ON RIGHT ZERO UNIONS OF COMMUTATIVE SEMIGROUPS

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Let $F = \{S_r : r \in R\}$ be a disjoint family of semigroups. One says that $F$ has a right zero union (RZU) if there exists a semigroup $S$ which is a disjoint union of the $S_r$ where each $S_r$ is a left ideal of $S$. This paper gives some theorems on RZU of commutative semigroups with special emphasis placed on commutative cancellative semigroups.

Suppose $S$ is an RZU of commutative cancellative semigroups. It is proven that $S$ has a quotient right abelian group; thus $S$ is left commutative and left cancellative. Conversely, it is proven that if a semigroup $S$ is left commutative and left cancellative, then $S$ is an RZU of commutative cancellative semigroups. Suppose $F$ is a family of commutative semigroups having an RZU; it is proven that a certain family of cancellative homomorphic images of $F$ also has an RZU. Finally, necessary and sufficient conditions are given for a family of commutative cancellative semigroups to have an RZU.

The study of RZU is a special case of the study of "bands of semigroups." R. Yoshida has studied the dual problem of left zero unions.

II. Some necessary conditions for RZU and an embedding result. A semigroup $S$ is left commutative if $xyz = yxz$ for all $x, y,$ and $z$ in $S$.

**Lemma 2.1.** The RZU of two commutative semigroups is left commutative.

**Proof.** The symmetric conditions $AB \subseteq B, BA \subseteq A, A$ and $B$ are commutative, are given. Let $a \in A$, and let $b, b_i \in B$. Now $abb_i = a(bb_i) = a(b,b) = (ab)b = b(ab) = bab_i$. Other cases are proven similarly.

**Definition 2.2.** Let $C$ be a commutative cancellative semigroup. The quotient group, $G$, of $C$ is the smallest group into which $C$ may be injected. If $C \subseteq T$, a group, then $G \cong \{st^{-1} : s, t \in C\}$. Note $G$ is abelian. (For more on quotient groups see [1].)

A right abelian group is the direct product of a right zero semigroup and an abelian group. A quotient right abelian group will have the same meaning as quotient group; namely, the smallest right abelian group into which a semigroup $S$ can be injected.

The next lemma is proven using the following result of Petrich
[2]: A semigroup $S$ is a semilattice of semigroups each of which is the Cartesian product of rectangular band and a group iff $S$ is a union of groups and its idempotents form a semigroup.

**Lemma 2.3.** Let $F = \{G_\alpha: \alpha \in A\}$ be a disjoint family of groups. Then $F$ has an RZU iff all the $G_\alpha$ are isomorphic. If the RZU exists then it is isomorphic to the right group $G \times A$, where $G_\alpha \cong G$, and where $A$ is considered as a right zero semigroup.

*Proof.* Let $S$ be an RZU of $F$. Certainly $S$ is union of groups. The idempotents of $S$ are exactly the $e_\alpha$, where $e_\alpha$ is the identity of $G_\alpha$. Since $e_\alpha$ is an identity and since $G_\alpha G_\beta \subseteq G_\beta$, we have $(e_\alpha e_\beta)(e_\alpha e_\beta) = e_\alpha (e_\beta (e_\alpha e_\beta)) = e_\alpha (e_\alpha e_\beta) = (e_\alpha e_\alpha) e_\beta = e_\alpha e_\beta = e_\beta$, for $e_\alpha e_\beta$ is the idempotent of $G_\beta$. Thus the idempotents of $S$ form a right zero semigroup. This semigroup is isomorphic to $A$, but also, by Petrich, to a semilattice union $\bigcup_{r \in r} L_r \times R_r$, and this implies that $|\Gamma| = 1$, $|L_r| = 1$, and $R_r = A$.

**Theorem 2.4.** Let $S$ be an RZU of $F = \{C_\alpha: \alpha \in A\}$, where $F$ is a disjoint family of commutative cancellative semigroups. Let $G_\alpha$ be the quotient group of $C_\alpha$. We consider the $G$ to be disjoint. Then all the $G_\alpha$ are isomorphic, and they have an RZU, $T$.

