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ON RIGHT ZERO UNIONS OF COMMUTATIVE SEMIGROUPS

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Let $F = \{S_r: r \in R\}$ be a disjoint family of semigroups. One says that F has a *right zero union* (RZU) if there exists a semigroup S which is a disjoint union of the S_r where each S_r is a left ideal of S. This paper gives some theorems on RZU of commutative semigroups with special emphasis placed on commutative cancellative semigroups.

Suppose S is an RZU of commutative cancellative semigroups. It is proven that S has a quotient right abelian group; thus S is left commutative and left cancellative. Conversely, it is proven that if a semigroup S is left commutative and left cancellative, then S is an RZU of commutative cancellative semigroups. Suppose F is a family of commutative semigroups having an RZU; it is proven that a certain family of cancellative homomorphic images of F also has an RZU. Finally, necessary and sufficient conditions are given for a family of commutative cancellative semigroups to have an RZU.

The study of RZU is a special case of the study of "bands of semigroups." R. Yoshida has studied the dual problem of left zero unions.

II. Some necessary conditions for RZU and an embedding result. A semigroup S is left commutative if $xyz = yxz$ for all x, y, and z in S.

LEMMA 2.1. *The RZU of two commutative semigroups is left commutative.*

Proof. The symmetric conditions $AB \subseteq B$, $BA \subseteq A$, A and B are commutative, are given. Let $a \in A$, and let $b, b_1 \in B$. Now $abb_1 = a(bb_1) = a(b_1b) = (ab_1)b = b(ab_1) = bab_1$. Other cases are proven similarly.

DEFINITION 2.2. Let C be a commutative cancellative semigroup. The quotient group, G, of C is the smallest group into which C may be injected. If $C \subseteq T$, a group, then $G \cong \{st^{-1}: s, t \in C\}$. Note G is abelian. (For more on quotient groups see [1].)

A right abelian group is the direct product of a right zero semigroup and an abelian group. A quotient right abelian group will have the same meaning as quotient group; namely, the smallest right abelian group into which a semigroup S can be injected.

The next lemma is proven using the following result of Petrich

[2]: A semigroup S is a semilattice of semigroups each of which is the Cartesian product of rectangular band and a group iff S is a union of groups and its idempotents form a semigroup.

LEMMA 2.3. *Let $F = \{G_\alpha : \alpha \in A\}$ be a disjoint family of groups. Then F has an RZU iff all the G_α are isomorphic. If the RZU exists then it is isomorphic to the right group $G \times A$, where $G_\alpha \cong G$, and where A is considered as a right zero semigroup.*

Proof. Let S be an RZU of F. Certainly S is union of groups. The idempotents of S are exactly the e_α, where e_α is the identity of G_α. Since e_α is an identity and since $G_\alpha G_\beta \subseteqq G_\beta$, we have $(e_\alpha e_\beta)(e_\alpha e_\beta) = e_\alpha(e_\beta(e_\alpha e_\beta)) = e_\alpha(e_\alpha e_\beta) = (e_\alpha e_\alpha)e_\beta = e_\alpha e_\beta = e_\beta$, for $e_\alpha e_\beta$ is the idempotent of G_β. Thus the idempotents of S form a right zero semigroup. This semigroup is isomorphic to A, but also, by Petrich, to a semilattice union $\bigcup_{\gamma \in \Gamma} L_\gamma \times R_\gamma$, and this implies that $|\Gamma| = 1$, $|L_\gamma| = 1$, and $R_\gamma = A$.

THEOREM 2.4. *Let S be an RZU of $F = \{C_\alpha : \alpha \in A\}$, where F is a disjoint family of commutative cancellative semigroups. Let G_α be the quotient group of C_α. We consider the G to be disjoint. Then all the G_α are isomorphic, and they have an RZU, T.*

T is isomorphic to $G \times A$, where $G_\alpha \cong G$, and where A is considered as a right zero semigroup.

Furthermore, T is the quotient right abelian group of S in the following sense. There exists an injection (isomorphism into) h from S into T. If $H \times R$ is any right abelian group into which S can be injected (by f, say), then there exists an injection $k\colon T \to H \times R$ such that the following diagram commutes:

$$\begin{tikzcd}
T = \cup G_\alpha \cong G \times A \arrow[dr, "k"] & \\
S \arrow[u, "h"] \arrow[r, "f"'] & H \times R
\end{tikzcd}$$

Figure 1

Proof. Let $F' = \{G_\alpha : \alpha \in A\}$, where G_α is the quotient group of C_α, and where $G_\alpha \cap G_\beta = \phi$ if $\alpha \neq \beta$. Each C_α may be injected as a set of generators into G_α. Let h_α be such an injection: $G_\alpha = \{h_\alpha(s)h_\alpha(t)^{-1}: s, t \in C_\alpha\}$.

