. *$\zeta = \phi^G$ for some character ϕ on H.*
2. *For each Sylow p-subgroup S of G, $\zeta|_S = m\lambda$ for some irreducible character λ on S where $(p, m) = 1$. Then G is solvable.*

Proof. A theorem of P. Hall ([3] 1.10 p. 662) asserts that a group is solvable if every Sylow subgroup has a complement, this theorem will be applied to G/H. Let p be a prime dividing $[G: 1]$, let P be the Sylow p-subgroup of H and S a Sylow p-subgroup of G. Since P is a characteristic subgroup of H, P is a normal subgroup of G and $P \subseteqq S$. By Clifford's Theorem ([3] 17.3 p. 565)

$$\zeta|_P = e(\rho_1 + \cdots + \rho_n)$$

where the ρ_i are inequivalent irreducible characters on P conjugate in G. We determine the number n. By hypothesis 2, $\zeta|_P = m\lambda|_P$ so $\lambda|_P = e/m(\rho_1 + \cdots + \rho_n)$ and the ρ_i are conjugate in S by Clifford's Theorem. Now $(\phi, \zeta|_H) \geqq 1$ so by relabeling we can say $(\rho_1, \phi|_P) \geqq 1$. We claim $\rho_1^S = \lambda$ so $e/m = 1$ and $n = [S: P]$. To verify the claim hypothesis 2 says the p-part of $\zeta(1)$ is $\lambda(1)$. Also, $\phi|_P = q\rho_1$ (since H is nilpotent) where $\rho_1(1)$ is a power of p so $\rho_1^S(1)$ divides the p-part of $\phi^G(1) = \zeta(1)$. Since λ is contained in ρ_1^S this implies $\lambda = \rho_1^S$ verifying the claim.

Now G acts on $\rho_1 \cdots \rho_n$ by conjugation and the inertia group H^* of the action of G on ρ_1 has index $n = \lambda(1)/\rho_1(1) = [S: P]$. Also H^* contains H since H is nilpotent so H^*/H is a p-complement in G/H. The Theorem of P. Hall completes the proof.

We next give a sufficient condition that ζ satisfy condition 2 of Theorem 1. (See [2] Theorem 2).

THEOREM 2. *Let ζ be an irreducible character on G and let Z be the center of G. If $\zeta(1)^2 = [G: Z]$ then for each Sylow p-subgroup S of G, $\zeta|_S = m\lambda$ for some irreducible character λ on S and $(p, m) = 1$.*

Proof. By Schur's lemma $\zeta|_Z = \zeta(1)\psi$ where ψ is a linear character on Z. Then by reciprocity $(\zeta, \psi^G) = (\zeta|_Z, \psi) = \zeta(1)$ so by counting degrees, $\zeta(1)\zeta = \psi^G$. Let S be a Sylow p-subgroup of G and let R be the subgroup of G generated by Z and S. Let λ be an irreducible character of R contained in ψ^R. By Schur's lemma $\lambda|_S$ remains irreducible because the elements of Z are represented by Sclars. Since λ is contained in ψ^R, $\lambda^G = m\zeta$ for some integer m. By counting degrees

$$m = [G: R]\lambda(1)/\zeta(1) .$$

Since λ is irreducible on S, $\lambda(1) = p^a$ for some a, $[G: R]$ is prime to p since R contains S. The p-part of $\zeta(1)^2$ is $[S: S \cap Z]$. Thus $\lambda(1)^2 = [S: S \cap Z]$ and $(\zeta, \lambda^G) = (\zeta|_R, \lambda) = (\zeta|_S, \lambda|_S) = [G: Z]/\lambda(1)^2$. Thus $\zeta|_S = m\lambda$ where m is the largest divisor of $\zeta(1)$ prime to p. We combine the first two results to obtain.

COROLLARY 1. *Let ζ be an irreducible character on the group G, and let H be a nilpotent normal subgroup of G. Assume $\zeta = \phi^G$ for some character ϕ on H and $\zeta(1)^2 = [G: Z]$ where Z is the center of G. Then G is solvable.*

The principal theorem of [1] is now an easy consequence of Corollary 1.