$T$ is isomorphic to $G \times A$, where $G_\alpha \cong G$, and where $A$ is considered as a right zero semigroup.

Furthermore, $T$ is the quotient right abelian group of $S$ in the following sense. There exists an injection (isomorphism into) $h$ from $S$ into $T$. If $H \times R$ is any right abelian group into which $S$ can be injected (by $f$, say), then there exists an injection $k: T \to H \times R$ such that the following diagram commutes:

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  T = \cup G_\alpha \cong G \times A
    \downarrow k
  S f \quad H \times R
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*Proof.* Let $F' = \{G_\alpha: \alpha \in A\}$, where $G_\alpha$ is the quotient group of $C_\alpha$, and where $G_\alpha \cap G_\beta = \phi$ if $\alpha \neq \beta$. Each $C_\alpha$ may be injected as a
set of generators into \( G_a \). Let \( h_a \) be such an injection: \( G_a = \{ h_a(s)h_a(t)^{-1}: s, t \in C_a \} \).

Let \( T = \bigcup_{a \in A} G_a \). We define a semigroup operation \( * \) on \( T \). With this operation \( T \) will be an RZU of \( F^a \). Let \( g = h_a(s)h_a(t)^{-1} \), and let \( l = h_b(u)h_b(v)^{-1} \).

Let \( g * l = h_\beta(s \circ u)h_\beta(t \circ v)^{-1} \), where \( \circ \) is the semigroup operation on \( S \).

Since \( s, t \in C_a \) and \( u, v \in C_\beta \), \((s \circ u)\) and \((t \circ v)\) are in \( C_\beta \). Thus these quantities are in the domain of \( h_\beta \). We now verify \( * \) is well defined.

Suppose \( g = h_a(s)h_a(t)^{-1} = h_a(a)h_a(b)^{-1} \), \( a, b \in C_a \), and \( l = h_\beta(u)h_\beta(v)^{-1} = h_\beta(c)h_\beta(d)^{-1} \), \( c, d \in C_\beta \). We would like to prove that: \( h_\beta(s \circ u)h_\beta(t \circ v)^{-1} = h_\beta(a \circ c)h_\beta(b \circ d)^{-1} \). Equivalently: \( h_\beta(s \circ u)h_\beta(b \circ d) = h_\beta(a \circ c)h_\beta(t \circ v) \), or \( h_\beta((s \circ u) \circ (b \circ d)) = h_\beta((a \circ c) \circ (t \circ v)) \). We now verify that \((s \circ u) \circ (b \circ d) = (a \circ c) \circ (t \circ v) \).

We are given \( g = h_a(s)h_a(t)^{-1} = h_a(a)h_a(b)^{-1} \). Equivalently: \( h_a(s)h_a(b) = h_a(a)h_a(t) \), or \( h_a(s \circ b) = h_a(a \circ t) \). Since \( h_a \) is 1-1 \( s \circ b = a \circ t \). Similarly \( u \circ d = c \circ v \). Multiply left and right hand sides together: \((s \circ b) \circ (u \circ d) = (a \circ t) \circ (c \circ v) \). These products are taken in the subsemigroup \( C_a \cup C_\beta \) of \( S \). By Lemma 2.1, \( C_a \cup C_\beta \) is left commutative. Thus \((s \circ u) \circ (b \circ d) = (a \circ c) \circ (t \circ v) \).

It is easily proven that \( * \) is associative, and that \( * \) restricted to any \( G_a \) is just the given group operation.

Since \( T \) is an RZU of groups, it follows from Lemma 2.3 that \( T \cong G \times A \).