Let $T = \bigcup_{\alpha \in A} G_\alpha$. We define a semigroup operation $*$ on T. With this operation T will be an RZU of F'. Let $g = h_\alpha(s)h_\alpha(t)^{-1}$, and let $l = h_\beta(u)h_\beta(v)^{-1}$.

Let $g*l = h_\beta(s\circ u)h_\beta(t\circ v)^{-1}$, where $\circ$ is the semigroup operation on S.

Since $s, t \in C_\alpha$ and $u, v \in C_\beta$, $(s\circ u)$ and $(t\circ v)$ are in C_β. Thus these quantities are in the domain of h_β. We now verify $*$ is well defined.

Suppose $g = h_\alpha(s)h_\alpha(t)^{-1} = h_\alpha(a)h_\alpha(b)^{-1}$, $a, b \in C_\alpha$, and $l = h_\beta(u)h_\beta(v)^{-1} = h_\beta(c)h_\beta(d)^{-1}$, $c, d \in C_\beta$. We would like to prove that: $h_\beta(s\circ u)h_\beta(t\circ v)^{-1} = h_\beta(a\circ c)h_\beta(b\circ d)^{-1}$. Equivalently: $h_\beta(s\circ u)h_\beta(b\circ d) = h_\beta(a\circ c)h_\beta(t\circ v)$, or $h_\beta((s\circ u)\circ(b\circ d)) = h_\beta((a\circ c)\circ(t\circ v))$. We now verify that $(s\circ u)\circ(b\circ d) = (a\circ c)\circ(t\circ v)$.

We are given $h_\alpha(s)h_\alpha(t)^{-1} = h_\alpha(a)h_\alpha(b)^{-1}$. Equivalently: $h_\alpha(s)h_\alpha(b) = h_\alpha(a)h_\alpha(t)$, or $h_\alpha(s\circ b) = h_\alpha(a\circ t)$. Since h_α is $1-1$: $s\circ b = a\circ t$. Similarly $u\circ d = c\circ v$. Multiply left and right hand sides together: $(s\circ b)\circ(u\circ d) = (a\circ t)\circ(c\circ v)$. These products are taken in the subsemigroup $C_\alpha \cup C_\beta$ of S. By Lemma 2.1, $C_\alpha \cup C_\beta$ is left commutative. Thus $(s\circ b)\circ(u\circ d) = (s\circ u)\circ(b\circ d)$, and $(a\circ t)\circ(c\circ v) = (a\circ c)\circ(t\circ v)$. Thus $(s\circ u)\circ(b\circ d) = (a\circ c)\circ(t\circ v)$.

It is easily proven that $*$ is associative, and that $*$ restricted to any G_α is just the given group operation.

Since T is an RZU of groups, it follows from Lemma 2.3 that $T \cong G \times A$.

The h of the diagram is to be an injection of $S = \bigcup_{\alpha \in A} C_\alpha$ into $\bigcup_{\alpha \in A} G_\alpha$. Recall that if $\alpha \neq \beta$ then $G_\alpha \cap G_\beta = \phi$ and $C_\alpha \cap C_\beta = \phi$. Define h by: h restricted to C_α is h_α. Since h_α is $1-1$ h is $1-1$. Let $x \in C_\alpha$, $y \in C_\beta$. We now prove that $h(x\circ y) = h(x)*h(y)$, or $h_\beta(x\circ y) = h_\alpha(x)*h_\beta(y)$. Now $h_\alpha(x) = h_\alpha(x\circ x)h_\alpha(x)^{-1}$ and $h_\beta(y) = h_\beta(y\circ y)h_\beta(y)^{-1}$. Thus $h_\alpha(x)*h_\beta(y) = h_\beta((x\circ x)\circ(y\circ y))h_\beta(x\circ y)^{-1}$. By Lemma 2.1, $(x\circ x)\circ(y\circ y) = (x\circ y)\circ(x\circ y)$. Thus

$$h_\alpha(x)*h_\beta(y) = h_\beta((x\circ y)\circ(x\circ y))h_\beta(x\circ y)^{-1} = h_\beta(x\circ y)h_\beta(x\circ y)h_\beta(x\circ y)^{-1} = h_\beta(x\circ y)\ .$$

Let f be an injection of S into another right abelian group $H \times R$. If $f(x) = (g, r)$ define $f(x)^{-1} = (g^{-1}, r)$. One proves that $f(x\circ y)^{-1} = f(x)^{-1}f(y)^{-1}$.