COROLLARY 2. *Let ζ be an irreducible character on the finite group G, and let A be an abelian normal subgroup of G. If $\zeta(1)^2 = [G: A]^2 = [G. Z]$ where Z is the center of G then G is solvable.*

Proof. Let ϕ be a linear constitutent of $\zeta|_A$. Then by reciprocity, ζ is a constitutent of ϕ^G. But $\zeta(1) = \phi^G(1) = [G: A]$ so $\phi^G = \zeta$. By Corollary 1, G is solvable.

We now verify some of the hypothesis of Theorem 1 in another situation. We begin by summarizing basic results relating ordinary representations, projective represention, and the Schur Multiplier. Our nontrivial assertions are the contents of 23.3, p. 629 of [3]. Let G be a finite group with center Z, assume n is the exponent of $[G, G] \cap Z$ and let $\bar{G} = G/Z$. Write

$$G = \bigcup_{g \in \bar{G}} ZR(g)$$

where $R(g)$ is an element in G corresponding to g. Then $R(g_1)R(g_2) = A(g_1, g_2)R(g_1g_2)$ where $A(g_1, g_2) \in Z$. Let $a \in [G, G] \cap Z$ order n and let θ be a linear character on Z which is faithful on the cyclic group generated by a. Define a 2-cycle α on $\bar{G}$ by

$$\alpha(g_1, g_2) = \theta(A(g_1, g_2)) .$$

Let K^* be the multiplicative group of the complex numbers. The element α represents in the Schur multipler $H^2(\bar{G}, K^*)$ has order n.

Form the projective group algebra $K\bar{G}_\alpha$ and let M be a left $K\bar{G}_\alpha$-module. For each $g \in \bar{G}$, left multiplication by g on M induces a K-linear transformation $T(g)$ of M and

$$T(g_1)T(g_2) = \alpha(g_1, g_2)T(g_1g_2) .$$

If $x \in G$ then $x = z_1R(g_1)$ where $z_1 \in Z$ and $g_1 \in \bar{G}$. Let left multiplication by x on M be the linear transformation $T^*(x) = \theta(z_1)T(g_1)$. If $y = z_2R(g_2) \in G$ *then*

$$xy = z_1z_2A(g_1, g_2)R(g_1g_2)$$

and

$$T^*(x)T^*(y) = \theta(z_1)T(g_1)\theta(z_2)T(g_2) = \theta(z_1z_2)\theta(A(g_1, g_2))T(g_1g_2) = T^*(xy) .$$

Thus M can be viewed as a KG-module. Notice that M is irreducible over KG if and only if M is irreducible over $K\bar{G}_\alpha$. Also, note that $T^*|_Z = T^*(1)$. This process can be reversed when M is a KG-module giving the G representation T^* if $T^*|_Z = T^*(1)\theta$ for the given linear character θ on Z. Define a linear character ψ on G by the equation $\psi(x) = \det(T^*(x))$. Since $a \in [G, G]$, $\psi(a) = 1$. But $\psi(a) = \theta(a)^m$ where $m = T^*(1)$ so n divides $T^*(1)$.

Let S be a Sylow p-subgroup of G and $\bar{S}$ the natural image of S in $\bar{G}$. The element the restriction of α to $\bar{S}$ represents in $H^2(\bar{S}, K^*)$ is realized by the equation $\alpha(y_1, y_2) = \theta(A(y_1, y_2))$ in the group SZ. By ([3] 16.21, p. 118) α represents an element whose order is n_p in $H^2(\bar{S}, K^*)$. In the correspondence of ([3] 23.3, p. 629) this implies θ is faithful on a cyclic group of order n_p in $[S, S] \cap Z$. Form the projective group algebra $K\bar{S}_\alpha$. Now M can be viewed as a $K\bar{S}_\alpha$-module, let $M = M_1 \oplus \cdots \oplus M_k$ where the M_i are irreducible $K\bar{S}_\alpha$ modules. As above, each M_i affords an ordinary representation T_i^* on SZ which is irreducible. The restriction of T_i^* to S is also irreducible since each T_i^* restricted to Z is $T_i^*(1)\theta$. Also, θ is faithful on a cyclic group of order n_p in $[S, S] \cap Z$ so arguing as before n_p must divide the degree of T_i^*.