The \( h \beta \) of the diagram is to be an injection of \( S = \bigcup_{a \in A} C_a \) into \( \bigcup_{a \in A} G_a \). Recall that if \( \alpha \neq \beta \) then \( C_\alpha \cap G_\beta = \phi \) and \( C_\alpha \cap C_\beta = \phi \). Define \( h \) by: \( h \) restricted to \( C_a \) is \( h_a \). Since \( h_a \) is 1-1 \( h \) is 1-1. Let \( x \in C_a, y \in C_\beta \). We now prove that \( h(x \circ y) = h(x) \circ h(y) \), or \( h_\beta(x \circ y) = h_\beta(x) \circ h_\beta(y) \). Now \( h_\alpha(x) = h_\alpha(x \circ x)h_\alpha(x)^{-1} \) and \( h_\beta(y) = h_\beta(y \circ y)h_\beta(y)^{-1} \). Thus \( h_\alpha(x) \circ h_\beta(y) = h_\beta((x \circ x) \circ (y \circ y))h_\beta(x \circ y)^{-1} \). By Lemma 2.1, \((x \circ x) \circ (y \circ y) = (x \circ y) \circ (x \circ y) \). Thus
\[
\begin{align*}
    h_\alpha(x) \circ h_\beta(y) &= h_\beta((x \circ y) \circ (x \circ y))h_\beta(x \circ y)^{-1} \\
    &= h_\beta(x \circ y)h_\beta(x \circ y)h_\beta(x \circ y)^{-1} = h_\beta(x \circ y).
\end{align*}
\]

Let \( f \) be an injection of \( S \) into another right abelian group \( H \times R \). If \( f(x) = (g, r) \) define \( f(x)^{-1} = (g^{-1}, r) \). One proves that \( f(x \circ y)^{-1} = f(x)^{-1}f(y)^{-1} \).

We now define \( k \) of the diagram. Let \( x \in G_a \). There exists \( s, t \in C_a \) such that \( x = h_a(s)h_a(t)^{-1} \). Define \( k(x) = f(s)f(t)^{-1} \).

We now verify that \( k \) is well defined. Suppose \( x = h_a(s)h_a(t)^{-1} = h_a(u)h_a(v)^{-1} \). Then \( h_a(s)h_a(v) = h_a(u)h_a(t) \), or \( h_a(s \circ v) = h_a(u \circ t) \). Since \( h_a \) is 1-1, \( s \circ v = u \circ t \). Now \( f(s \circ v) = f(u \circ t) \), or \( f(s)f(v) = f(u)f(t) \). We now show that \( f(s)f(t)^{-1} = f(u)f(v)^{-1} \).

Let \( \pi \) be the projection of \( H \times R \) onto \( R \), the right zero semi-
group. Since $C_a$ is commutative, $\pi f(C_a)$ is commutative, but then $|\pi f(C_a)| = 1$. Thus $f(C_a) \subseteq H \times \{\alpha'\} = T_{\alpha'}$ for some $\alpha'$ in $R$.

Since $s, t, u, v$ are in $C_a$, $f(s), f(t), f(u), f(v), f(t)^{-1}$, and $f(v)^{-1}$ are all in $T_{\alpha'}$. Since $T_{\alpha'}$ is commutative, $f(s)f(v) = f(u)f(t)$ implies $f(s)f(t)^{-1} = f(u)f(v)^{-1}$.

We now verify that the diagram is commutative. Let $s \in C_a$. Then $h(s) = h_a(s) = h_a(s \cdot s)h_a(s)^{-1}$. $k(h(s)) = f(s \cdot s)f(s)^{-1} = f(s)f(s)f(s)^{-1} = f(s)$.

We now verify that $k$ is a homomorphism. Let $x = h_a(s)h_a(t)^{-1}$, $y = h_b(u)h_b(v)^{-1}$. Then $k(xy) = k(h_a(s)h_b(t \cdot v)^{-1}) = f(s \cdot u)f(t \cdot v)^{-1} = f(s)f(u)f(t)^{-1}f(v)^{-1}$. Since a right abelian group is left commutative, $k(x \cdot y) = f(s)f(u)f(t)^{-1}f(v)^{-1} = f(s)f(t)^{-1}f(u)f(v)^{-1} = k(x)k(y)$.