We now define k of the diagram. Let $x \in G_\alpha$. There exists $s, t \in C_\alpha$ such that $x = h_\alpha(s)h_\alpha(t)^{-1}$. Define $k(x) = f(s)f(t)^{-1}$.

We now verify that k is well defined. Suppose $x = h_\alpha(s)h_\alpha(t)^{-1} = h_\alpha(u)h_\alpha(v)^{-1}$. Then $h_\alpha(s)h_\alpha(v) = h_\alpha(u)h_\alpha(t)$, or $h_\alpha(s\circ v) = h_\alpha(u\circ t)$. Since h_α is $1-1$, $s\circ v = u\circ t$. Now $f(s\circ v) = f(u\circ t)$, or $f(s)f(v) = f(u)f(t)$. We now show that $f(s)f(t)^{-1} = f(u)f(v)^{-1}$.

Let π be the projection of $H \times R$ onto R, the right zero semigroup. Since C_α is commutative, $\pi f(C_\alpha)$ is commutative, but then $|\pi f(C_\alpha)| = 1$. Thus $f(C_\alpha) \subseteqq H \times \{\alpha'\} = T_{\alpha'}$ for some α' in R.

Since s, t, u, v are in C_α, $f(s)$, $f(t)$, $f(u)$, $f(v)$, $f(t)^{-1}$, and $f(v)^{-1}$ are all in $T_{\alpha'}$. Since $T_{\alpha'}$ is commutative, $f(s)f(v) = f(u)f(t)$ implies $f(s)f(t)^{-1} = f(u)f(v)^{-1}$.

We now verify that the diagram is commutative. Let $s \in C_\alpha$. Then $h(s) = h_\alpha(s) = h_\alpha(s \circ s)h_\alpha(s)^{-1}$. $k(h(s)) = f(s \circ s)f(s)^{-1} = f(s)f(s)f(s)^{-1} = f(s)$.

We now verify that k is a homomorphism. Let $x = h_\alpha(s)h_\alpha(t)^{-1}$, $y = h_\beta(u)h_\beta(v)^{-1}$. Then $k(x*y) = k(h_\beta(s \circ u)h_\beta(t \circ v)^{-1}) = f(s \circ u)f(t \circ v)^{-1} = f(s)f(u)f(t)^{-1}f(v)^{-1}$. Since a right abelian group is left commutative, $k(x*y) = f(s)f(u)f(t)^{-1}f(v)^{-1} = f(s)f(t)^{-1}f(u)f(v)^{-1} = k(x)k(y)$.

We now prove k is $1-1$. We first prove k restricted to G_α is $1-1$. Let $x = h_\alpha(s)h_\alpha(t)^{-1}$, $y = h_\alpha(u)h_\alpha(v)^{-1}$. Assume $k(x) = k(y)$. Then $f(s)f(t)^{-1} = f(u)f(v)^{-1}$. Since s, t, u, v, are in C_α, $f(s)$, $f(t)$, $f(u)$, $f(v)$, $f(t)^{-1}$ $f(v)^{-1}$ are in $f(C_\alpha) = T_{\alpha'}$ a commutative semigroup. Thus $f(s)f(t)^{-1} = f(u)f(v)^{-1}$ implies $f(s)f(v) = f(u)f(t)$, or $f(s \circ v) = f(u \circ t)$. Since f is $1-1$, $s \circ v = u \circ t$. Now $h(s \circ v) = h(u \circ t)$, or $h_\alpha(s)h_\alpha(v) = h_\alpha(u)h_\alpha(t)$. Thus $x = y$.