LEMMA 1. *Let G be a finite group with center Z, let $a \in [G, G] \cap Z$ of order n, and let θ be a linear character on Z faithful on the cyclic group generated by a. Then*

(1) *$\theta^G = \sum_{i=1}^{s} \zeta_i(1)\zeta_i$ where $n \mid \zeta_i(1)$ and the ζ_i are inequivalent irreducible characters of G.*

(2) *If ζ is an irreducible character on G with $(\theta^G, \zeta) \geqq 1$ and S is a Sylow p-subgroup of G then $\zeta|_S = \sum_{j=1}^{l} b_j\lambda_j$ where $n_p \mid \lambda_j(1)$, the λ_j are inequivalent irreducible characters on S, and the b_j are positive integers.*

Proof. Let ζ be an irreducible character on G. By Schur's lemma $\zeta|_Z = \zeta(1)\psi$ for a linear character ψ on Z. Now $(\zeta, \psi^G) = (\zeta|_Z, \psi) = \zeta(1)$. This shows $\theta^G = \sum_{i=1}^{s} \zeta_i(1)\zeta_i$ where the ζ_i are inequivalent irreducible characters of G. If T_i is the representation affording ζ_i then $\det T_i$ is a linear character on G. Since $a \in [G, G]$, $1 =$

$\det(T_i(a)) = \det[\theta(a)T_i(1)] = \theta(a)^{\zeta_i(1)}$. Therefore $n \mid \zeta_i(1)$.

To prove (2) we need the analysis which preceded the lemma. Let T^* be the ordinary representation on G which affords ζ and T the corresponding projective representation on $\bar{G}$. In this situaion we showed $T^*|_S = T_1^* + \cdots + T_k^*$ where the T_i^* are irreducible and their degree are divisible by n_p. Let $\lambda_1, \cdots, \lambda_l$ be a full set of inequivalent characters afforded by the $T_1^*, \cdots, T_k^*$. Then $\zeta|_S = \sum_{i=1}^{l} b_j \lambda_j$ where b_j is the multiplicity of λ_j in $\zeta|_S$ and $\lambda_j(1)$ is the degree of some T_i^* and so is divisible by n_p. We can now prove

THEOREM 3. *Let G be group with center Z. Let H be a normal nilpotent subgroup of G and let $d = \max\{\rho(1) \mid \rho$ is an irreducible character of $H\}$. If $[G, G] \cap Z$ contains an element of order $d[G: H]$ then there is an irreducible character ζ on G so that $\zeta = \phi^G$ for some character ϕ on H, and for each Sylow p-subgroup S of G, $\zeta|_S = \sum_{i=1}^{n} b_i \lambda_i$ where $\lambda_i(1) = \zeta(1)_p$. If $n=1$ for each p then G is solvable.*

Proof. Let $n = d[G: H]$ and let $a \in [G, G] \cap Z$ of order n. Let θ be a linear character on Z which is faithful on the cyclic group generated by a. By the first part of LEMMA 1

$$\theta^G = \sum_{i=1}^{s} \zeta_i(1)\zeta_i$$

where $n \mid \zeta_i(1)$ and the ζ_i are inequivalent irreducible characters of G. We will show each of the ζ_i satisfy the conclusion of the Theorem. By 17.9 p. 570 of [3], n is the largest possible degree of an irreducible character on G so $n = \zeta_i(1) (i = 1, \cdots, s)$ and H is a maximal nilpotent normal subgroup of G so $Z \subseteqq H$. Now $\theta^G(1) = [G: Z]$ so $[G: Z] = sn^2$ where s is the number of inequivalent ζ_i in θ^G. By Clifford's Theorem (17.3 p. 565, [3])

$$\zeta_i|_H = e(\phi_1^i + \cdots + \phi_m^i)$$

where the $\phi_i^i (j = 1, \cdots, m)$ are inequivalent irreducible characters on H conjugate in G. Now ζ_i is a constitutent of $(\phi_j^i)^G$ and $(\phi_j^i)^G(1) \leqq d[G: H] = \zeta_i(1)$ so for each j, $\phi_j^i(1) = d$ and $(\phi_j^i)^G = \zeta_i$. This verifies the first conclusion of Theorem 3 for each $i (i = 1, 2, \cdots, s)$.