We now prove $k$ is a homomorphism. Let $x = h_a(s)h_a(t)^{-1}$, $y = h_b(u)h_b(v)^{-1}$. Assume $k(x) = k(y)$. Then $f(s)f(t)^{-1} = f(u)f(v)^{-1}$. Since $s, t, u, v$ are in $C_a$, $f(s), f(t), f(u), f(v), f(t)^{-1}f(v)^{-1}$ are in $f(C_a) = T_{\alpha'}$ a commutative semigroup. Thus $f(s)f(t)^{-1} = f(u)f(v)^{-1}$ implies $f(s)f(v) = f(u)f(t)$, or $f(s \cdot v) = f(u \cdot t)$. Since $f$ is $1 - 1$, $s \cdot v = u \cdot t$. Now $h(s \cdot v) = h(u \cdot t)$, or $h_a(s)h_a(v) = h_a(u)h_a(t)$. Thus $x = y$.

Let $x = h_a(s)h_a(t)^{-1}$, $y = h_b(u)h_b(v)^{-1}$. Assume $k(x) = k(y)$. We prove that $x = y$. Since $k$ restricted to $G_a$ is $1 - 1$, this will prove $x = y$. Now $f(s)f(t)^{-1} = f(u)f(v)^{-1}$, where $s, t \in C_a$ and $u, v \in C_b$. We proved $f(C_a) \subseteq H \times \{\alpha'\}$; similarly, $f(C_b) \subseteq H \times \{\beta'\}$. Since $f(s)f(t)^{-1} = f(u)f(v)^{-1}$, $\alpha' = \beta'$. If $\alpha \neq \beta$ then $f$ would be an injection of the noncommutative semigroup $C_a \cup C_b$ into the commutative semigroup $H \times \{\alpha'\}$. Thus $\alpha = \beta$.

**Corollary 2.5.** Let $S$ be an RZU of $F = \{C_{\alpha}: \alpha \in A\}$, where $F$ is a disjoint family of commutative cancellative semigroups. Then $S$ is left cancellative and left commutative.

**Proof.** By Theorem 2.4, $S$ can be thought of as a subsemigroup of a right abelian group. Every subsemigroup of a right abelian group is left cancellative and left commutative.

**Theorem 2.6.** If a semigroup $S$ is left commutative and left cancellative, then $S$ has a quotient right abelian group.

**Proof.** Define a relation $\rho$ on $S$ by $xy$ if and only if there exist $c, d \in S$ such that $cx = dy$. We prove that $\rho$ is an $r$-congruence on $S(S/\rho$ is a right zero semigroup), and each congruence class is commutative cancellative. Thus $S$ is an RZU of commutative cancellative semigroups and the result follows from the previous theorem.
Now \( \rho \) is certainly reflexive and symmetric.

Suppose \( x\rho y \) and \( y\rho z \). There exist \( a, b, c, d \) in \( S \) such that: \( ax = by \) and \( cy = dz \). Now \( cax = cby \), and \( bey = bdz \). By left commutativity, \( cby = bey \). Thus \( cax = bdz \), or \( x\rho z \). Easily, \( \rho \) is right compatible. Left compatibility follows from left commutativity.

Now \( x\rho y \), for let \( c \) be arbitrary, and let \( d = cx \); then \( cxy = dy \). Thus \( \rho \) is an \( r \)-congruence.

We now prove that each congruence class is commutative. Since \( S \) is left cancellative, each congruence class will be commutative and cancellative.

Let \( x\rho y \). We have \( cx = dy \). Thus \( cxy = dycx \). By left commutativity \( cdxy = cdyx \). By left cancellativity \( xy = yx \). Easily any congruence class of an \( r \)-congruence is a semigroup.

**Remark.** Since each congruence class of \( \rho \) is commutative, \( \rho \) is the smallest \( r \)-congruence on \( S \).

Every subsemigroup of a right abelian group is left commutative and left cancellative. Thus the last theorem characterizes subsemigroups of right abelian groups.

**Lemma 2.7.** Let \( S \) be a left commutative semigroup. Define \( \eta \) on \( S \) by: \( x\eta y \) if and only if there is an element \( b \) in \( S \) such that \( bx = by \). Then \( \eta \) is the smallest left cancellative congruence on \( S \).