Let $x = h_\alpha(s)h_\alpha(t)^{-1}$, $y = h_\beta(u)h_\beta(v)^{-1}$. Assume $k(x) = k(y)$. We prove that $\alpha = \beta$. Since k restricted to G_α is $1-1$, this will prove $x = y$. Now $f(s)f(t)^{-1} = f(u)f(v)^{-1}$, where $s, t \in C_\alpha$ and $u, v \in C_\beta$. We proved $f(C_\alpha) \subseteqq H \times \{\alpha'\}$; similarly, $f(C_\beta) \subseteqq H \times \{\beta'\}$. Since $f(s)f(t)^{-1} = f(u)f(v)^{-1}$, $\alpha' = \beta'$. If $\alpha \neq \beta$ then f would be an injection of the noncommutative semigroup $C_\alpha \cup C_\beta$ into the commutative semigroup $H \times \{\alpha'\}$. Thus $\alpha = \beta$.

COROLLARY 2.5. *Let S be an RZU of $F = \{C_\alpha : \alpha \in A\}$, where F is a disjoint family of commutative cancellative semigroups. Then S is left cancellative and left commutative.*

Proof. By Theorem 2.4, S can be thought of as a subsemigroup of a right abelian group. Every subsemigroup of a right abelian group is left cancellative and left commutative.

THEOREM 2.6. *If a semigroup S is left commutative and left cancellative, then S has a quotient right abelian group.*

Proof. Define a relation ρ on S by $x\rho y$ if and only if there exist $c, d \in S$ such that $cx = dy$. We prove that ρ is an r-congruence on $S(S/\rho$ is a right zero semigroup), and each congruence class is commutative cancellative. Thus S is an RZU of commutative cancellative semigroups and the result follows from the previous theorem.

Now ρ is certainly reflexive and symmetric.

Suppose $x\rho y$ and $y\rho z$. There exist a, b, c, d in S such that: $ax = by$ and $cy = dz$. Now $cax = cby$, and $bcy = bdz$. By left commutativity, $cby = bcy$. Thus $cax = bdz$, or $x\rho z$. Easily, ρ is right compatible. Left compatibility follows from left commutativity.

Now $xy\rho y$, for let c be arbitrary, and let $d = cx$; then $cxy = dy$. Thus ρ is an r-congruence.

We now prove that each congruence class is commutative. Since S is left cancellative, each congruence class will be commutative and cancellative.

Let $x\rho y$. We have $cx = dy$. Thus $cxdy = dycx$. By left commutativity $cdxy = cdyx$. By left cancellativity $xy = yx$. Easily any congruence class of an r-congruence is a semigroup.

REMARK. Since each congruence class of ρ is commutative, ρ is the smallest r-congruence on S.

Every subsemigroup of a right abelian group is left commutative and left cancellative. Thus the last theorem characterizes subsemigroups of right abelian groups.

LEMMA 2.7. *Let S be a left commutative semigroup. Define η on S by: $x\eta y$ if and only if there is an element b in S such that $bx = by$. Then η is the smallest left cancellative congruence on S.*

Proof. Using left commutativity one proves η is a congruence. It is also easy to prove that S/η is left cancellative.

Let f be a homomorphism of S onto a left cancellative semigroup S'. Suppose $x\eta y$, or $ax = ay$ for some a in S; then $f(ax) = f(ay)$, or $f(a)f(x) = f(a)f(y)$. Since S' is left cancellative $f(x) = f(y)$. Let ρ be the congruence induced by f. If $x\eta y$ then $x\rho y$, or $n \subseteqq \rho$.

We now consider constructing an RZU of a family of homomorphic images given that the original family has an RZU.

THEOREM 2.8. *Let S be an RZU of $\{C_\alpha: \alpha \in A\}$, where C_α are commutative semigroups. Let η_α be the smallest left cancellative congruence defined on C_α. Then the family $\{C_\alpha/\eta_\alpha: \alpha \in A\}$ has an RZU.*

Proof. Let $\eta_\alpha[x]$ be a congruence class of C_α, and let $\eta_\beta[y]$ be a congruence class of C_β. Define $\eta_\alpha[x] \circ \eta_\beta[y] = \eta_\beta[xy]$. ($xy$ is taken in S.) If the operation is well defined, then it is associative, and it defines an RZU of the C_α/η_α.

Suppose $\eta_\alpha[x] = \eta_\alpha[a]$, and $\eta_\beta[y] = \eta_\beta[b]$. We would like to show that $\eta_\beta[ab] = \eta_\beta[xy]$. Since $\eta_\alpha[x] = \eta_\alpha[a]$ there exists d in C_α such that $dx = da$. Similarly, there exists w in C_β such that $wy = wb$. Now $dxwy = dawb$. All elements lie in the RZU of C_α and C_β. We invoke Lemma 2.1. By left commutativity, $dwxy = dwab$. Thus $\eta_\beta[xy] = \eta_\beta[ab]$, because $dw \in C_\beta$ as are xy and ab.