Let S be a Sylow p-subgroup of G. By the second part of LEMMA 1,

$$\zeta_i|_S = \sum_{j=1}^{l} b_j \lambda_j^i$$

where the λ_j^i are inequivalent irreducible characters on S and n_p divides $\lambda_j^i(1)$. Since H is nilpotent, $d_p = \max\{\gamma(1) \mid \gamma$ is an irreducible character on $P\}$. If λ is an irreducible constitutent of $\lambda_j^i|_P$ then

$\gamma^S(1) \leqq \lambda_j^i(1)$ so $\gamma^S = \lambda_j^i$ and $\lambda_j^i(1) = d_p[S\colon P] = n_p$. This verifies the second conclusion of Theorem 3. If $n = 1$ for each p then $\zeta|_S = b_1\lambda_1$ and $\zeta(1) = b_1\lambda_1(1)$. But $\zeta(1)_p = \lambda_1(1)_p$ so $(p, b_1) = 1$ and by Theorem 1, G is solvable. This completes the proof.

For an example to show the necessity of Condition 2 in Theorem 1 let H be any group of order n and $J_n(H)$ the group algebra of H over the ring J_n of integers modulo n. Let $A = J_n(H)$ viewed as an additive group and let H act as a group of automorphisms of A by

$$h(ax) = ahx \text{ (regular representation) } x, h \in H, a \in J_n .$$

Let G be the semi-direct product of A by H with respect to this action. Let ϕ be the linear character defined on A by $\phi(\sum_{h\in H} a_h h) = \xi^a$ where ξ is a primitive n^{th} root of 1 and a is an integer representing the coefficient in J_n of the identity e of H. One checks that $[G, A] \cap Z = Z$ where Z, the center of G, is $\{\sum a_h h \mid a_h = a_k \text{ all } h, k \in H\}$ and has exponent n. Also ϕ is distinct from all its conjugates so $\phi^G = \zeta$ is irreducible. Yet G need not be solvable. The problem is that the restriction of ζ to a Sylow subgroup does not behave properly. For example, if $H = A_5$ (the simple group of order 60), and S is the Sylow 5-subgroup of G then $\zeta|_S = \sum_{i=1}^{12} \lambda_i$ where the λ_i are 12 distinct irreducible characters on S of degree 5.

If G is a finite group with center Z and ζ is a faithful irreducible character on G with $\zeta|_S = m\lambda$ for some Sylow subgroup S and irreducible character λ on S then the center of S is $Z \cap S$. The proof of this observation also proves

THEOREM 4. *The group G is nilpotent if and only if for each irreducible character ζ on G and each Sylow subgroup S of G, $\zeta|_S = m\lambda$ for some irreducible character λ on S.*

Proof. Assume G is nilpotent, let ζ be an irreducible character on G and S a Sylow subgroup. Then S is normal in G so by Clifford's Theorem

$$\zeta|_S = e(\phi_1 + \cdots + \phi_m)$$

with the ϕ_i distinct conjugate irreducible characters on S. If $g \in G$ then $g = g_1g_2$ where g_1 centralizes S and $g_2 \in S$. Then, $\phi_1^g = \phi_1^{g_1g_2} = \phi^{g_2} = \phi$. So $m = 1$.

Conversely, let S be a Sylow subgroup of G and let a be an element of the center of S. Let ζ be an irreducible character on G, then $\zeta|_S = m\lambda$ where λ is an irreducible character on S. Let $Z(S)$ be the center of S. Then by Schur's lemma, $\lambda|_{Z(S)} = \lambda(1)\theta$ for some linear character on $Z(S)$. Thus $\zeta(a) = \zeta(1)\theta(a)$ so a is an element of the center of $G/\ker \zeta$. Since this is true for all irreducible characters on G, a is an element of the center of G. If $\langle a \rangle$ is the central subgroup of G generated by a then the irreducible characters of $G/\langle a \rangle$ correspond to the irreducible characters of G with kernel $\langle a \rangle$. Thus $G/\langle a \rangle$ satisfies the same hypothesis G does so by induction G is nilpotent.

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