**Proof.** Using left commutativity one proves \( \eta \) is a congruence. It is also easy to prove that \( S/\eta \) is left cancellative.

Let \( f \) be a homomorphism of \( S \) onto a left cancellative semigroup \( S' \). Suppose \( x\eta y \), or \( ax = ay \) for some \( a \) in \( S \); then \( f(ax) = f(ay) \), or \( f(a)f(x) = f(a)f(y) \). Since \( S' \) is left cancellative \( f(x) = f(y) \). Let \( \rho \) be the congruence induced by \( f \). If \( x\eta y \) then \( x\rho y \), or \( n \subseteq \rho \).

We now consider constructing an RZU of a family of homomorphic images given that the original family has an RZU.

**Theorem 2.8.** Let \( S \) be an RZU of \( \{C_a : \alpha \in A\} \), where \( C_a \) are commutative semigroups. Let \( \eta_a \) be the smallest left cancellative congruence defined on \( C_a \). Then the family \( \{C_a/\eta_a : \alpha \in A\} \) has an RZU.

**Proof.** Let \( \eta_a[x] \) be a congruence class of \( C_a \), and let \( \eta_b[y] \) be a congruence class of \( C_b \). Define \( \eta_a[x] \circ \eta_b[y] = \eta_b[xy] \). (\( xy \) is taken in \( S \).) If the operation is well defined, then it is associative, and it defines an RZU of the \( C_a/\eta_a \).

Suppose \( \eta_a[x] = \eta_a[a] \), and \( \eta_b[y] = \eta_b[b] \). We would like to show
that $\gamma_\beta[ab] = \gamma_\beta[xy]$. Since $\gamma_\beta[a] = \gamma_\beta[a]$ there exists $d$ in $C_\beta$ such that $dx = da$. Similarly, there exists $w$ in $C_\beta$ such that $wy = wb$. Now $dxwy = dawb$. All elements lie in the RZU of $C_\alpha$ and $C_\beta$. We invoke Lemma 2.1. By left commutativity, $dwx = dwab$. Thus $\gamma_\beta[xy] = \gamma_\beta[ab]$, because $dw \in C_\beta$ as are $\gamma$ and $\alpha \delta$.

Since $\{C_\alpha/\gamma_\alpha: \alpha \in A\}$ has an RZU, by Theorem 2.4, the quotient groups of the $C_\alpha/\gamma_\alpha$ are isomorphic. This imposes another necessary condition for a family of commutative semigroups to have an RZU. If $|A| = 2$, using Lemma 2.1, then for $\eta$ of Lemma 2.7: $\eta = \eta_1 \cup \eta_2$, $S/\eta = C_\eta/\gamma_1 \cup C_\eta/\gamma_2$ RZU.

III. Necessary and sufficient conditions on commutative cancellative semigroups to have an RZU. This section begins by relating the translational semigroup of a commutative cancellative semigroup $A$ with the quotient group of $A$.

DEFINITION 3.1. Let $A$ be a commutative semigroup. A function $f$, from $A$ into $A$, is called a translation of $A$ if $f(ab) = f(a)b$ for all $a$ and $b$ in $A$. $T(A)$ will denote the semigroup of all translations on $A$. Let $i$ be the mapping from $A$ into $T(A)$ given by $i(a) = f_a$, where $f_a$ is the inner translation induced by $a \in A$: $f_a(x) = ax$, for all $x$ in $A$. $i(A)$ is the semigroup of all inner translations on $A$.

Let $A$ be a commutative cancellative semigroup. Let $G$ be the quotient group of $A$. Recall $G$ is abelian. $A$ may be injected into $G$ as a set of generators. Using this fact we relate $G$ to $T(A)$.

The following lemmas are easily proven.

LEMMA 3.2. Let $A$ be injected by $j$ as a set of generators into $G$. Let $f \in T(A)$. Define $f^*$ on $j(A)$ by $f^*(j(a)) = j(f(a))$. $f^*$ can be extended to a translation on $G$ as follows: if $g \in G$ there exists $j(a_2)$ and $j(a_3)$ such that $g = j(a_3)j(a_2)^{-1}$. Define $f^*(g) = f^*(j(a_3))j(a_2)^{-1}$.