Since $\{C_\alpha/\eta_\alpha : \alpha \in A\}$ has an RZU, by Theorem 2.4, the quotient groups of the C_α/η_α are isomorphic. This imposes another necessary condition for a family of commutative semigroups to have an RZU. If $|A| = 2$, using Lemma 2.1, then for η of Lemma 2.7: $\eta = \eta_1 \cup \eta_2$, $S/\eta = C_1/\eta_1 \cup C_2/\eta_2$ RZU.

III. Necessary and sufficient conditions on commutative cancellative semigroups to have an *RZU*. This section begins by relating the translational semigroup of a commutative cancellative semigroup A with the quotient group of A.

DEFINITION 3.1. Let A be a commutative semigroup. A function f, from A into A, is called a translation of A if $f(ab) = f(a)b$ for all a and b in A. $T(A)$ will denote the semigroup of all translations on A. Let i be the mapping from A into $T(A)$ given by $i(a) = f_a$, where f_a is the inner translation induced by $a \in A$: $f_a(x) = ax$, for all x in A. $i(A)$ is the semigroup of all inner translations on A.

Let A be a commutative cancellative semigroup. Let G be the quotient group of A. Recall G is abelian. A may be injected into G as a set of generators. Using this fact we relate G to $T(A)$.

The following lemmas are easily proven.

LEMMA 3.2. *Let A be injected by j as a set of generators into G. Let $f \in T(A)$. Define f^* on $j(A)$ by $f^*(j(a)) = j(f(a))$. f^* can be extended to a translation on G as follows: if $g \in G$ there exists $j(a_1)$ and $j(a_2)$ such that $g = j(a_1)j(a_2)^{-1}$. Define $f^*(g) = f^*(j(a_1))\, j(a_2)^{-1}$.*

LEMMA 3.3. *Let $i: A \to T(A)$ given by: $i(a) = f_a$. Let h: $T(A) \to$*

$$\begin{tikzcd} T(A) \arrow[r, "h"] & T(G) \arrow[r, "k"] & G \\ A \arrow[u, "i"] \arrow[urr, "j"'] & & \end{tikzcd}$$

Figure 2

$T(G)$ given by: $h(f) = f^*$. Let k: $T(G) \to G$ given by: $k(f^*) = f^*(1)$, where 1 is the identity of G. The above diagram commutes in the sense that $j(a) = k(h(i(a)))$ for all $a \in A$. Each map is injective; k is onto.

COROLLARY 3.4. *Let A be a commutative cancellative semigroup. $T(A)$ is a commutative cancellative semigroup. If $f \in T(A)$ then f is $1-1$ on A.*

Proof. Since kh injects $T(A)$ into an abelian group, $T(A)$ is commutative and cancellative. Let $f \in T(A)$. Suppose that $f(a_1) = f(a_2)$. Then $j(f(a_1)) = j(f(a_2))$, or $f^*(j(a_1)) = f^*(j(a_2))$. j is injective; also every translation on a group is $1-1$. Thus $j(a_1) = j(a_2)$, and $a_1 = a_2$, or f is $1-1$.

LEMMA 3.5. *Let A be a commutative cancellative semigroup. Let G be the quotient group of A. Let j be an injection of A into G as a set of generators. Define $TG(A) = \{g \in G: gj(A) \subseteqq j(A)\}$. Under the injection kh of Lemma 3.3, $T(A) \cong TG(A)$. Also $i(A)$ is equal to $h^{-1}k^{-1}(j(A))$.*

Proof. Let $g \in TG(A)$. Define f on A by $f(a) = j^{-1}(gj(a))$, $a \in A$. Then $f \in T(A)$, and $kh(f) = g$. Thus $TG(A) \subseteqq kh(T(A))$. To prove the reverse inclusion, *let* $f \in T(A)$. Since f^* is a translation, and $f^*(j(a)) = j(f(a))$, we have $f^*(1)j(a) = f^*(1 \cdot j(a)) = f^*(j(a)) = j(f(a))$. Thus $f^*(1)j(A) \subseteqq j(A)$, or $kh(f) \in TG(A)$. The remaining part of the lemma is proven by $kh(i(A)) = j(A)$ (Lemma 3.3) and the fact that kh is injective.