LEMMA 3.3. Let $i: A \to T(A)$ given by $i(a) = f_a$. Let $h: T(A) \to T(G)$ given by $h(f) = f^*$. Let $k: T(G) \to G$ given by $k(g) = g$. Figure 2
Let $h(f) = f^*$. Let $k: T(G) \rightarrow G$ given by $k(f^*) = f^*(1)$, where $1$ is the identity of $G$. The above diagram commutes in the sense that $j(a) = k(h(i(a)))$ for all $a \in A$. Each map is injective; $k$ is onto.

**Corollary 3.4.** Let $A$ be a commutative cancellative semigroup. $T(A)$ is a commutative cancellative semigroup. If $f \in T(A)$ then $f$ is $1 - 1$ on $A$.

**Proof.** Since $kh$ injects $T(A)$ into an abelian group, $T(A)$ is commutative and cancellative. Let $f \in T(A)$. Suppose that $f(a_1) = f(a_2)$. Then $j(f(a_1)) = j(f(a_2))$, or $f^*(j(a_1)) = f^*(j(a_2))$. $j$ is injective; also every translation on a group is $1 - 1$. Thus $j(a_1) = j(a_2)$, and $a_1 = a_2$, or $f$ is $1 - 1$.

**Lemma 3.5.** Let $A$ be a commutative cancellative semigroup. Let $G$ be the quotient group of $A$. Let $j$ be an injection of $A$ into $G$ as a set of generators. Define $TG(A) = \{g \in G: gj(A) \subseteq j(A)\}$. Under the injection $kh$ of Lemma 3.3, $T(A) \equiv TG(A)$. Also $i(A)$ is equal to $h^{-1}k^{-1}(j(A))$.

**Proof.** Let $g \in TG(A)$. Define $f$ on $A$ by $f(a) = j^{-1}(gj(a))$, $a \in A$. Then $f \in T(A)$, and $kh(f) = g$. Thus $TG(A) \subseteq kh(T(A))$. To prove the reverse inclusion, let $f \in T(A)$. Since $f^*$ is a translation, and $f^*(j(a)) = j(f(a))$, we have $f^*(1)j(a) = f^*(1 \cdot j(a)) = f^*(j(a)) = j(f(a))$. Thus $f^*(1)j(A) \subseteq j(A)$, or $kh(f) \in TG(A)$. The remaining part of the lemma is proven by $kh(i(A)) = j(A)$ (Lemma 3.3) and the fact that $kh$ is injective.

**Theorem 3.6.** Let $F = \{S_\alpha: \alpha \in \Gamma\}$ be a disjoint family of commutative cancellative semigroups. Let $\alpha \in \Gamma$, and let $P(\alpha)$ be the following statement: there exists $T_\alpha = \{f_\beta: \beta \in \Gamma\}$, a family of injections (isomorphisms, into), where $f_\beta: S_\beta \rightarrow T(S_\alpha)$ for all $\beta$ in $\Gamma$, and where $f_\gamma(S_\gamma)f_\beta(S_\beta) \subseteq f_\gamma(S_\gamma) \cap f_\beta(S_\beta)$ for all $\gamma$ and $\beta$ in $\Gamma$. The following are equivalent:

(a) $F$ has an RZU.
(b) For any $\alpha_\beta \in \Gamma$, $P(\alpha_\beta)$ holds.
(c) For some $\alpha_\beta \in \Gamma$, $P(\alpha_\beta)$ holds.

Furthermore, in (b) and (c) we may take $f_{\alpha_\beta}$ to be $i$, the natural map of $S_{\alpha_\beta}$ onto the inner translations of $S_{\alpha_\beta}$.