THEOREM 3.6. *Let $F = \{S_\alpha: \alpha \in \Gamma\}$ be a disjoint family of commutative cancellative semigroups. Let $\alpha \in \Gamma$, and let $P(\alpha)$ be the following statement: there exists $T_\alpha = \{f_\beta: \beta \in \Gamma\}$, a family of injections (isomorphisms, into), where f_β: $S_\beta \to T(S_\alpha)$ for all β in Γ, and where $f_\gamma(S_\gamma)f_\beta(S_\beta) \subseteqq f_\gamma(S_\gamma) \cap f_\beta(S_\beta)$ for all γ and β in Γ. The following are equivalent:*

(a) *F has an RZU.*

(b) *For any $\alpha_0 \in \Gamma$, $P(\alpha_0)$ holds.*

(c) *For some $\alpha_0 \in \Gamma$, $P(\alpha_0)$ holds.*

Furthermore, in (b) and (c) we may take f_{α_0} to be i, the natural map of S_{α_0} onto the inner translations of S_{α_0}.

Proof. We first prove (a) implies (b). Let S be an RZU of F, and let α_0 be a fixed but arbitrary member of Γ. For each x in S, let f_x be the mapping of S_{α_0} into S_{α_0} given by $f_x(a) = xa$ for all a in S_{α_0}. The range of f_x is contained in S_{α_0} because S_{α_0} is a left ideal of S. The following are true:

(1) $f_x \in T(S_{\alpha_0})$.

(2) Let f be the mapping from S into $T(S_{\alpha_0})$ given by $f(x) = f_x$. f is a homomorphism and f restricted to any S_α is $1-1$. Note that f restricted to S_{α_0} is the map i.

(3) $f(S_\alpha)$ is an ideal of $f(S)$ for all α in Γ.

(1) is easily checked as is the first part of (2). Let α be an arbitrary member of Γ. We now prove that f restricted to S_α is $1-1$. Let a and b be members of S_α. Suppose $f(a) = f(b)$. Then $ax = bx$ for all $x \in S_{\alpha_0}$. But then $axa = bxa$ for all $x \in S_{\alpha_0}$. Let $x_0 \in S_{\alpha_0}$. We have $a(x_0a) = b(x_0a)$. Now $a, b \in S_\alpha$, and $(x_0a) \in S_\alpha$ because S_α is a left ideal of S. Since S_α is cancellative $a = b$. We now prove (3) by Corollary 3.4, $T(S_{\alpha_0})$ is commutaive. Thus $f(S)$ is commutative. Each S_α is a left ideal of S. Since f is a homomorphism, $f(S_\alpha)$ is a left ideal of $f(S)$. But all left ideals of a commutative semigroup are ideals.

For each α in Γ, let f_α be the restriction of f to S_α. Then f_α: $S_\alpha \to T(S_{\alpha_0})$. f_α is an injection by (2). By (3) $f_\alpha(S_\alpha)$ and $f_\beta(S_\beta)$ are ideals of $f(S)$. Thus $f_\alpha(S_\alpha)f_\beta(S_\beta) \subseteqq f_\alpha(S_\alpha) \cap f_\beta(S_\beta)$. This completes the proof of (a) implies (b).

Trivially (b) implies (c). We now prove (c) implies (a). Let $p(\alpha_0)$ hold. Define a binary operation on F as follows: Let $x \in S_\alpha$ and $y \in S_\beta$.

$$x \circ y = f_\beta^{-1}(f_\alpha(x)f_\beta(y))$$

$$y \circ x = f_\alpha^{-1}(f_\beta(y)f_\alpha(x))$$

where $f_\alpha, f_\beta \in T_{\alpha_0}$. The operation is well defined because $f_\alpha(x)f_\beta(y) \in f_\alpha(S_\alpha)f_\beta(S_\beta) \subseteqq f_\alpha(S_\alpha) \cap f_\beta(S_\beta)$. Thus $f_\alpha(x)f_\beta(y) \in f_\beta(S_\beta)$, and we may apply f_β^{-1}. Similarly $f_\beta(y)f_\alpha(x) \in f_\alpha(S_\alpha)$. The operation restricted to any S_α is the semigroup operation already given on S_α. Let $x, y \in S_\alpha$. Then $x \circ y = f_\alpha^{-1}(f_\alpha(x)f_\alpha(y)) = f_\alpha^{-1}(f_\alpha(xy)) = xy$. This is true because f_α is an injection. If the operation is associative, it certainly defines an RZU of F.

Let $x \in S_\alpha$, $y \in S_\beta$, and $z \in S_\gamma$. Then $(x \circ y) \circ z = (f_\beta^{-1}(f_\alpha(x)f_\beta(y))) \circ z = f_\gamma^{-1}(f_\beta(f_\beta^{-1}(f_\alpha(x)f_\beta(y)))f_\gamma(z)) = f_\gamma^{-1}((f_\alpha(x)f_\beta(y))f_\gamma(z))$. Similarly $x \circ (y \circ z) = f_\gamma^{-1}(f_\alpha(x)(f_\beta(y)f_\gamma(z)))$. Now $(x \circ y) \circ z = x \circ (y \circ z)$ since $f_\alpha(x)(f_\beta(y)f_\gamma(z)) = (f_\alpha(x)f_\beta(y))f_\gamma(z)$. The above product is taken in the semigroup $T(S_{\alpha_0})$, and is in $f_\gamma(S_\gamma)$.

REMARK. Let $(\alpha, \beta) \in \Gamma \times \Gamma$. Because $f_\alpha(S_\alpha)$ and $f_\beta(S_\beta)$ are subsets of the commutative semigroup $T(S_{\alpha_0})$, $f_\alpha(S_\alpha)f_\beta(S_\beta) \subseteqq f_\alpha(S_\alpha) \cap f_\beta(S_\beta)$ implies the same condition for the pair (β, α). Thus we need only consider one condition.

We restate Theorem 3.6 for two semigroups as follows: $F = \{A, B\}$ has an RZU if and only if there exists an injection f from B into

$T(A)$ such that $f(B)i(A) \subseteqq f(B) \cap i(A)$.

COROLLARY 3.7. *Let $F = \{S_\alpha : \alpha \in A\}$ be a disjoint family of commutative cancellative semigroups. If for some $\alpha_0 \in A$ each S_α is isomorphic to an ideal of S_{α_0} then F has an RZU.*

Proof. To say S_α is isomorphic to an ideal of S_{α_0} means there exists $h_\alpha : S_\alpha \to S_{\alpha_0}$, where h_α is an injection, and $h_\alpha(S_\alpha)$ is an ideal of S_{α_0}. Let $f_\alpha : S_\alpha \to T(S_{\alpha_0})$, given by $f_\alpha = i_{\alpha_0} \circ h_\alpha$, where $i_{\alpha_0} : S_{\alpha_0} \to T(S_{\alpha_0})$, given by $i_{\alpha_0}(x) = f_x$. $\{f_\alpha : \alpha \in A\}$ satisfies (c) of Theorem 3.6 because $f_\alpha(S_\alpha)$ is an ideal of $i_{\alpha_0}(S_{\alpha_0})$.

COROLLARY 3.8. *Let A and B be two disjoint commutative cancellative semigroups having an RZU. If A is a group then B is a group, and $A \cong B$.*

Proof. Every translation of a group is inner; thus $T(A) = i(A)$. Now $i(A)$ is the regular representation of A; thus $i(A) \cong A$. By Theorem 3.6, there exists an injection f of B into $T(A)$ such that $f(B)i(A) \subseteqq f(B) \cap i(A)$. f is an injection into $i(A)$. Since $T(A)$ is commutative, $f(B)$ is an ideal of $i(A)$. But a group has no proper ideals. Thus $f(B) \cong i(A) = A$. Since f is an injection $B \cong A$.

We now give an interpretation of Theorem 3.6 in terms of quotient groups. Let A be a commutative cancellative semigroup. Let j be an injection of A as a set of generators into G, the quotient group of A. Let f be the isomorphism from $T(A)$ onto $TG(A)$ ($TG(A)$ of Lemma 3.5; $f = kh$ of Lemma 3.3). Let B be a commutative cancellative semigroup having an RZU with A. Let h be an injection of B into $T(A)$ such that $h(B)i(A) \subseteqq h(B) \cap i(A)$. Compose the maps h and f. We have $(fh)(B)j(A) \subseteqq (fh)(B) \cap j(A)$. Evidently, B is isomorphic to B', a subsemigroup of $TG(A)$ such that $B'j(A) \subseteqq B' \cap j(A)$. Conversely, an isomorphic copy of such a B' will have an RZU with A. Thus we have a way of finding all commutative cancellative semigroups having an RZU with A.

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