**Proof.** We first prove (a) implies (b). Let $S$ be an RZU of $F$, and let $\alpha_\beta$ be a fixed but arbitrary member of $\Gamma$. For each $x$ in $S$, let $f_x$ be the mapping of $S_{\alpha_\beta}$ into $S_{\alpha_\beta}$ given by $f_x(a) = xa$ for all $a$ in $S_{\alpha_\beta}$. The range of $f_x$ is contained in $S_{\alpha_\beta}$ because $S_{\alpha_\beta}$ is a left ideal of $S$. The following are true:
Let $f$ be the mapping from $S$ into $T(S_a)$ given by $f(x) = f_x$. $f$ is a homomorphism and $f$ restricted to any $S_a$ is 1–1. Note that $f$ restricted to $S_a$ is the map $i_a$.

$f(S_a)$ is an ideal of $f(S)$ for all $a$ in $\Gamma$. (1) is easily checked as is the first part of (2). Let $a$ be an arbitrary member of $\Gamma$. We now prove that $f$ restricted to $S_a$ is 1–1. Let $a$ and $b$ be members of $S_a$. Suppose $f(a) = f(b)$. Then $ax = bx$ for all $x \in S_a$. But then $axa = bxa$ for all $x \in S_a$. Let $x_a \in S_a$. We have $a(x,a) = b(x,a)$. Now $a, b \in S_a$, and $(x,a) \in S_a$ because $S_a$ is a left ideal of $S$. Since $S_a$ is cancellative $a = b$. We now prove (3) by Corollary 3.4, $T(S_a)$ is commutative. Thus $f(S)$ is commutative. Each $S_a$ is a left ideal of $S$. Since $f$ is a homomorphism, $f(S_a)$ is a left ideal of $f(S)$. But all left ideals of a commutative semigroup are ideals.

For each $a$ in $\Gamma$, let $f_a$ be the restriction of $f$ to $S_a$. Then $f_a: S_a \rightarrow T(S_a)$. $f_a$ is an injection by (2). By (3) $f_a(S_a)$ and $f_b(S_b)$ are ideals of $f(S)$. Thus $f_a(S_a)f_b(S_b) \subseteq f_a(S_a) \cap f_b(S_b)$. This completes the proof of (a) implies (b).

Trivially (b) implies (c). We now prove (c) implies (a). Let $p(\alpha, \beta)$ hold. Define a binary operation on $F$ as follows: Let $x \in S_a$ and $y \in S_b$.

$$x \circ y = f_{\beta}^{-1}(f_a(x)f_{\beta}(y))$$

$$y \circ x = f_{\alpha}^{-1}(f_\beta(y)f_a(x))$$

where $f_a, f_\beta \in T(S_a)$. The operation is well defined because $f_a(x)f_{\beta}(y) \in f_a(S_a)f_\beta(S_b) \subseteq f_a(S_a) \cap f_\beta(S_b)$. Thus $f_a(x)f_{\beta}(y) \in f_\beta(S_b)$, and we may apply $f_{\beta}^{-1}$. Similarly $f_\beta(y)f_a(x) \in f_a(S_a)$. The operation restricted to any $S_a$ is the semigroup operation already given on $S_a$. Let $x, y \in S_a$. Then $x \circ y = f_{\alpha}^{-1}(f_a(x)f_{\beta}(y)) = f_{\alpha}^{-1}(f_a(xy)) = xy$. This is true because $f_a$ is an injection. If the operation is associative, it certainly defines an RZU of $F$.

Let $x \in S_a$, $y \in S_b$, and $z \in S_b$. Then $(x \circ y) \circ z = (f_{\beta}^{-1}(f_a(x)f_{\beta}(y))) \circ z = f_{\beta}^{-1}(f_\beta(f_{\alpha}^{-1}(f_a(x)f_{\beta}(y)))f_{\beta}(z)) = f_{\beta}^{-1}((f_a(x)f_{\beta}(y))f_{\beta}(z))$. Similarly $x \circ (y \circ z) = f_{\beta}^{-1}(f_a(x)(f_{\beta}(y)f_{\beta}(z)))$. Now $(x \circ y) \circ z = x \circ (y \circ z)$ since $f_a(x)(f_{\beta}(y)f_{\beta}(z)) = (f_a(x)f_{\beta}(y))f_{\beta}(z)$. The above product is taken in the semigroup $T(S_a)$, and is in $f_a(S_a)$.

REMARK. Let $(\alpha, \beta) \in \Gamma \times \Gamma$. Because $f_a(S_a)$ and $f_\beta(S_b)$ are subsets of the commutative semigroup $T(S_a)$, $f_a(S_a)f_\beta(S_b) \subseteq f_a(S_a) \cap f_\beta(S_b)$ implies the same condition for the pair $(\beta, \alpha)$. Thus we need only consider one condition.

We restate Theorem 3.6 for two semigroups as follows: $F = \{A, B\}$ has an RZU if and only if there exists an injection $f$ from $B$ into
T(A) such that \( f(B)i(A) \subseteq f(B) \cap i(A) \).

**Corollary 3.7.** Let \( F = \{ S_{\alpha} : \alpha \in A \} \) be a disjoint family of commutative cancellative semigroups. If for some \( \alpha_0 \in A \) each \( S_{\alpha} \) is isomorphic to an ideal of \( S_{\alpha_0} \) then \( F \) has an RZU.

**Proof.** To say \( S_{\alpha} \) is isomorphic to an ideal of \( S_{\alpha_0} \) means there exists \( h_{\alpha} : S_{\alpha} \rightarrow S_{\alpha_0} \), where \( h_{\alpha} \) is an injection, and \( h_\alpha(S_{\alpha}) \) is an ideal of \( S_{\alpha_0} \). Let \( f_\alpha : S_{\alpha} \rightarrow T(S_{\alpha_0}) \), given by \( f_\alpha = i_{\alpha_0} \circ h_\alpha \), where \( i_{\alpha_0} : S_{\alpha_0} \rightarrow T(S_{\alpha_0}) \), given by \( i_{\alpha_0}(x) = f_\alpha \). \( \{ f_\alpha : \alpha \in A \} \) satisfies (c) of Theorem 3.6 because \( f_\alpha(S_{\alpha}) \) is an ideal of \( i_{\alpha_0}(S_{\alpha_0}) \).

**Corollary 3.8.** Let \( A \) and \( B \) be two disjoint commutative cancellative semigroups having an RZU. If \( A \) is a group then \( B \) is a group, and \( A \cong B \).

**Proof.** Every translation of a group is inner; thus \( T(A) = i(A) \). Now \( i(A) \) is the regular representation of \( A \); thus \( i(A) \cong A \). By Theorem 3.6, there exists an injection \( f \) of \( B \) into \( T(A) \) such that \( f(B)i(A) \subseteq f(B) \cap i(A) \). \( f \) is an injection into \( i(A) \). Since \( T(A) \) is commutative, \( f(B) \) is an ideal of \( i(A) \). But a group has no proper ideals. Thus \( f(B) \cong i(A) = A \). Since \( f \) is an injection \( B \cong A \).

We now give an interpretation of Theorem 3.6 in terms of quotient groups. Let \( A \) be a commutative cancellative semigroup. Let \( j \) be an injection of \( A \) as a set of generators into \( G \), the quotient group of \( A \). Let \( f \) be the isomorphism from \( T(A) \) onto \( TG(A) \) (\( TG(A) \) of Lemma 3.5; \( f = kh \) of Lemma 3.3). Let \( B \) be a commutative cancellative semigroup having an RZU with \( A \). Let \( h \) be an injection of \( B \) into \( T(A) \) such that \( h(B)i(A) \subseteq h(B) \cap i(A) \). Compose the maps \( h \) and \( f \). We have \( (fh)(B)j(A) \subseteq (fh)(B) \cap j(A) \). Evidently, \( B \) is isomorphic to \( B' \), a subsemigroup of \( TG(A) \) such that \( B'j(A) \subseteq B' \cap j(A) \). Conversely, an isomorphic copy of such a \( B' \) will have an RZU with \( A \). Thus we have a way of finding all commutative cancellative semigroups having an RZU with \( A \).

**References**


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